Design of Control Systems with Multiple Memoryless Nonlinearities for Inputs Restricted in Magnitude and Slope

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ABSTRACT
This paper develops a methodology for designing a control system composed of a linear time-invariant system interconnecting with multiple decoupled time-invariant memoryless nonlinearities. The design problem is to determine parameters of the system such that its outputs and the nonlinearity inputs always remain within prescribed bounds for all exogenous inputs whose magnitude and slope satisfy certain bounding conditions. By using Schauder fixed point theorem, we show that a design associated with a linear system is also a solution of the problem. Based on this, we further develop surrogate design criteria in the form of the inequalities that can readily be solved in practice. Sufficient conditions for the solvability of such inequalities are given for deadzone and saturation. To show the usefulness and the effectiveness of the methodology, a design example of a load frequency control system with time delay is carried out where deadzone and saturation are taken into account.

KEYWORDS
Control systems design; nonlinear control systems; memoryless nonlinearity; peak output; method of inequalities; BIBO stability.

1. Introduction

In many practical applications, some system variables are required to be kept within their prescribed bounds (or tolerances) for all time and for all inputs (or disturbances) that happen or are likely to happen. This requirement has prompted many researchers (e.g., Birch & Jackson, 1959; Horowitz, 1962; Bongiorno, 1967; Zakian, 1979b, 1986, 1987b, 1996, 2005; Lane, 1992, 1995; Rutland, 1994b; Reinelt, 2000; Silpsrikul & Arunsawatwong, 2010; Arunsawatwong & Chuman, 2017, and also the references therein) to investigate – for the case of linear time-invariant (LTI) systems – the following design criteria:

$$\hat{z}_i \le \varepsilon_i, \quad i = 1, 2, \ldots, m \tag{1}$$

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where $\hat{z}_i$ is a performance measure defined as

$$\hat{z}_i \triangleq \sup_{f \in \mathcal{F}} \|z_i\|_\infty, \tag{2}$$

the notation z_i denotes an output (or response) of the system under consideration which is caused by an exogenous input f, $\mathcal{F}$ is a set of continuous functions that satisfy certain norm-bounding conditions, ε_i is the largest value of $\|z_i\|_\infty$ that can be accepted, and $\|x\|_\infty$ denotes the infinity norm of a function x. The set $\mathcal{F}$ can be regarded as the set of all *possible inputs* (defined as inputs that happen or are likely to happen in practice); for this reason, $\mathcal{F}$ is called the *possible set* (Zakian, 1996, 2005). The performance measure $\hat{z}_i$ is called the *peak output* of z_i for the possible set $\mathcal{F}$. Notice that the criteria (1) are often used by practical engineers for monitoring the performance of control systems.

Once the peak outputs $\hat{z}_i$ (or their upper bounds $\tilde{z}_i$) can be evaluated efficiently, the criteria (1) (or $\tilde{z}_i \leq \varepsilon_i$) become useful design inequalities that can be solved by numerical methods, which is in keeping with the method of inequalties (Zakian & Al-Naib, 1973; Zakian, 1979b, 1996, 2005).

Important case studies (Rutland, 1992, 1994a; Lane, 1992, 1995; Whidborne, 1993, 2005; Ono & Inooka, 2009; Arunsawatwong, 2005; Silpsrikul & Arunsawatwong, 2010; Arunsawatwong & Kalvibool, 2016), where linear dynamical models are used, manifest that if the problem of designing control systems is formulated by using the criteria (1), then the problem is expressed in a realistic and useful manner. For more details regarding the criteria (1), readers are referred to Zakian (1996, 2005); Silpsrikul and Arunsawatwong (2010) and the references therein.

For LTI systems whose input and whose output are related by a convolution integral, Silpsrikul and Arunsawatwong (2010) develop a unified method for computing peak outputs defined in (2) for a class of possible sets described by

$$\mathcal{F} = \mathcal{P}_i \cap \dot{\mathcal{P}}_j \tag{3}$$

where integers $i, j \in \{0, 1, 2\}$ are specified; the sets $\mathcal{P}_i$ and $\dot{\mathcal{P}}_j$ are defined as

$$\mathcal{P}_i \triangleq \{f : \|f\|_k \leq M_k,\ k \in \mathcal{I}_i\}, \quad \dot{\mathcal{P}}_j \triangleq \{f : \|\dot{f}\|_k \leq D_k,\ k \in \mathcal{I}_j\},$$
$$\mathcal{I}_0 \triangleq \{2, \infty\}, \quad \mathcal{I}_1 \triangleq \{\infty\}, \quad \mathcal{I}_2 \triangleq \{2\};$$

the positive constants M_k, D_k are given by designers; and $\|x\|_k$ ($k = 2, \infty$) denotes the k-norm of a function x. Notice that the possible sets $\mathcal{P}_i \cap \dot{\mathcal{P}}_j$ contain time functions whose magnitude and whose slope satisfy norm-bounding conditions. The reasons for imposing the restrictions on both the magnitude and the slope of inputs can be found in Zakian (1996, 2005) and also Silpsrikul and Arunsawatwong (2010). In this method, the original infinite-dimensional convex optimization problem is approximated as a large-scale convex programme defined in a Euclidean space with sparse matrices, which can be solved efficiently by available tools. Moreover, it is worth noting that the description in (3) includes two notable cases that had been considered previously by many researchers: namely,

(I) $\mathcal{P}_1 \cap \dot{\mathcal{P}}_1 = \{f : \|f\|_\infty \leq M_\infty, \|\dot{f}\|_\infty \leq D_\infty\}$ (e.g., Birch & Jackson, 1959; Horowitz, 1962; Bongiorno, 1967; Zakian, 1979b; Lane, 1992; Reinelt, 2000),

(II) $\mathcal{P}_2 \cap \dot{\mathcal{P}}_2 = \{f : \|f\|_2 \leq M_2, \|\dot{f}\|_2 \leq D_2\}$ (e.g., Lane, 1995).

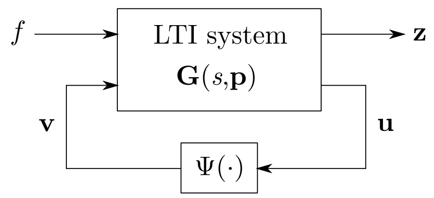


Figure 1. System with multiple nonlinearities where $\Psi(\mathbf{u}) = [\psi_1(u_1), \psi_2(u_2), \ldots, \psi_n(u_n)]^T$ and $f \in \mathcal{F}$

For detailed discussion on this and a complete list of related references, see Silpsrikul and Arunsawatwong (2010). Hence, for the whole class of possible sets in (3), not only can the method be used for computing the peak outputs $\hat{z}_i$ but it also enables ones to design linear control systems to fulfil the criteria (1) in conjunction with the method of inequalities.

In this paper, we consider a more general and practical control system as shown in Figure 1, where: $\mathbf{G}(s, \mathbf{p})$ is the transfer matrix of an LTI system; $\mathbf{p} \triangleq [p_1, p_2, \ldots, p_N]^T$ is a vector of design parameters (p_i may be, for example, the coefficients in a controller transfer function); $\Psi \triangleq [\psi_1, \psi_2, \ldots, \psi_n]^T$ is the vector of n decoupled time-invariant and memoryless nonlinearities that are continuous functions; f is an exogenous input known to the extent that it belongs to a set $\mathcal{F}$; $\mathcal{F}$ is one of the possible sets described in (3); $\mathbf{z} \triangleq [z_1, z_2, \ldots, z_m]^T$ is the vector of m outputs of interest; $\mathbf{u} \triangleq [u_1, u_2, \ldots, u_n]^T$ is the vector of the nonlinearity inputs; $\mathbf{v} \triangleq [v_1, v_2, \ldots, v_n]^T$ is the vector of the nonlinearity outputs. The details of the system model are given in equations (8).

The aim of the paper is to develop a methodology for determining a vector $\mathbf{p} \in \mathbb{R}^N$ satisfying the following design criteria.

$$\hat{z}_i(\mathbf{p}) \leq \varepsilon_i, \quad i = 1, 2, \ldots, m \tag{4a}$$
$$\hat{u}_j(\mathbf{p}) \leq \sigma_j, \quad j = 1, 2, \ldots, n \tag{4b}$$

where: $\varepsilon_i > 0$ and $\sigma_j > 0$ represent the bounds or tolerances given by designers; $\mathcal{F}$ is one of the possible sets given in (3); $\hat{z}_i(\mathbf{p})$ and $\hat{u}_j(\mathbf{p})$ are respectively the peak outputs of z_i and u_j for the set $\mathcal{F}$, defined as

$$\hat{z}_i(\mathbf{p}) \triangleq \sup_{f \in \mathcal{F}} \|z_i\|_\infty \quad \text{and} \quad \hat{u}_j(\mathbf{p}) \triangleq \sup_{f \in \mathcal{F}} \|u_j\|_\infty. \tag{5}$$

Any vector $\mathbf{p}$ that satisfies (4) is called a *design solution*. From Figure 1, it is clear that the responses z_i and u_j depend on both f and $\mathbf{p}$. For simplicity of notation, we simply write z_i and u_j throughout the paper.

Since computing $\hat{z}_i(\mathbf{p})$ and $\hat{u}_j(\mathbf{p})$ is intractable, we derive a sufficient condition for (4) in the form of the following inequalities.

$$\hat{z}_i^*(\mathbf{p}) \leq \varepsilon_i, \quad i = 1, 2, \ldots, m \tag{6a}$$
$$\hat{u}_j^*(\mathbf{p}) \leq \sigma_j, \quad j = 1, 2, \ldots, n \tag{6b}$$

where $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ are the peak outputs of a certain linear system (to be called a nominal system in Section 3 and thereafter) and thereby can readily be obtained by known results for linear systems (see Section 3 for details). Consequently, inequalities (6) can be solved by numerical methods and hence will be used to obtain a solution of the criteria (4).

It should be noted that in connection with the present paper, preliminary investigations were carried out for the case of single-loop unity-feedback systems consisting of an LTI controller $G_c(s, \mathbf{p})$, a time-invariant nonlinearity ψ and an LTI plant $G_p(s)$ where the design problem is to determine a design parameter vector $\mathbf{p}$ such that for the possible set $\mathcal{P}_1 \cap \dot{\mathcal{P}}_1$, the peak outputs of the error and the controller output do not exceed prescribed bounds. Specifically, Mai, Arunsawatwong, and Abed (2010a, 2010b, 2011) considered the case of nonlinearities ψ that are continuous and memoryless, whereas Nguyen and Arunsawatwong (2013, 2014) investigated the case of backlash nonlinearities ψ that are represented by an uncertain band model containing multi-valued functions.

The methodology developed in this paper can be seen as a generalization of the previous results in Mai et al. (2010a, 2010b) so that control systems with general configuration and multiple nonlinearities can be treated. More specifically, the main contributions of the present paper are as follows.

- We derive a sufficient condition for the design criteria (4) for possible sets consisting of bounded and continuous functions, which is presented in Theorem 2.6. This is the key result for subsequent development in the paper.
- By applying Theorem 2.6 to the case of the possible sets described in (3), we derive inequalities (6). This result is presented in Theorem 3.1.
- Conditions for the solvability of inequalities (6b) are given in connection with deadzone and saturation nonlinearities. A sufficient condition for the solvability of (6b) associated with deadzones is provided in Proposition 5.2, whereas a sufficient condition for the solvability of (6b) associated with one saturation is provided in Theorem 5.4. Both conditions are used to facilitate the numerical solution of (6).
- A design example of a load frequency control (LFC) system is carried out where deadzone and saturation nonlinearities are explicitly taken into account. The numerical results show that the controller obtained by neglecting the nonlinearities may lead to unacceptable results or instability, whereas the controller obtained by using the developed methodology (if exists) provides satisfactory results for the nonlinear LFC system.

The structure of the paper is as follows. Section 2 presents the main theoretical result, which provides a sufficient condition for the criteria (4). Section 3 derives the design inequalities (6), which will be used for obtaining a solution of (4) for the possible sets in (3). Section 4 presents a useful inequality for ensuring the finiteness of $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ in connection with a known class of LTI systems; with this inequality, algorithms for computing a solution of (6) are able to start from an arbitrary point in the parameter space $\mathbb{R}^N$. Section 5 presents conditions for the solvability of inequalities (6b) for deadzone and saturation nonlinearities. In Section 6, the design example of the LFC system is presented so as to illustrate the usefulness of the developed method. Finally, conclusions and discussion are given in Section 7.

2. Main Theoretical Result

This section presents the main theoretical result, which is presented in Theorem 2.6. The result provides a foundation for the development in Section 3.

The following notations are used throughout this paper. Let $\mathbb{R}_+ \triangleq [0, \infty)$. For a function $x : \mathbb{R}_+ \to \mathbb{R}$, let the norms $\|x\|_1$, $\|x\|_2$ and $\|x\|_\infty$ be defined as $\|x\|_1 = \int_0^\infty |x(t)|\mathrm{d}t$,

$\|x\|_2 = \left[\int_0^\infty x^2(t)\mathrm{d}t\right]^{1/2}$ and $\|x\|_\infty = \sup_{t\geq 0}|x(t)|$. Let $\mathbb{L}^i$ $(i = 1, 2, \infty)$ denote the set of functions $x : \mathbb{R}_+ \to \mathbb{R}$ such that $\|x\|_i < \infty$. Define $\mathbb{L}_n^\infty \triangleq \underbrace{\mathbb{L}^\infty \times \mathbb{L}^\infty \times \ldots \times \mathbb{L}^\infty}_{n}$. For any $\mathbf{x} \triangleq [x_1, x_2, \ldots, x_n]^T \in \mathbb{L}_n^\infty$, define the norm

$$||\mathbf{x}|| \triangleq \sum_{i=1}^{n} ||x_i||_\infty. \tag{7}$$

For any $\mathbf{x} : \mathbb{R}_+ \to \mathbb{R}^n$ and for a fixed $T > 0$, define the truncated function $\mathbf{x}_T$ as follows.

$$\mathbf{x}_T(t) \triangleq \begin{cases} \mathbf{x}(t), & 0 \leq t \leq T \\ \mathbf{0}, & t > T \end{cases}.$$

For $X \subseteq \mathbb{L}_n^\infty$, define the truncated set X_T as $X_T \triangleq \{\mathbf{x}_T : \mathbf{x} \in X\}$. For any operator $\mathcal{H} : X \subseteq \mathbb{L}_n^\infty \to \mathbb{L}_n^\infty$ and for any set $M \subseteq X$, let $\mathcal{H}(M)$ denote the image of M under the operator $\mathcal{H}$, i.e., $\mathcal{H}(M) = \{\mathcal{H}(\mathbf{x}) : \mathbf{x} \in M\}$. The symbol $*$ denotes the convolution; i.e., for $x : \mathbb{R}_+ \to \mathbb{R}$ and $y : \mathbb{R}_+ \to \mathbb{R}$,

$$(x * y)(t) = \int_0^t x(t-\tau)y(\tau)\mathrm{d}\tau, \quad t > 0.$$

Now consider the mathematical description of the control system in Figure 1. To this end, let $\mathbf{G}(s, \mathbf{p})$ be the transfer matrix from $[f, \mathbf{v}^T]^T$ to $[\mathbf{z}^T, \mathbf{u}^T]^T$ where

$$\begin{gathered} \mathbf{G}(s,\mathbf{p}) = \begin{bmatrix} \mathbf{G}_f^z(s,\mathbf{p}) & \mathbf{G}_v^z(s,\mathbf{p}) \\ \mathbf{G}_f^u(s,\mathbf{p}) & \mathbf{G}_v^u(s,\mathbf{p}) \end{bmatrix}, \\ \mathbf{G}_f^z(s,\mathbf{p}) \triangleq [G_f^{zi}(s,\mathbf{p})]_{m\times 1}, \quad \mathbf{G}_v^z(s,\mathbf{p}) \triangleq [G_{vk}^{zi}(s,\mathbf{p})]_{m\times n}, \\ \mathbf{G}_f^u(s,\mathbf{p}) \triangleq [G_f^{uj}(s,\mathbf{p})]_{n\times 1}, \quad \mathbf{G}_v^u(s,\mathbf{p}) \triangleq [G_{vk}^{uj}(s,\mathbf{p})]_{n\times n}. \end{gathered}$$

And let $g_f^{zi}(\mathbf{p}), g_{vk}^{zi}(\mathbf{p}), g_f^{uj}(\mathbf{p})$ and $g_{vk}^{uj}(\mathbf{p})$ denote the inverse Laplace transforms of $G_f^{zi}(s,\mathbf{p})$, $G_{vk}^{zi}(s,\mathbf{p})$, $G_f^{uj}(s,\mathbf{p})$ and $G_{vk}^{uj}(s,\mathbf{p})$, respectively. Notice that the notation $G_{yk}^{xi}(s,\mathbf{p})$ denotes the transfer function from input y_k to response x_i. Then the control system is described by

$$\left.\begin{aligned} z_i &= g_f^{zi}(\mathbf{p}) * f + \sum_{k=1}^{n} g_{vk}^{zi}(\mathbf{p}) * v_k, \quad i = 1, 2, \ldots, m \\ u_j &= g_f^{uj}(\mathbf{p}) * f + \sum_{k=1}^{n} g_{vk}^{uj}(\mathbf{p}) * v_k, \quad j = 1, 2, \ldots, n \end{aligned}\right\} \tag{8a}$$

$$v_j = \psi_j(u_j), \quad j = 1, 2, \ldots, n \tag{8b}$$

where $f \in \mathcal{F}$. Notice that equations (8a) and (8b) describe the LTI system and the nonlinearities of the system, respectively.

Assumption 1. *The LTI system* (8a) *is non-anticipative and has zero initial conditions.*

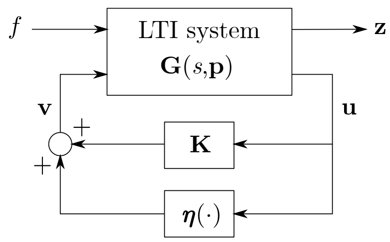


Figure 2. Equivalent system of the original nonlinear system (8) where $f \in \mathcal{F}$

Assumption 2. *Each nonlinearity $\psi_j : \mathbb{R} \to \mathbb{R}$ $(j = 1, 2, \ldots, n)$ is a continuous and time-invariant nonlinear function.*

Assumption 3. *There are unique $\mathbf{z} : \mathbb{R}_+ \to \mathbb{R}^m$ and $\mathbf{u} : \mathbb{R}_+ \to \mathbb{R}^n$ that satisfy* (8) *for every input $f \in \mathcal{F}$.*

In the following, the nonlinear functions ψ_i are decomposed as

$$\psi_j(u_j) = K_j u_j + \eta_j(u_j), \quad j = 1, 2, \ldots, n \tag{9}$$

where $K_j \in \mathbb{R}$ and $\eta_j : \mathbb{R} \to \mathbb{R}$. Then the system (8) becomes equivalent to the system shown in Figure 2 where

$$\mathbf{K} \triangleq \operatorname{diag}(K_1, K_2, \ldots, K_n) \quad \text{and} \quad \boldsymbol{\eta}(\mathbf{u}) \triangleq [\eta_1(u_1), \eta_2(u_2), \ldots, \eta_n(u_n)]^T.$$

Notice that, as a consequence of Assumption 2, $\boldsymbol{\eta}(\mathbf{u})$ is bounded whenever $\mathbf{u}$ is bounded.

Oldak, Baril, and Gutman (1994) used the decomposition (9) in connection with quantitative feedback theory for designing control systems with hard nonlinearities that can be expressed as

$$\psi_j(u_j) = K_j u_j + \eta_j(u_j),\ j = 1, 2, \ldots, n \quad \text{where} \quad |\eta_j(u_j)| \leq C < \infty \ \forall\, u_j \in \mathbb{R}. \tag{10}$$

Equations (10) include many time-invariant nonlinearities found in practice such as deadzone, saturation, dry friction, backlash, etc. Note, however, that the class of nonlinearities considered here is different from that in Oldak et al. (1994). For example, asymmetric deadzone and polynomial functions satisfy Assumption 2 but not condition (10).

Now let $\mathcal{U}$ denote the set of additional input vectors, defined as

$$\mathcal{U} \triangleq \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_n, \quad \mathcal{U}_j \triangleq \{w_j \in \mathbb{L}^\infty : \|w_j\|_\infty \leq \sigma_j\} \tag{11}$$

where $A \times B$ is the Cartesian product of sets A and B. By replacing each $\eta_j(u_j)$ with

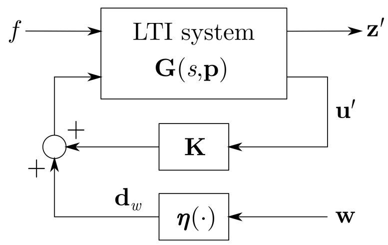


Figure 3. Auxiliary system of the original nonlinear system (8) where $f \in \mathcal{F}$ and $\mathbf{w} \in \mathcal{U}$

$\eta_j(w_j)$ for $w_j \in \mathcal{U}_j$, the equivalent system in Figure 2 becomes

$$z'_i = g_f^{zi}(\mathbf{p}) * f + \sum_{k=1}^{n} g_{vk}^{zi}(\mathbf{p}) * \big(K_k u'_k + d_{wk}\big), \quad i = 1, 2, \ldots, m \tag{12a}$$

$$u'_j = g_f^{uj}(\mathbf{p}) * f + \sum_{k=1}^{n} g_{vk}^{uj}(\mathbf{p}) * \big(K_k u'_k + d_{wk}\big), \quad j = 1, 2, \ldots, n \tag{12b}$$

$$d_{wj} = \eta_j(w_j), \quad j = 1, 2, \ldots, n \tag{12c}$$

where $f \in \mathcal{F}$ and $\mathbf{w} \triangleq [w_1, w_2, \ldots, w_n]^T \in \mathcal{U}$. The system (12) is depicted in Figure 3 where $\mathbf{z}' \triangleq [z'_1, z'_2, \ldots, z'_m]^T$, $\mathbf{u}' \triangleq [u'_1, u'_2, \ldots, u'_n]^T$ and $\mathbf{d}_w \triangleq [d_{w1}, d_{w2}, \ldots, d_{wn}]^T$, and hereafter is called the *auxiliary system*. Define the peak outputs $\hat{z}'_i(\mathbf{p})$ and $\hat{u}'_j(\mathbf{p})$ for the system (12) as follows.

$$\hat{z}'_i(\mathbf{p}) \triangleq \sup_{f \in \mathcal{F}, \mathbf{w} \in \mathcal{U}} \|z'_i\|_\infty \quad \text{and} \quad \hat{u}'_j(\mathbf{p}) \triangleq \sup_{f \in \mathcal{F}, \mathbf{w} \in \mathcal{U}} \|u'_j\|_\infty. \tag{13}$$

Consider the system (12) and let $\mathbf{H}(s, \mathbf{p})$ be the transfer matrix from $[f, \mathbf{d}_w^T]^T$ to $[\mathbf{z}'^T, \mathbf{u}'^T]^T$. Then it is easy to verify that

$$\begin{gathered} \mathbf{H}(s,\mathbf{p}) \triangleq \begin{bmatrix} \mathbf{H}_f^z(s,\mathbf{p}) & \mathbf{H}_d^z(s,\mathbf{p}) \\ \mathbf{H}_f^u(s,\mathbf{p}) & \mathbf{H}_d^u(s,\mathbf{p}) \end{bmatrix}, \\ \mathbf{H}_f^z(s,\mathbf{p}) \triangleq [H_f^{zi}(s,\mathbf{p})]_{m\times 1}, \quad \mathbf{H}_d^z(s,\mathbf{p}) \triangleq [H_{dk}^{zi}(s,\mathbf{p})]_{m\times n}, \\ \mathbf{H}_f^u(s,\mathbf{p}) \triangleq [H_f^{uj}(s,\mathbf{p})]_{n\times 1}, \quad \mathbf{H}_d^u(s,\mathbf{p}) \triangleq [H_{dk}^{uj}(s,\mathbf{p})]_{n\times n}, \end{gathered} \tag{14}$$

$$\mathbf{H}_f^z(s,\mathbf{p}) = \mathbf{G}_f^z(s,\mathbf{p}) + \mathbf{G}_v^z(s,\mathbf{p})\mathbf{K}[I - \mathbf{G}_v^u(s,\mathbf{p})\mathbf{K}]^{-1}\mathbf{G}_f^u(s,\mathbf{p}), \tag{15a}$$

$$\mathbf{H}_d^z(s,\mathbf{p}) = \mathbf{G}_v^z(s,\mathbf{p})[I - \mathbf{K}\mathbf{G}_v^u(s,\mathbf{p})]^{-1}, \tag{15b}$$

$$\mathbf{H}_f^u(s,\mathbf{p}) = [I - \mathbf{G}_v^u(s,\mathbf{p})\mathbf{K}]^{-1}\mathbf{G}_f^u(s,\mathbf{p}), \tag{15c}$$

$$\mathbf{H}_d^u(s,\mathbf{p}) = [I - \mathbf{G}_v^u(s,\mathbf{p})\mathbf{K}]^{-1}\mathbf{G}_v^u(s,\mathbf{p}). \tag{15d}$$

Furthermore, let $h_f^{zi}(\mathbf{p})$, $h_{dk}^{zi}(\mathbf{p})$, $h_f^{uj}(\mathbf{p})$ and $h_{dk}^{uj}(\mathbf{p})$ denote the inverse Laplace transforms of $H_f^{zi}(s,\mathbf{p})$, $H_{dk}^{zi}(s,\mathbf{p})$, $H_f^{uj}(s,\mathbf{p})$ and $H_{dk}^{uj}(s,\mathbf{p})$, respectively.

Theorem 2.1. [BIBO Stability (see, e.g., Chen, 1984)]. *Consider a single-input single-output LTI system whose input $x : \mathbb{R}_+ \to \mathbb{R}$ and output $y : \mathbb{R}_+ \to \mathbb{R}$ are related by*

$$y(t) = (g * x)(t)$$

where g is the impulse response of the system and all initial conditions at $t = 0$ are zero. The system is bounded-input bounded-output (BIBO) stable if and only if

$$\int_0^\infty |g(t)|\mathrm{d}t < \infty.$$

In connection with Theorem 2.1, the following definition will be used in the paper for convenience.

Definition 2.2. A transfer function $G(s)$ is said to be BIBO stable if

$$\int_0^\infty |g(t)|\mathrm{d}t < \infty$$

where g is the inverse Laplace transform of $G(s)$.

Assumption 4. *The transfer functions $H_f^{uj}(s, \mathbf{p})$ $(j = 1, 2, \ldots, n)$ are proper and BIBO stable and the transfer functions $H_{dk}^{uj}(s, \mathbf{p})$ $(j, k = 1, 2, \ldots, n)$ are strictly proper and BIBO stable.*

Next the main result is stated in Theorem 2.6 and, in essence, is obtained by using the Schauder fixed point theorem in the time domain. Prior to this work, Baños and Horowitz (2004) had used the fixed point theorem in the time domain to develop design methods. However, it is important to note that we consider the control system (8) in connection with the design criteria (4), whereas the problem settings considered by Baños and Horowitz (2004) are different.

Theorem 2.3. [Schauder theorem (Zeidler, 1986, p. 56)]. *Suppose that Ω is a nonempty, closed, bounded and convex subset of a Banach space. Every compact operator $\Phi : \Omega \to \Omega$ has a fixed point.*

The following results will be used in the proof of Theorem 2.6. It may be noted that Lemma 2.4 is a key tool in proving the theorem.

Lemma 2.4. *Let $X \subset \mathbb{L}_n^\infty$. For every $T > 0$, let $\mathcal{H}$ denote the affine operator defined over X_T as*

$$\left.\begin{aligned} \mathcal{H}\mathbf{x}(t) &= [\mathcal{H}_1\mathbf{x}(t), \mathcal{H}_2\mathbf{x}(t), \ldots, \mathcal{H}_n\mathbf{x}(t)]^T \\ \mathcal{H}_j\mathbf{x}(t) &= \sum_{k=1}^n (h_k^j * x_k)(t) + r_j(t) \end{aligned}\right\}, \quad \forall\, t \in [0, T] \tag{16}$$

where $h_k^j : \mathbb{R}_+ \to \mathbb{R}$ are given, and $r_j : [0, T] \to \mathbb{R}$ are continuous and satisfy $\|r_j\|_\infty \leq M_r$ for some $M_r < \infty$. If $h_k^j \in \mathbb{L}^1$ for all j, k, then $\mathcal{H}$ is compact.

Proof. See Appendix A. □

Proposition 2.5. *Let X, Y and Z be Banach spaces. Let $\mathcal{Q} : D_1 \subset X \to Y$ be a continuous operator, and let $\mathcal{H} : Y \to Z$ be an operator that is compact over $D_2 \subset Y$. If $\mathcal{Q}(D_1) \subset D_2$, then the composite operator $\mathcal{H}\mathcal{Q} : D_1 \to Z$ is compact.*

Proof. See Appendix A. □

We are now ready to state the main result of this section.

Theorem 2.6. *Consider the nonlinear system* (8) *and assume that the possible set* $\mathcal{F}$ *is a set of bounded and continuous functions. Let Assumptions 1–4 hold. Then the original design criteria* (4) *are satisfied if, for the auxiliary system* (12)*, the following inequalities hold.*

$$\hat{z}_i'(\mathbf{p}) \le \varepsilon_i, \quad i = 1, 2, \ldots, m, \tag{17a}$$
$$\hat{u}_j'(\mathbf{p}) \le \sigma_j, \quad j = 1, 2, \ldots, n. \tag{17b}$$

Proof. First, we prove that if condition (17b) holds, then the design criteria (4b) are satisfied. Let $f \in \mathcal{F}$ be a fixed input and let (17b) hold. Consider the auxiliary system (12). From (12b), (12c), (15c) and (15d), it can be verified that for $j = 1, 2, \ldots, n$,

$$u_j' = \sum_{k=1}^{n} h_{dk}^{uj}(\mathbf{p}) * \eta_k(w_k) + h_f^{uj}(\mathbf{p}) * f \triangleq \Phi_j(\mathbf{w}) \tag{18}$$

for any input vector $\mathbf{w}$. Let $T > 0$ be fixed. Truncating to both sides of (18) yields

$$u_{j,T}' = \left(\sum_{k=1}^{n} h_{dk}^{uj}(\mathbf{p}) * \eta_k(w_{k,T}) + h_f^{uj}(\mathbf{p}) * f \right)_T \triangleq \Phi_{j,T}(\mathbf{w}_T). \tag{19}$$

In connection with (19), define $\Phi_T(\mathbf{w}_T) \triangleq [\Phi_{1,T}(\mathbf{w}_T), \ldots, \Phi_{n,T}(\mathbf{w}_T)]^T$.

Now let $\mathbf{w} \in \mathcal{U}$. Then (17b) implies that $\mathbf{u}' \in \mathcal{U}$ and thus $\mathbf{u}_T' \in \mathcal{U}_T$. Hence, $\Phi_T(\mathcal{U}_T) \subseteq \mathcal{U}_T$; i.e., Φ_T maps $\mathcal{U}_T$ into itself. Then we use the Schauder theorem to show that Φ_T has a fixed point in $\mathcal{U}_T$. To this end, write $\Phi_T = \mathcal{H}\mathcal{Q}$ where $\mathcal{Q}\mathbf{x} = \boldsymbol{\eta}(\mathbf{x})$,

$$\mathcal{H}\mathbf{x} = [\mathcal{H}_1\mathbf{x}, \mathcal{H}_2\mathbf{x}, \ldots, \mathcal{H}_n\mathbf{x}]^T \quad \text{and} \quad \mathcal{H}_j\mathbf{x} = \left(\sum_{k=1}^{n} h_{dk}^{uj}(\mathbf{p}) * x_k + h_f^{uj}(\mathbf{p}) * f \right)_T.$$

Since $H_f^{uj}(s, \mathbf{p})$ $(j = 1, 2, \ldots, n)$ are BIBO stable and f is bounded and continuous, we can verify that $r_j \triangleq \left(h_f^{uj}(\mathbf{p}) * f \right)_T$ $(j = 1, 2, \ldots, n)$ are continuous and satisfy $\|r_j\|_\infty \le M_r$ for some $M_r < \infty$. Furthermore, the condition that $H_{dk}^{uj}(s, \mathbf{p})$ $(j, k = 1, 2, \ldots, n)$ are strictly proper and BIBO stable implies that $h_{dk}^{uj}(\mathbf{p}) \in \mathbb{L}^1$. Consequently, it follows from Lemma 2.4 that $\mathcal{H}$ is compact over $\mathcal{U}_T$. Because of this and by the continuity of $\mathcal{Q}$, it follows from Proposition 2.5 that Φ_T is compact over $\mathcal{U}_T$, which is clearly a nonempty, closed, bounded and convex set. By applying Theorem 2.3, one can see that Φ_T always has a fixed point $\mathbf{u}_T^\dagger \triangleq [u_{1,T}^\dagger, u_{2,T}^\dagger, \ldots, u_{n,T}^\dagger]^T \in \mathcal{U}_T$ such that

$$\mathbf{u}_T^\dagger = \Phi_T(\mathbf{u}_T^\dagger). \tag{20}$$

From (19) and (20), it follows immediately that for each j,

$$u_{j,T}^\dagger = \left(h_f^{uj}(\mathbf{p}) * f + \sum_{k=1}^{n} h_{dk}^{uj}(\mathbf{p}) * \eta_k(u_{k,T}^\dagger) \right)_T.$$

Consequently, by using (15c) and (15d), one can verify that for each j,

$$\begin{aligned} u^{\dagger}_{j,T} &= \left(g^{uj}_{f}(\mathbf{p}) * f + \sum_{k=1}^{n} g^{uj}_{vk}(\mathbf{p}) * \{K_k u^{\dagger}_{k,T} + \eta_k(u^{\dagger}_{k,T})\} \right)_T \\ &= \left(g^{uj}_{f}(\mathbf{p}) * f + \sum_{k=1}^{n} g^{uj}_{vk}(\mathbf{p}) * \psi_k(u^{\dagger}_{k,T}) \right)_T \end{aligned} \tag{21}$$

where the second equality follows by the decomposition in (9). By the uniqueness of $\mathbf{u}$ in Assumption 3, it follows from (21) that $\mathbf{u}^{\dagger}_T$ is identical to $\mathbf{u}_T$ of the original nonlinear system (8). Thus, $\mathbf{u}_T \in \mathcal{U}_T$ since $\mathbf{u}^{\dagger}_T \in \mathcal{U}_T$. Because the above arguments hold for any $f \in \mathcal{F}$ and any $T > 0$, we conclude that the criteria (4b) are satisfied.

Second, we prove that condition (17a) and the consequence of condition (17b) imply that the design criteria (4a) are satisfied. Let (17a) hold and let $\mathbf{z}^{\dagger} \triangleq [z^{\dagger}_{1,T}, z^{\dagger}_{2,T}, \ldots, z^{\dagger}_{m,T}]^T$ denote the vector of the associated outputs of the auxiliary system (12) when $\mathbf{w} = \mathbf{u}^{\dagger}_T$. Following (12b), (12c) and (21), one can verify that $\mathbf{u}'_T = \mathbf{u}^{\dagger}_T$ when $\mathbf{w} = \mathbf{u}^{\dagger}_T$. Consequently, from (12a) and (12c), it follows that for each i,

$$\begin{aligned} z^{\dagger}_{i,T} &= \left(g^{zi}_{f}(\mathbf{p}) * f + \sum_{k=1}^{n} g^{zi}_{vk}(\mathbf{p}) * \{K_k u^{\dagger}_{k,T} + \eta_k(u^{\dagger}_{k,T})\} \right)_T \\ &= \left(g^{zi}_{f}(\mathbf{p}) * f + \sum_{k=1}^{n} g^{zi}_{vk}(\mathbf{p}) * \psi_k(u^{\dagger}_{k,T}) \right)_T \end{aligned} \tag{22}$$

where the second equality follows by using (9) again. Then, by Assumption 3, one can see that $\mathbf{z}^{\dagger}_T$ is identical to $\mathbf{z}_T$ of the original nonlinear system (8). Moreover, condition (17a) implies that $\|z^{\dagger}_{i,T}\|_\infty \le \varepsilon_i$ $(i = 1, 2, \ldots, m)$. For these reasons, it follows that $\|z_{i,T}\|_\infty \le \varepsilon_i$ $(i = 1, 2, \ldots, m)$. Finally, since the above arguments hold for any $f \in \mathcal{F}$ and any $T > 0$, the criteria (4a) are satisfied. □

Regarding the possible set $\mathcal{F}$, Theorem 2.6 uses only the assumption that $\mathcal{F}$ comprises bounded and continuous functions. For this reason, it should be noted that apart from the possible sets $\mathcal{F}$ in (3), Theorem 2.6 is also applicable to other possible sets $\mathcal{F}$ as long as all inputs in $\mathcal{F}$ are bounded and continuous.

From Theorem 2.6, we see that when a solution of inequalities (17) (associated with the auxiliary system (12) subject to f and $\mathbf{d}_w$) is found, it is also a solution of the criteria (4) (associated with the nonlinear system (8) subject to f). However, the peak outputs $\hat{z}'_i(\mathbf{p})$ and $\hat{u}'_j(\mathbf{p})$ are in general not easily obtainable, because $\mathbf{d}_w$ is a nonlinear function of $\mathbf{w}$ for $\mathbf{w} \in \mathcal{U}$. Therefore, solving inequalities (17) is not convenient, and more tractable design inequalities will be developed in Section 3.

3. Practical Design Inequalities and Evaluation of Associated Peak Outputs

Based on Theorem 2.6, this section develops a practical sufficient condition for (4) in connection with the possible sets $\mathcal{F}$ in (3).

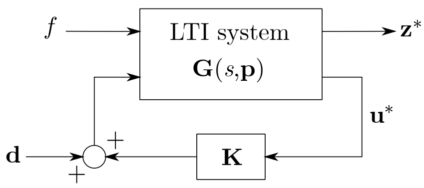


Figure 4. Nominal system where $f \in \mathcal{F}$ and $\mathbf{d} \in \mathcal{D}$

Consider the auxiliary system (12). Let $\mathcal{D}_w$ denote the set of all input vectors $\mathbf{d}_w$ for $\mathbf{w} \in \mathcal{U}$. Then the set $\mathcal{D}_w$ is given by

$$\mathcal{D}_w = \mathcal{D}_{w1} \times \mathcal{D}_{w2} \times \ldots \times \mathcal{D}_{wn} \quad \text{where} \quad \mathcal{D}_{wj} = \{d_{wj} = \eta_j(w_j) : \; w_j \in \mathcal{U}_j\}. \tag{23}$$

From (13), it follows that the peak outputs $\hat{z}_i'(\mathbf{p})$ and $\hat{u}_j'(\mathbf{p})$ can be rewritten as

$$\hat{z}_i'(\mathbf{p}) = \sup_{f \in \mathcal{F}, \mathbf{d}_w \in \mathcal{D}_w} \|z_i'\|_\infty \quad \text{and} \quad \hat{u}_j'(\mathbf{p}) = \sup_{f \in \mathcal{F}, \mathbf{d}_w \in \mathcal{D}_w} \|u_j'\|_\infty. \tag{24}$$

Now let $\mathcal{D}$ denote the set of additional input vectors, defined as

$$\mathcal{D} \triangleq \mathcal{D}_1 \times \mathcal{D}_2 \times \ldots \times \mathcal{D}_n, \quad \mathcal{D}_j \triangleq \{d_j \in \mathbb{L}^\infty : \|d_j\|_\infty \le \mathcal{M}_j\} \tag{25}$$

where the bounds $\mathcal{M}_j$ are given by

$$\mathcal{M}_j \triangleq \sup_{|x| \le \sigma_j} |\eta_j(x)|, \quad j = 1, 2, \ldots, n. \tag{26}$$

Notice that, for each j, the bound $\mathcal{M}_j$ is always finite since η_j is continuous. Furthermore, since $d_{wj} = \eta_j(w_j)$, it follows from (26) that $\|d_{wj}\|_\infty \le \mathcal{M}_j$ for all j. Thus, it is clear from (23) and (25) that $\mathcal{D}_w \subset \mathcal{D}$.

Replacing each d_{wj} in the auxiliary system (12) with $d_j \in \mathcal{D}_j$ yields the following LTI system.

$$\begin{aligned} z_i^* &= g_f^{zi}(\mathbf{p}) * f + \sum_{k=1}^{n} g_{vk}^{zi}(\mathbf{p}) * (K_k u_k^* + d_k), \quad i = 1, 2, \ldots, m \\ u_j^* &= g_f^{uj}(\mathbf{p}) * f + \sum_{k=1}^{n} g_{vk}^{uj}(\mathbf{p}) * (K_k u_k^* + d_k), \quad j = 1, 2, \ldots, n \end{aligned} \tag{27}$$

where $f \in \mathcal{F}$ and $\mathbf{d} \triangleq [d_1, d_2, \ldots, d_n]^T \in \mathcal{D}$. The system (27) is depicted in Figure 4 where $\mathbf{z}^* \triangleq [z_1^*, z_2^*, \ldots, z_m^*]^T$ and $\mathbf{u}^* \triangleq [u_1^*, u_2^*, \ldots, u_n^*]^T$, and hereafter is called the *nominal system*. The only difference between the auxiliary system (12) and the nominal system (27) is that $\mathbf{d}_w$ is replaced with $\mathbf{d}$. So the transfer matrix of the system (27) from $[f, \mathbf{d}^T]^T$ to $[\mathbf{z}^{*T}, \mathbf{u}^{*T}]^T$ is identical to $\mathbf{H}(s, \mathbf{p})$ described in (14) and (15).

Define the peak outputs $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ for the nominal system (27) as follows.

$$\hat{z}_i^*(\mathbf{p}) \triangleq \sup_{f \in \mathcal{F}, \mathbf{d} \in \mathcal{D}} \|z_i^*\|_\infty \quad \text{and} \quad \hat{u}_j^*(\mathbf{p}) \triangleq \sup_{f \in \mathcal{F}, \mathbf{d} \in \mathcal{D}} \|u_j^*\|_\infty. \tag{28}$$

Then we state the sufficient condition for the criteria (4) in connection with the peak outputs $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$.

Theorem 3.1. *Consider the nonlinear system* (8) *and let the possible set* $\mathcal{F}$ *be given by* (3). *Let Assumptions 1–4 hold. Then the original design criteria* (4) *are satisfied if, for the nominal system* (27)*, the following inequalities hold.*

$$\hat{z}_i^*(\mathbf{p}) \le \varepsilon_i, \quad i = 1, 2, \ldots, m \tag{29a}$$

$$\hat{u}_j^*(\mathbf{p}) \le \sigma_j, \quad j = 1, 2, \ldots, n \tag{29b}$$

where the peak outputs $\hat{z}_i^*(\mathbf{p})$ *and* $\hat{u}_j^*(\mathbf{p})$ *are given by*

$$\hat{z}_i^*(\mathbf{p}) = \phi_f^{zi}(\mathbf{p}) + \sum_{k=1}^{n} \mathcal{M}_k \|h_{dk}^{zi}(\mathbf{p})\|_1, \quad i = 1, 2, \ldots, m, \tag{30a}$$

$$\hat{u}_j^*(\mathbf{p}) = \phi_f^{uj}(\mathbf{p}) + \sum_{k=1}^{n} \mathcal{M}_k \|h_{dk}^{uj}(\mathbf{p})\|_1, \quad j = 1, 2, \ldots, n, \tag{30b}$$

$$\phi_f^{zi}(\mathbf{p}) \triangleq \sup\{\|z_i^*\|_\infty : f \in \mathcal{F}, \mathbf{d} = \mathbf{0}\}, \quad \phi_f^{uj}(\mathbf{p}) \triangleq \sup\{\|u_j^*\|_\infty : f \in \mathcal{F}, \mathbf{d} = \mathbf{0}\}. \tag{31}$$

Proof. Since $\mathcal{D}_w \subset \mathcal{D}$, it readily follows that $\hat{z}_i'(\mathbf{p}) \le \hat{z}_i^*(\mathbf{p})$ $(i = 1, 2, \ldots, m)$ and $\hat{u}_j'(\mathbf{p}) \le \hat{u}_j^*(\mathbf{p})$ $(j = 1, 2, \ldots, n)$, provided that $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ are all finite. Then it follows from Theorem 2.6 that if inequalities (29) are satisfied, then the criteria (4) are also satisfied.

Now it remains to prove the formulae (30a) and (30b). By using (15), the nominal system (27) can be described by

$$\begin{aligned} z_i^* &= h_f^{zi}(\mathbf{p}) * f + \sum_{k=1}^{n} h_{dk}^{zi}(\mathbf{p}) * d_k, \quad i = 1, 2, \ldots, m \\ u_j^* &= h_f^{uj}(\mathbf{p}) * f + \sum_{k=1}^{n} h_{dk}^{uj}(\mathbf{p}) * d_k, \quad j = 1, 2, \ldots, n \end{aligned}. \tag{32}$$

Thus, by the system linearity, the peak outputs $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ are given by

$$\hat{z}_i^*(\mathbf{p}) = \phi_f^{zi}(\mathbf{p}) + \sum_{k=1}^{n} \phi_{dk}^{zi}(\mathbf{p}), \quad \hat{u}_j^*(\mathbf{p}) = \phi_f^{uj}(\mathbf{p}) + \sum_{k=1}^{n} \phi_{dk}^{uj}(\mathbf{p}) \tag{33}$$

where $\phi_{dk}^{zi}(\mathbf{p})$ and $\phi_{dk}^{uj}(\mathbf{p})$ are the peak outputs of z_i^* and u_j^*, respectively, that are associated with the input d_k while other inputs being zero. Furthermore, by using a known result (Pfeiffer, 1955), it follows that

$$\phi_{dk}^{zi}(\mathbf{p}) = \mathcal{M}_k \|h_{dk}^{zi}(\mathbf{p})\|_1, \quad \phi_{dk}^{uj}(\mathbf{p}) = \mathcal{M}_k \|h_{dk}^{uj}(\mathbf{p})\|_1. \tag{34}$$

From (33) and (34), the theorem readily follows. □

Remark 1. From the first part of the proof of Theorem 2.6, one can see that condition (17b) implies the satisfaction of the criteria (4b). Consequently, since $\hat{u}_j'(\mathbf{p}) \le \hat{u}_j^*(\mathbf{p})$

($j = 1, 2, \ldots, n$), it readily follows that condition (29b) also implies that the criteria (4b) are satisfied. $\square$

Remark 2. The numbers $\phi_f^{zi}(\mathbf{p})$ and $\phi_f^{uj}(\mathbf{p})$ are the peak outputs of z_i^* and u_j^*, respectively, in response to the input f with $\mathbf{d} = \mathbf{0}$. For the possible sets $\mathcal{F}$ in (3), the peak outputs $\phi_f^{zi}(\mathbf{p})$ and $\phi_f^{uj}(\mathbf{p})$ are computed by using Silpsrikul and Arunsawatwong's (2010) method, in which associated convex optimization problems can be solved by using solvers available in open sources (for example, SeDuMi and SDPT3). References to such solvers can be found in Silpsrikul and Arunsawatwong (2010). $\square$

Remark 3. Following Theorem 2.6, one can conclude that the peak outputs $\hat{z}_i'(\mathbf{p})$ and $\hat{u}_j'(\mathbf{p})$ are upper bounds of the peak outputs $\hat{z}_i(\mathbf{p})$ and $\hat{u}_j(\mathbf{p})$, respectively, when inequalities (17) are satisfied. Moreover, it follows from Theorem 3.1 that the peak outputs $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ are also upper bounds of $\hat{z}_i(\mathbf{p})$ and $\hat{u}_j(\mathbf{p})$, respectively, when inequalities (29) are satisfied. Furthermore, it is clear that $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ are upper bounds of $\hat{z}_i'(\mathbf{p})$ and $\hat{u}_j'(\mathbf{p})$, respectively, provided that $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ are finite. $\square$

From Theorem 3.1, it is clear that once $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ are readily computable, inequalities (29) provide a practical condition for checking whether the criteria (4) are satisfied for a given $\mathbf{p}$. More importantly, such inequalities can be used in conjunction with a numerical algorithm to search for a solution of the criteria (4) in the space $\mathbb{R}^N$; see Section 4.2 for further details. For this reason, inequalities (29) are appropriately called the surrogate design criteria for (4).

In this work, an algorithm called the moving-boundaries-process (MBP) is used for determining a solution of inequalities; the details of the MBP algorithm can be found in Zakian (2005); Zakian and Al-Naib (1973). It may be noted that other algorithms for solving inequalities may also be used; for more details, see Chapters 7 and 8 of Zakian (2005) and the references therein.

Following Theorem 3.1 and the above discussion, an algorithm for computing the peak outputs $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ for the nominal system (27) is stated as follows.

Algorithm 1. Consider the nonlinear system (8) as stated in Theorem 3.1 and the possible set $\mathcal{F}$ in (3). Let a design parameter vector $\mathbf{p}$ be given such that the nominal system (27) is BIBO stable. Let $\sigma_j > 0$ ($j = 1, 2, \ldots, n$) be given.

Step 1. Determine the transfer matrix $\mathbf{H}(s, \mathbf{p})$ defined in (14) and (15).

Step 2. Compute the impulse responses $h_f^{zi}(\mathbf{p})$, $h_{dk}^{zi}(\mathbf{p})$, $h_f^{uj}(\mathbf{p})$ and $h_{dk}^{uj}(\mathbf{p})$ from the transfer functions $H_f^{zi}(s, \mathbf{p})$, $H_{dk}^{zi}(s, \mathbf{p})$, $H_f^{uj}(s, \mathbf{p})$ and $H_{dk}^{uj}(s, \mathbf{p})$, respectively.

Step 3. With the obtained responses $h_f^{zi}(\mathbf{p})$ and $h_f^{uj}(\mathbf{p})$, compute $\phi_f^{zi}(\mathbf{p})$ and $\phi_f^{uj}(\mathbf{p})$ defined in (31) by using Silpsrikul and Arunsawatwong's (2010) method.

Step 4. With the given ψ_j and σ_j, compute $\mathcal{M}_j$ by using (9) and (26).

Step 5. With the responses $h_{dk}^{zi}(\mathbf{p})$ and $h_{dk}^{uj}(\mathbf{p})$ obtained from Step 2 and the bounds $\mathcal{M}_j$ obtained from Step 4, compute $\phi_{d_k}^{zi}(\mathbf{p})$ and $\phi_{d_k}^{uj}(\mathbf{p})$ given in the formula (34) by using a numerical integration method (e.g., the trapezoidal rule).

Step 6. Compute $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ given in the formula (33).

4. Determination of a Solution in the Space of Design Parameters

In this section, we point out the connection between the finiteness of the peak outputs $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ for the possible sets in (3) and the BIBO stability of the nominal system (27). Then we explain how to use numerical search methods to determine a solution $\mathbf{p}$ of inequalities (29) in the space $\mathbb{R}^N$ of design parameters, for a known class of LTI systems whose BIBO stability is equivalent to the condition that all characteristic roots have negative real parts.

4.1. Connection between finiteness of peak outputs and BIBO stability

Below, we show that the BIBO stability of the nominal system (27) implies the finiteness of the peak outputs $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ for the possible sets in (3).

Proposition 4.1. *Consider the nominal system* (27)*. Let $\mathcal{F}$ be given by* (3) *and $\mathcal{D}$ by* (25)*. Then the peak outputs $\hat{z}_i^*(\mathbf{p})$ $(i = 1, 2, \ldots, m)$ and $\hat{u}_j^*(\mathbf{p})$ $(j = 1, 2, \ldots, n)$ are finite if the following two conditions hold.*

- (a) *For $i = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$, the transfer functions $H_f^{zi}(s, \mathbf{p})$ and $H_{dk}^{zi}(s, \mathbf{p})$ are BIBO stable.*
- (b) *For $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n$, the transfer functions $H_f^{uj}(s, \mathbf{p})$ and $H_{dk}^{uj}(s, \mathbf{p})$ are BIBO stable.*

Proof. It is easy to see from (32) that $\mathbf{z}^* \in \mathbb{L}_m^\infty$ for any $f \in \mathbb{L}^\infty$ and $\mathbf{d} \in \mathbb{L}_n^\infty$ if and only if condition (a) holds, and that $\mathbf{u}^* \in \mathbb{L}_n^\infty$ for any $f \in \mathbb{L}^\infty$ and $\mathbf{d} \in \mathbb{L}_n^\infty$ if and only if condition (b) holds. Since $\mathcal{F} \subset \mathbb{L}^\infty$ (Silpsrikul & Arunsawatwong, 2010, Proposition 2.1) and since $\mathcal{D} \subset \mathbb{L}_n^\infty$, it follows from equations (28) that $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ are finite for all i, j if conditions (a) and (b) hold. □

From the above, it is clear that for the nominal system (27), the finiteness of the peak outputs is in close connection with BIBO stability, which is a basic concept that has been used widely in control engineering.

4.2. Procedure for computing a solution of inequalities (29)

From Section 3, it follows that the peak outputs $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ map $\mathbb{R}^N$ to the extended half line $[0, \infty]$. In general, there are points $\mathbf{p} \in \mathbb{R}^N$ (possibly forming large regions in $\mathbb{R}^N$) such that $\hat{z}_i^*(\mathbf{p}) = \infty$ for some i or $\hat{u}_j^*(\mathbf{p}) = \infty$ for some j (Zakian, 1987a); in which case, the values of $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ (as well as their gradients) are not defined. Consequently, numerical search methods can fail to locate a solution of inequalities (29) if they start at such points.

Following previous work (Zakian & Al-Naib, 1973; Zakian, 1979b, 1986, 1987a, 2005), it is readily appreciated that in seeking a solution of inequalities (29) in $\mathbb{R}^N$, a search algorithm in general needs to start from a point $\mathbf{p} \in \mathbb{R}^N$ such that

$$\begin{aligned} \hat{z}_i^*(\mathbf{p}) &< \infty, \quad i = 1, 2, \ldots, m \\ \hat{u}_j^*(\mathbf{p}) &< \infty, \quad j = 1, 2, \ldots, n. \end{aligned} \tag{35}$$

In connection with (35), we define a stability region $\mathbb{S}$ as

$$\mathbb{S} \triangleq \{\mathbf{p} \in \mathbb{R}^N : \hat{z}_i^*(\mathbf{p}) < \infty \,\forall\, i \;\text{ and }\; \hat{u}_j^*(\mathbf{p}) < \infty \,\forall\, j\}.$$

Any point $\mathbf{p} \in \mathbb{S}$ is called a *stability point.* Once a stability point is obtained, the algorithm is then used to search for a solution of inequalities (29) inside the region $\mathbb{S}$.

In fact, there are many classes of LTI systems whose BIBO stability is equivalent to the condition that all the characteristic roots have negative real parts. Below a condition for ensuring the finiteness of $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ for such systems is stated.

Proposition 4.2. *Consider the nominal system* (27) *whose characteristic function is denoted by* $W(s)$*. Suppose that the system is BIBO stable if and only if all roots of the equation* $W(s) = 0$ *have negative real parts. Let* $\mathcal{F}$ *be described by* (3) *and* $\mathcal{D}$ *by* (25)*. Then the peak outputs* $\hat{z}_i^*(\mathbf{p})$ $(i = 1, 2, \ldots, m)$ *and* $\hat{u}_j^*(\mathbf{p})$ $(j = 1, 2, \ldots, n)$ *are finite if*

$$\alpha(\mathbf{p}) < 0, \quad \alpha(\mathbf{p}) \triangleq \max\{\operatorname{Re} s : W(s) = 0\}. \tag{36}$$

Proof. Let condition (36) hold. Clearly, (36) is equivalent to the fact that all the roots of $W(s) = 0$ have negative real parts; hence it readily follows that the nominal system (27) is BIBO stable. From this, one can see that all the elements of $\mathbf{H}(s, \mathbf{p})$ are BIBO stable transfer functions. Therefore, it follows from Proposition 4.1 that all the peak outputs $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ are finite. □

The number $\alpha(\mathbf{p})$ is known as the *abscissa of stability* of the characteristic function $W(s)$. Practical algorithms for computing the abscissa of stability, which makes use of repeated stability tests, are developed by: (i) Zakian (1979a) for the case of rational systems, (ii) Arunsawatwong (1996) for retarded delay differential systems (RDDSs) (see, e.g., Hale & Verduyn-Lunel, 1993), and (iii) Arunsawatwong and Nguyen (2009) for retarded fractional delay differential systems (RFDDSs) (see, e.g., Bonnet & Partington, 2000, 2001). Because of the availability of such algorithms, inequality (36) can be used to determine a stability point by numerical methods for rational systems, RDDSs and RFDDSs.

In conjunction with the method of inequalities, inequality (36) is replaced with

$$\alpha(\mathbf{p}) \le -\varepsilon_0 \quad (0 < \varepsilon_0 \ll 1) \tag{37}$$

where the bound $-\varepsilon_0$ is introduced to ensure that the system is stable as long as the magnitude of error in the computed value of $\alpha(\mathbf{p})$ is less than ε_0. Note that inequality (37) was introduced by Zakian and Al-Naib (1973) and, since then, has been used by a number of researchers in designing control systems by the method of inequalities (e.g., Rutland, 1992, 1994a; Lane, 1992, 1995; Whidborne, 1993, 2005; Arunsawatwong, 2005; Arunsawatwong & Nguyen, 2009; Silpsrikul & Arunsawatwong, 2010; Arunsawatwong & Kalvibool, 2016); more references can be found in Zakian (2005).

For the class of the nominal system considered in Proposition 4.2, a stability point can easily be obtained by using a search algorithm to solve inequality (37). Once a stability point is found, a design solution is obtained by solving the set of inequalities (29) and (37). With inequality (37), the solution is being sought in the set

$$\bar{\mathbb{S}} \triangleq \{\mathbf{p} \in \mathbb{R}^N : \alpha(\mathbf{p}) < 0\}. \tag{38}$$

For this class of the nominal system, we easily see that $\bar{\mathbb{S}} \subset \mathbb{S}$. However, when $\mathcal{F} = \mathcal{P}_1 \cap \dot{\mathcal{P}}_1$, both sets turn out to be identical; for the detail on this, see Proposition 2.2 in Silpsrikul and Arunsawatwong (2010).

According to Proposition 4.2 and the above discussion, a procedure for determining a solution of inequalities (29) involves two phases of computation as follows.

Algorithm 2. Consider the nonlinear system (8) and the nominal system (27) as stated in Theorem 3.1 and Proposition 4.2, respectively. Let the numbers $\varepsilon_i > 0$ $(i = 1, 2, \ldots, m)$, $\sigma_j > 0$ $(j = 1, 2, \ldots, n)$ and ε_0 be given where $0 < \varepsilon_0 \ll 1$.

Phase 1. Given a starting point $\mathbf{p}_0 \in \mathbb{R}^N$, determine a stability point $\mathbf{p}_1$ by solving inequality (37).

Phase 2. By starting from $\mathbf{p}_1$, determine a design solution $\mathbf{p}$ by solving inequalities (29) and (37).

5. Solvability of Design Inequalities for Deadzone and Saturation Nonlinearities

It is well known (Zakian & Al-Naib, 1973; Zakian, 1979b, 1996, 2005) that not every design problem that is cast as a set of inequalities has a solution; thus, the designer has to face the possibility that the design problem has no solution. In solving inequalities (29) by numerical methods (which is usually a non-convex problem in the space $\mathbb{R}^N$), when a search algorithm cannot locate a solution of such inequalities after a large number of iterations, a question often arising is whether or not the inequalities have a solution. When no solution exists, the designer has to reformulate the inequalities by simply increasing some bounds ε_i or σ_j or, in some cases, using a different controller structure with non-decreasing complexity so that the resultant inequalities have a solution.

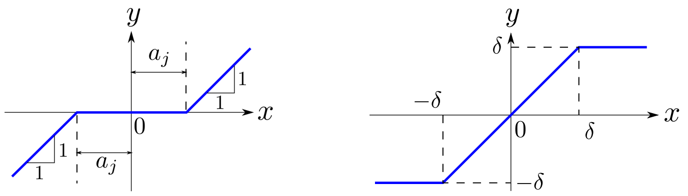
From a computational viewpoint, it is useful to know a condition under which inequalities (29a) and (29b) are guaranteed to have a solution for sufficiently large bounds ε_i and σ_j, respectively. Following Zakian (1979b, 1996, 2005); Zakian and Al-Naib (1973), it is appreciated that once such a solution is found, designers can tighten some of the bounds and solve the inequalities again in order to obtain a better design.

In this connection, the following definition is used for the sake of convenience.

Definition 5.1. Consider inequalities (29a) and (29b). An inequality $\hat{z}_i^*(\mathbf{p}) \leq \varepsilon_i$ is said to be solvable if it has a solution for sufficiently large ε_i. Similarly, an inequality $\hat{u}_j^*(\mathbf{p}) \leq \sigma_j$ is said to be solvable if it has a solution for sufficiently large σ_j.

From the definition of the stability region $\mathbb{S}$, it follows that if there exists a stability point $\mathbf{p}^*$ (i.e., a point $\mathbf{p}^* \in \mathbb{S}$), then $\hat{z}_i^*(\mathbf{p}^*)$ and $\hat{u}_j^*(\mathbf{p}^*)$ are finite for all i and j. From this, we arrive at the following two facts.

- Consider inequalities (29a). From (30a) and the fact that $\hat{z}_i^*(\mathbf{p})$ does not depend on ε_i, it follows that for each i, the inequality $\hat{z}_i^*(\mathbf{p}^*) \leq \varepsilon_i$ holds for a sufficiently large ε_i, i.e., each inequality is solvable. Hence, we conclude that if there exists a stability point $\mathbf{p}^*$, then inequalities (29a) are solvable.
- Consider inequalities (29b). In contrast to the former, it can be seen from (26) and (30b) that $\hat{u}_j^*(\mathbf{p})$ depends on σ_j for all $\mathbf{p}$. In this case, we cannot conclude that for each j, the inequality $\hat{u}_j^*(\mathbf{p}^*) \leq \sigma_j$ holds for a sufficiently large σ_j. To



(a) Deadzone (b) Saturation

Figure 5. Characteristics of deadzone and saturation nonlinearities

answer whether or not the existence of a stability point $\mathbf{p}^*$ implies the solvability of inequalities (29b), more analysis needs to be done by taking into account the characteristics of nonlinearities.

In this section, we provide conditions for ensuring that inequalities (29b) are solvable for the cases of deadzone and saturation, in connection with the class of the nominal system considered in Proposition 4.2. Such conditions will be used in solving the design problem considered in Section 6.

5.1. Deadzone nonlinearity

Consider the nonlinear system (8). Let $n_d \le n$ and assume that ψ_j $(j = 1, 2, \ldots, n_d)$ are deadzones described by

$$\psi_j(x) = \begin{cases} x + a_j, & x < -a_j \\ 0, & |x| \le a_j \\ x - a_j, & x > a_j \end{cases} \tag{39}$$

where $a_j > 0$ (see Figure 5(a)). Let the deadzones ψ_j be decomposed as in (9) with $K_j = 1$. Then the functions η_j $(j = 1, 2, \ldots, n_d)$ are given by

$$\eta_j(x) = \begin{cases} a_j, & x < -a_j \\ -x, & |x| \le a_j \\ -a_j, & x > a_j \end{cases}.$$

Assume now that there exists a stability point $\mathbf{p}^*$. Then $\hat{u}_j^*(\mathbf{p}^*)$ are finite for all j. Using (26) and (30b), one can verify that

$$\hat{u}_j^*(\mathbf{p}^*) \le \phi_f^{uj}(\mathbf{p}^*) + \sum_{\substack{k=1 \\ k \ne j}}^{n} \mathcal{M}_k \|h_{dk}^{uj}(\mathbf{p}^*)\|_1 + a_j \|h_{dj}^{uj}(\mathbf{p}^*)\|_1, \quad j = 1, 2, \ldots, n_d. \tag{40}$$

In this case, since the right-hand side of (40) is finite and independent of σ_j, we conclude from Definition 5.1 that the inequalities

$$\hat{u}_j^*(\mathbf{p}) \le \sigma_j, \quad j = 1, 2, \ldots, n_d \tag{41}$$

are solvable.

From the above, we are ready to state a sufficient condition for the solvability of inequalities (41) for the class of the nominal system considered in Proposition 4.2.

Proposition 5.2. *Consider the nonlinear system* (8) *and let Assumptions 1 and 2 hold. Let* ψ_j $(j = 1, 2, \ldots, n_d)$ *be the deadzone functions given by* (39) *and be decomposed as in* (9) *with* $K_j = 1$*. Then for the class of the nominal system considered in Proposition 4.2, inequalities* (41) *are solvable if there exists a vector* $\mathbf{p}^*$ *such that* $\alpha(\mathbf{p}^*) < 0$.

Proof. The proof readily follows from Proposition 4.2 and the above discussion. □

From Proposition 5.2, it is evident that condition (37) provides a practical sufficiency test for the solvability of inequalities (41). That is, if a numerical solution of inequality (37) is found, then $\bar{\mathbb{S}}$ is nonempty and hence inequalities (41) are solvable.

5.2. Saturation nonlinearity

Consider the nonlinear system (8) and assume that ψ_n is the only saturation and described by

$$\psi_n(x) = \begin{cases} -\delta, & x < -\delta \\ x, & |x| \leq \delta \\ \delta, & x > \delta \end{cases} \tag{42}$$

where $\delta > 0$ is the saturation level (see Figure 5(b)). Let ψ_n be decomposed as in (9) with $K_n = 1$. Then it readily follows that

$$\eta_n(x) = \begin{cases} -\delta - x, & x < -\delta \\ 0, & |x| \leq \delta \\ \delta - x, & x > \delta \end{cases} . \tag{43}$$

Consider the peak output $\hat{u}_n^*(\mathbf{p})$ and define

$$A(\mathbf{p}) \triangleq \phi_f^{un}(\mathbf{p}) + \sum_{k=1}^{n-1} \mathcal{M}_k \|h_{dk}^{un}(\mathbf{p})\|_1. \tag{44}$$

From (30b) and the definition of $A(\mathbf{p})$, it is clear that if $\hat{u}_n^*(\mathbf{p})$ is finite, then $A(\mathbf{p}) < \hat{u}_n^*(\mathbf{p})$ and

$$\hat{u}_n^*(\mathbf{p}) = A(\mathbf{p}) + \mathcal{M}_n \|h_{dn}^{un}(\mathbf{p})\|_1. \tag{45}$$

Using (26) and (43), one can see that for any $\sigma_n > \delta$, $\mathcal{M}_n = \sigma_n - \delta$ and, consequently, expression (45) can be rewritten as

$$\hat{u}_n^*(\mathbf{p}) = A(\mathbf{p}) + (\sigma_n - \delta)\|h_{dn}^{un}(\mathbf{p})\|_1. \tag{46}$$

Now consider the situation when the peak output $\hat{u}_n^*(\mathbf{p})$ is greater than δ for all $\mathbf{p} \in \mathbb{S}$. This is the case in which the bounds M_k and D_k of the possible set $\mathcal{F}$ in (3) are relatively large; for an example of this, see Section 6. As a result, the inequality

$$\hat{u}_n^*(\mathbf{p}) \leq \sigma_n \tag{47}$$

has no solution for any $\sigma_n \leq \delta$. Regarding the case in which inequality (47) has a solution for some $\sigma_n \leq \delta$, see comments in Remark 4.

The following lemma, which will be used to establish Theorem 5.4, is provided.

Lemma 5.3. *Consider the nonlinear system* (8) *and let Assumptions 1 and 2 hold. Let ψ_n be a saturation function given by* (42) *and be decomposed as in* (9) *with $K_n = 1$. Let $\mathbf{p}^*$ be a stability point. If $\|h_{dn}^{un}(\mathbf{p}^*)\|_1 < 1$, then the inequality*

$$\hat{u}_n^*(\mathbf{p}^*) \leq \sigma_n \tag{48}$$

holds for sufficiently large $\sigma_n > \delta$. Furthermore, provided that $A(\mathbf{p}^*) > \delta$, inequality (48) *holds for sufficiently large $\sigma_n > \delta$ if and only if $\|h_{dn}^{un}(\mathbf{p}^*)\|_1 < 1$.*

Proof. Let $\|h_{dn}^{un}(\mathbf{p}^*)\|_1 < 1$ and let $\sigma_n > \delta$. Since $\mathbf{p}^*$ is a stability point, both $A(\mathbf{p}^*)$ and $\hat{u}_n^*(\mathbf{p}^*)$ are finite. Since $\sigma_n > \delta$, the peak output $\hat{u}_n^*(\mathbf{p}^*)$ is given by (46) and inequality (48) becomes

$$A(\mathbf{p}^*) - \delta\|h_{dn}^{un}(\mathbf{p}^*)\|_1 \leq (1 - \|h_{dn}^{un}(\mathbf{p}^*)\|_1)\sigma_n. \tag{49}$$

Since $1 - \|h_{dn}^{un}(\mathbf{p}^*)\|_1 > 0$, dividing both sides of (49) by $1 - \|h_{dn}^{un}(\mathbf{p}^*)\|_1$ yields

$$\bar{\sigma}_n \leq \sigma_n \quad \text{where} \quad \bar{\sigma}_n \triangleq \frac{A(\mathbf{p}^*) - \delta\|h_{dn}^{un}(\mathbf{p}^*)\|_1}{1 - \|h_{dn}^{un}(\mathbf{p}^*)\|_1}. \tag{50}$$

So it follows that (48) is satisfied if and only if so is (50). Hence, we conclude that if $\|h_{dn}^{un}(\mathbf{p}^*)\|_1 < 1$, then (48) is satisfied for any $\sigma_n \geq \bar{\sigma}_n$ where $\sigma_n > \delta$.

Now it remains to prove that whenever $A(\mathbf{p}^*) > \delta$, the condition $\|h_{dn}^{un}(\mathbf{p}^*)\|_1 < 1$ is also necessary for (48) to hold for some $\sigma_n > \delta$. Let $A(\mathbf{p}^*) > \delta$ and let (48) be satisfied for a given $\sigma_n > \delta$. Then, by using (46), it follows that

$$\begin{aligned} A(\mathbf{p}^*) + (\sigma_n - \delta)\|h_{dn}^{un}(\mathbf{p}^*)\|_1 &\leq \sigma_n \\ \delta + (\sigma_n - \delta)\|h_{dn}^{un}(\mathbf{p}^*)\|_1 &< \sigma_n. \end{aligned}$$

Consequently, it follows that $\|h_{dn}^{un}(\mathbf{p}^*)\|_1 < 1$. Hence, the proof is complete. $\square$

From Lemma 5.3, one can see that to ensure that inequality (48) holds for sufficiently large $\sigma_n > \delta$, both conditions that $\mathbf{p}^*$ is a stability point and that $\|h_{dn}^{un}(\mathbf{p}^*)\|_1 < 1$ are needed. Furthermore, if $\mathbf{p}^*$ is a stability point and if $A(\mathbf{p}^*) > \delta$, then the condition $\|h_{dn}^{un}(\mathbf{p}^*)\|_1 < 1$ turns out to be necessary for inequality (48) to hold for sufficiently large $\sigma_n > \delta$.

For the rest of this section, we investigate the solvability of inequality (47) in connection with the class of the nominal system considered in Proposition 4.2.

Theorem 5.4. *Consider the nonlinear system* (8) *and let Assumptions 1 and 2 hold. Let ψ_n be a saturation function given by* (42) *and be decomposed as in* (9) *with $K_n = 1$. Suppose that $\bar{\mathbb{S}}$ is nonempty. Then for the class of the nominal system considered in Proposition 4.2, the following two statements hold.*

(i) *Inequality* (47) *is solvable if there exists a $\mathbf{p}^* \in \bar{\mathbb{S}}$ such that $\|h_{dn}^{un}(\mathbf{p}^*)\|_1 < 1$.*

(ii) *Provided that $A(\mathbf{p}) > \delta$ for any $\mathbf{p} \in \bar{\mathbb{S}}$, inequality* (47) *is solvable if and only if there exists a $\mathbf{p}^* \in \bar{\mathbb{S}}$ such that $\|h_{dn}^{un}(\mathbf{p}^*)\|_1 < 1$.*

Proof. (i) Let there exist a $\mathbf{p}^* \in \bar{\mathbb{S}}$ such that $\|h_{dn}^{un}(\mathbf{p}^*)\|_1 < 1$. From Proposition 4.2, the point $\mathbf{p}^*$ is also a stability point. Consequently, it follows from Lemma 5.3 that $\hat{u}_n^*(\mathbf{p}^*) \leq \sigma_n$ for sufficiently large $\sigma_n > \delta$. Hence, we conclude from Definition 5.1 that (47) is solvable.
(ii) Since the sufficiency readily follows from (i), it only remains to prove the necessity. Let $A(\mathbf{p}) > \delta$ for any $\mathbf{p} \in \bar{\mathbb{S}}$ and assume that there exists a $\mathbf{p}^* \in \bar{\mathbb{S}}$ such that $\hat{u}_n^*(\mathbf{p}^*) \leq \sigma_n$ for sufficiently large $\sigma_n > \delta$. Since $\mathbf{p}^*$ is a stability point and since $A(\mathbf{p}^*) > \delta$, it can be seen from Lemma 5.3 that $\|h_{dn}^{un}(\mathbf{p}^*)\|_1 < 1$. Hence, the proof is complete. □

In contrast to the case of deadzone, Theorem 5.4 suggests that for the class of the nominal system considered, inequality (47) is solvable if there exists a $\mathbf{p}^*$ satisfying both $\alpha(\mathbf{p}^*) < 0$ and $\|h_{dn}^{un}(\mathbf{p}^*)\|_1 < 1$. Moreover, if both $A(\mathbf{p}) > \delta$ and $\|h_{dn}^{un}(\mathbf{p})\|_1 \geq 1$ for any $\mathbf{p} \in \bar{\mathbb{S}}$, then inequality (47) is not solvable. For this reason, in conjunction with the method of inequalities, the inequality

$$H(\mathbf{p}) \triangleq \|h_{dn}^{un}(\mathbf{p})\|_1 - 1 \leq -\gamma \quad (0 < \gamma \ll 1) \tag{51}$$

is used together with inequality (37) for finding a solution of (47) for the cases in which all solutions of (47) exist for $\sigma_n > \delta$.

From Theorem 5.4 and the above discussion, it is evident that when the nonlinear system (8) contains one saturation element, the condition $\|h_{dn}^{un}(\mathbf{p})\|_1 < 1$ plays an important role in the solvability of inequality (47), which is a part of the surrogate criteria (29). Hence, in this case, a procedure for determining a solution of (29) involves three phases of computation as follows.

Algorithm 3. Consider the nonlinear system (8) and the nominal system (27) as stated in Theorem 5.4 and Proposition 4.2, respectively. Let the numbers $\varepsilon_i > 0$ $(i = 1, 2, \ldots, m)$, $\sigma_j > 0$ $(j = 1, 2, \ldots, n)$, ε_0 and γ be given where $0 < \varepsilon_0 \ll 1$ and $0 < \gamma \ll 1$.

Phase 1. Given a starting point $\mathbf{p}_0 \in \mathbb{R}^N$, determine a stability point $\mathbf{p}_1$ by solving inequality (37).

Phase 2. By starting from $\mathbf{p}_1$, determine a point $\mathbf{p}_2$ by solving inequalities (37) and (51).

Phase 3. By starting from $\mathbf{p}_2$, determine a design solution $\mathbf{p}$ by solving inequalities (29), (37) and (51).

Remark 4. If there exists a stability point $\mathbf{p}^*$ such that $\hat{u}_j^*(\mathbf{p}^*) \leq \sigma_j$ $(j = 1, 2, \ldots, n-1)$ and $\hat{u}_n^*(\mathbf{p}^*) \leq \delta$ (cf. inequalities (29b)), then it follows from Remark 1 that $\hat{u}_n(\mathbf{p}^*) \leq \hat{u}_n^*(\mathbf{p}^*) \leq \delta$. With the point $\mathbf{p}^*$, the response u_n never exceeds the saturation level $\pm\delta$ for any input $f \in \mathcal{F}$, thereby implying that the saturation never occurs in the original system. In this case, the peak outputs $\hat{z}_i^*(\mathbf{p}^*)$ and $\hat{u}_j^*(\mathbf{p}^*)$ should therefore be recalculated by neglecting ψ_n so that the conservatism in $\hat{z}_i^*(\mathbf{p}^*)$ and $\hat{u}_j^*(\mathbf{p}^*)$ can be reduced (i.e., using (30) with $\mathcal{M}_n = 0$).

The result obtained in this subsection is applicable to the case in which only one of the nonlinearities ψ_j is a saturation. A condition for the solvability of inequalities (29b) for the case of more than one saturation nonlinearities can be a topic for future investigation.

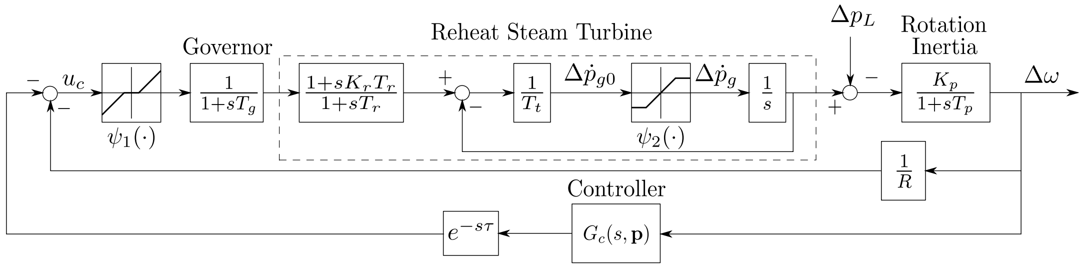


Figure 6. Block diagram of the LFC system with the GDZ and GRC

Power system gain	$K_p = 120.00$ Hz/pu
Power system time constant	$T_p = 20.00$ s
High pressure turbine fraction	$K_r = 0.50$
Reheat time constant	$T_r = 10.00$ s
Speed governor time constant	$T_g = 0.08$ s
Steam chest time constant	$T_t = 0.30$ s
Self regulation of the governor	$R = 2.40$ Hz/pu
Deadzone width	$a = 3.5 \times 10^{-3}$ pu
Saturation level of GRC	$\delta = 1.667 \times 10^{-3}$ pu/s
Delay time	$\tau = 1$ s

Table 1. Parameters of the LFC system in Figure 6

6. Numerical Example

This section demonstrates the usefulness of the developed method in designing a controller for an LFC system where deadzone and saturation nonlinearities are explicitly taken into account.

In power system operation, any imbalance between power generation and load demand causes the system frequency to deviate from its nominal value. When the frequency deviation is too large, it can cause severe problems to the system. The objective of the LFC is to maintain the frequency deviation in the presence of load variations within an acceptable range for all time during operation. For details on LFC, see, e.g., Kundur (1994).

6.1. LFC system model

The model of the LFC system considered is shown in Figure 6 where the system variables are as follows:

- Δp_L is the incremental load in pu,
- $\Delta\omega$ is the frequency deviation in Hz,
- u_c is the control signal in pu,
- $\Delta\dot{p}_{g0}$ is the incremental power generation rate before the saturation in pu/s,
- $\Delta\dot{p}_g$ is the incremental power generation rate in pu/s.

Furthermore, the values of the system parameters are given in Table 1.

Let ψ_1 represent the governor deadzone (GDZ) in the LFC model (see, e.g., Bevrani, 2014) where ψ_1 is described by (39) and its deadzone width a is 3.5×10^{-3} pu. Because of the generation rate constraint (GRC) in the reheat steam turbine, the saturation nonlinearity ψ_2 is included in the model to represent the upper and lower limits of the power generation rate, as well. The characteristic of ψ_2 is described by (42) where, according to Nanda and Kaul (1978), the GRC of 0.1 pu/min (corresponding to the saturation level δ given in Table 1) is used.

In order to demonstrate that the design methodology developed in this work is also applicable to control system described by non-rational transfer functions, the time delay term $e^{-s\tau}$, which represents communication delays, is included in the model where τ is assumed to be 1 s. Note in passing that after deregulation of the electric power industry, an open communication network is needed to support an increasing variety of ancillary services. Under this situation, many researchers (e.g., Yu & Tomsovic, 2004; Jiang, Yao, Wu, Wen, & Cheng, 2011) have considered time delays in the LFC model.

6.2. Problem formulation

Assume that the power system is subjected to persistent load disturbances (Arunsawatwong & Kalvibool, 2016) where the incremental load Δp_L is considered as any persistent disturbance satisfying

$$\|\Delta p_L\|_\infty \leq 0.05 \text{ pu} \quad \text{and} \quad \|\Delta \dot{p}_L\|_\infty \leq 1.57 \times 10^{-3} \text{ pu/s}.$$

For this reason, the possible set $\mathcal{F}$ to be used in this example is characterized by

$$\mathcal{F} = \mathcal{P}_1 \cap \dot{\mathcal{P}}_1 = \{f : \|f\|_\infty \leq M_\infty,\ \|\dot{f}\|_\infty \leq D_\infty\}$$

where $M_\infty = 0.05$ pu and $D_\infty = 1.57 \times 10^{-3}$ pu/s.

The main design objective is to determine the controller transfer function $G_c(s, \mathbf{p})$ for which the following specifications are satisfied.

1) The system is required to be stable so that the system responses are bounded.
2) According to ENTSO-E (2013), the frequency deviation $\Delta\omega$ is required to stay strictly within the range ± 200 mHz for all time and for all $\Delta p_L \in \mathcal{F}$.

One can see from Figure 6 that the variables u_c and $\Delta\dot{p}_{g0}$ are the nonlinearity inputs and the variable Δp_L is the disturbance. Furthermore, the variable $\Delta\omega$ is the system output under consideration. Then the system can be represented as the system in Figure 1 where

$$\mathbf{z} = [z] \triangleq \Delta\omega, \quad \mathbf{u} = [u_1, u_2]^T \triangleq [u_c, \Delta\dot{p}_{g0}]^T, \quad v_2 \triangleq \Delta\dot{p}_g, \quad f \triangleq \Delta p_L.$$

Accordingly, the principal design criteria are expressed as

$$\hat{z}(\mathbf{p}) \leq 200 \text{ mHz}, \quad \hat{u}_1(\mathbf{p}) \leq \sigma_1 \text{ pu}, \quad \hat{u}_2(\mathbf{p}) \leq \sigma_2 \text{ pu/s} \tag{52}$$

where $\hat{z}(\mathbf{p})$, $\hat{u}_1(\mathbf{p})$ and $\hat{u}_2(\mathbf{p})$ are the peak outputs for the possible set $\mathcal{F}$. For the purpose of demonstration, the bounds σ_1 and σ_2 are chosen to be

$$\sigma_1 = 0.1 \text{ pu} \quad \text{and} \quad \sigma_2 = 5 \times 10^{-2} \text{ pu/s}. \tag{53}$$

It should be noted that a controller satisfying the criteria (52) with simple structure is usually preferred. In this example, by starting searching a design solution from the simplest controller structure (i.e., $G_c(s,\mathbf{p}) = p_1$ where $\mathbf{p} \triangleq p_1 \in \mathbb{R}$) and progressively increasing the order of $G_c(s,\mathbf{p})$, the following PID structure is chosen.

$$G_c(s,\mathbf{p}) = p_1 \left(1 + \frac{1}{p_2 s} + \frac{p_3 s}{1 + p_4 s}\right) \tag{54}$$

where $\mathbf{p} \triangleq [p_1, p_2, p_3, p_4]^T \in \mathbb{R}^4$ is to be determined. It is worth noting that for a more complex controller structure, a PID controller with anti-windup procedure could be designed as well (if this is allowed) by using the developed methodology.

In the following, the peak outputs of the control system are computed with Algorithm 1, in which the impulse responses of the system are required. For the system with time-delay, the impulse responses are obtained by using Zakian I_{MN} recursions for delay differential systems, which are efficient and reliable even if the system is very stiff. See Arunsawatwong (1998) for the details of the recursions and their properties.

6.3. Design by neglecting nonlinearities

In this subsection, we demonstrate what can go wrong when the nonlinearities ψ_1 and ψ_2 are neglected in the design formulation. To this end, the nonlinearities ψ_1 and ψ_2 are replaced with a constant gain $C = 1$ and, consequently, the approximated linear system is obtained.

The characteristic function of the approximated system is given by

$$\begin{aligned}
&W(s) = P_0(s) + P_1(s)e^{-\tau s} \quad \text{where} \;\; P_0(s) = p_2 s(1 + p_4 s)P_2(s), \\
&P_1(s) = p_1\big\{p_2(p_3 + p_4)s^2 + (p_2 + p_4)s + 1\big\}K_p(1 + sK_rT_r), \\
&P_2(s) = (1 + sT_t)(1 + sT_p)(1 + sT_r)(1 + sT_g) + K_p(1 + sK_rT_r)/R.
\end{aligned} \tag{55}$$

In this case, the system is BIBO stable and hence the peak outputs $\hat{z}(\mathbf{p})$, $\hat{u}_1(\mathbf{p})$ and $\hat{u}_2(\mathbf{p})$ are finite if

$$\alpha(\mathbf{p}) \le -10^{-6} \quad \text{where} \;\; \alpha(\mathbf{p}) \triangleq \max\{\operatorname{Re} s : W(s) = 0\}. \tag{56}$$

Accordingly, the design problem is to determine a design parameter vector $\mathbf{p} \in \mathbb{R}^4$ that satisfies inequality (56) and the following design inequalities:

$$\hat{z}(\mathbf{p}) \le 200 \text{ mHz}, \quad \hat{u}_1(\mathbf{p}) \le 0.1 \text{ pu}, \quad \hat{u}_2(\mathbf{p}) \le 5 \times 10^{-2} \text{ pu/s}. \tag{57}$$

After a number of iterations, the MBP algorithm locates a solution $\mathbf{p}$ satisfying inequalities (56) and (57), which yields the following controller.

$$G_c(s,\mathbf{p}) = 2.137 \times 10^{-3} \left(1 + \frac{1}{0.1514 s} + \frac{11.36 s}{1 + 65.20 s}\right). \tag{58}$$

And the corresponding values of $\alpha(\mathbf{p})$ and the peak outputs of the approximated linear system are given by

$$\begin{gathered}
\alpha(\mathbf{p}) = -1.526 \times 10^{-3}, \\
\hat{z}(\mathbf{p}) = 102.0 \text{ mHz}, \quad \hat{u}_1(\mathbf{p}) = 5.825 \times 10^{-2} \text{ pu}, \quad \hat{u}_2(\mathbf{p}) = 1.890 \times 10^{-3} \text{ pu/s}.
\end{gathered}$$

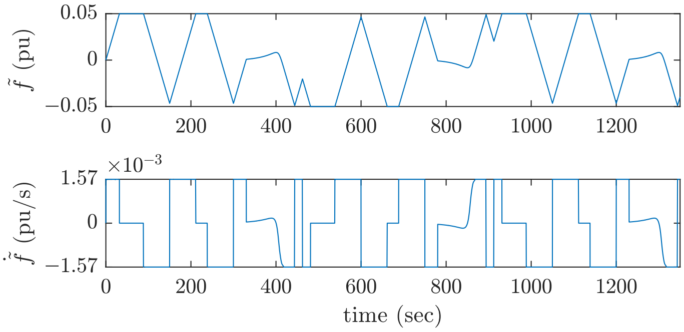


Figure 7. Waveforms of the test input $\tilde{f} \in \mathcal{F}$ and its derivative

To verify the design result, a simulation of the approximated LFC system is carried out for the test input $\tilde{f} \in \mathcal{F}$. The waveforms of $\tilde{f}$ and its derivative are shown in Figure 7. And the waveforms of the resultant system responses are shown in Figure 8 where the peak magnitudes of z, u_1 and u_2 are 100.2 mHz, 5.789×10^{-2} pu and 1.840×10^{-3} pu/s, respectively.

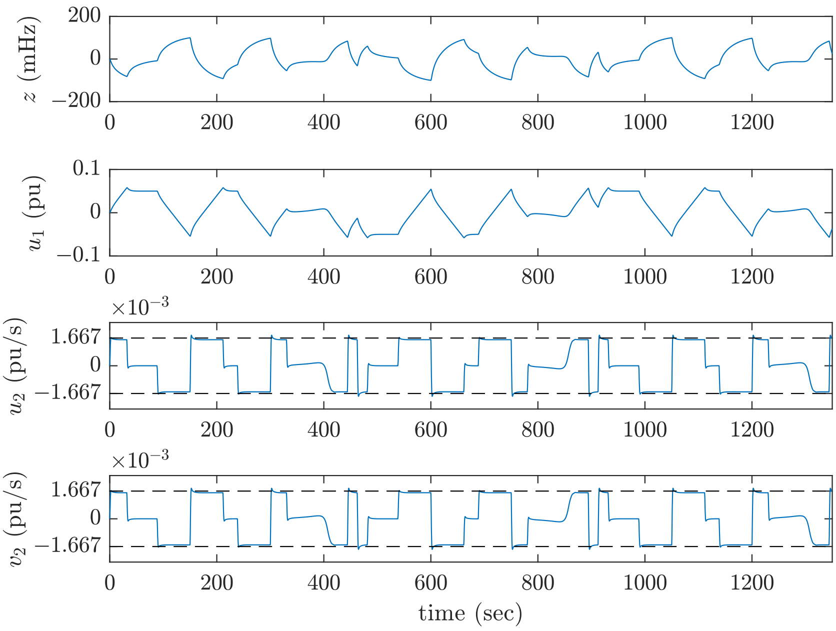


Figure 8. Responses of the approximated LFC system due to the test input $\tilde{f}$ with the controller (58)

To investigate the actual performance for the original LFC system using the controller (58), a nonlinear simulation is carried out with the test input $\tilde{f}$. The waveforms of the nonlinear system responses are displayed in Figure 9 and, in this case, the peak magnitude of z is about 150 times of the bound 200 mHz. The numerical results clearly show that with the controller (58), the performance of the nonlinear LFC system is very poor and therefore unacceptable.

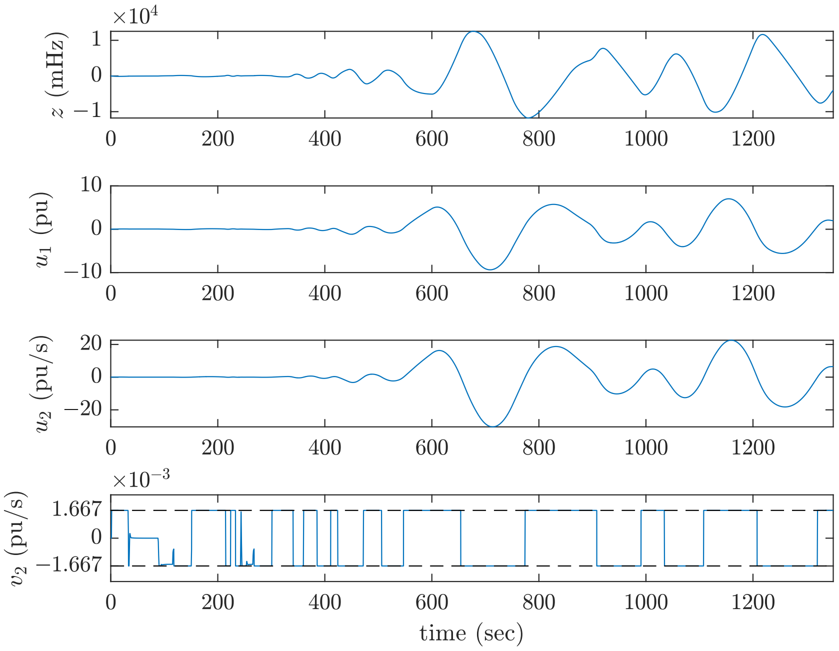


Figure 9. Responses of the nonlinear LFC system due to the test input $\tilde{f}$ with the controller (58)

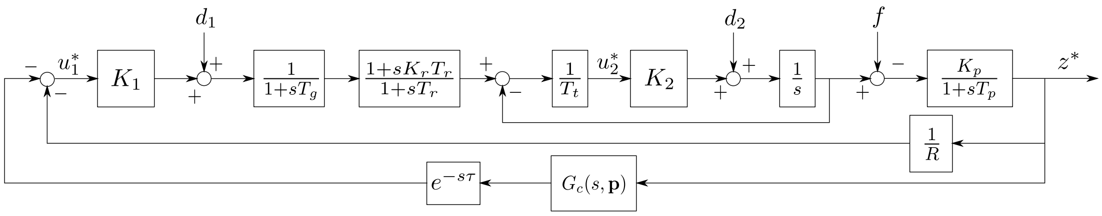


Figure 10. Nominal system of the LFC system in Figure 6

6.4. Design using the original nonlinear model

In this subsection, the nonlinearities ψ_1 and ψ_2 are explicitly taken into account. By replacing $\psi_j(u_j)\,(j = 1, 2)$ in the original LFC system with $K_j u_j + d_j$, we obtain the nominal system that is displayed in Figure 10. It is easy to verify that $H_{dk}^{uj}(s, \mathbf{p})$ $(j, k = 1, 2)$ are strictly proper. Consequently, for any proper $G_c(s, \mathbf{p})$, Assumption 4 is always satisfied whenever the nominal system is BIBO stable.

In this example, the fixed gains K_1 and K_2 are chosen to be $K_1 = K_2 = 1$. Thus, from (26) and (53), it follows that the bounds $\mathcal{M}_1$ and $\mathcal{M}_2$ are given by

$$\mathcal{M}_1 = \min(\sigma_1, a) = 3.5 \times 10^{-3} \quad \text{and} \quad \mathcal{M}_2 = \max(0, \sigma_2 - \delta) = 4.8323 \times 10^{-2}. \quad (59)$$

Moreover, the characteristic function of the nominal system is identical to $W(s)$, which is given in (55).

By applying Theorem 3.1, a solution $\mathbf{p}$ satisfying the criteria (52) is obtained by solving the following inequalities.

$$\hat{z}^*(\mathbf{p}) \leq 200 \text{ mHz}, \quad \hat{u}_1^*(\mathbf{p}) \leq 0.1 \text{ pu}, \quad \hat{u}_2^*(\mathbf{p}) \leq 5 \times 10^{-2} \text{ pu/s}. \quad (60)$$

By Proposition 4.2 and Theorem 5.4, it follows that a vector $\mathbf{p}$ satisfying inequalities (60) needs to satisfy the following two constraints.

$$\alpha(\mathbf{p}) \le -10^{-6}, \tag{61}$$
$$H(\mathbf{p}) \le -10^{-6}. \tag{62}$$

Inequality (61) ensures the BIBO stability of the nominal system and the finiteness of $\hat{z}^*(\mathbf{p})$, $\hat{u}_1^*(\mathbf{p})$ and $\hat{u}_2^*(\mathbf{p})$. Furthermore, together with (61), inequality (62) ensures that the inequality $\hat{u}_2^*(\mathbf{p}) \le \sigma_2$ is solvable.

By using the MBP algorithm in conjunction with Algorithm 3, we obtain a solution $\mathbf{p}$ of inequalities (60)–(62), which results in

$$G_c(s, \mathbf{p}) = 5.333 \times 10^{-3} \left(1 + \frac{1}{3.496 \times 10^3 s} + \frac{2.782s}{1 + 3.741 \times 10^{-3} s}\right). \tag{63}$$

And the corresponding values of $\alpha(\mathbf{p})$, $H(\mathbf{p})$ and the peak outputs of the nominal system are given by

$$\alpha(\mathbf{p}) = -6.104 \times 10^{-5}, \quad H(\mathbf{p}) = -0.2560,$$
$$\hat{z}^*(\mathbf{p}) = 195.5 \text{ mHz}, \quad \hat{u}_1^*(\mathbf{p}) = 8.150 \times 10^{-2} \text{ pu}, \quad \hat{u}_2^*(\mathbf{p}) = 5.000 \times 10^{-2} \text{ pu/s}.$$

Hence, by Theorem 3.1, the criteria (52) are satisfied.

For design verification, a simulation is carried out for the nonlinear LFC system with the controller (63) and the test input $\tilde{f}$. The waveforms of the resultant system responses are displayed in Figure 11 where the peak magnitudes of z, u_1 and u_2 are 142.0 mHz, 5.997×10^{-2} pu and 3.983×10^{-2} pu/s, respectively. Evidently, the controller (63) provides satisfactory performance for the nonlinear LFC system.

From this example, it can be seen that by neglecting nonlinearities, the so-obtained design formulation may be inaccurate and could lead to a design solution that may not be acceptable. On the other hand, the design methodology developed in this work offers a more accurate design formulation for this kind of problem and, whenever a design solution is found, always yields satisfactory results. Therefore, the value of the developed methodology is evident.

7. Conclusions and Discussion

This article has developed a practical and systematic methodology for designing the nonlinear system (8) so as to ensure that the outputs of interest z_i ($i = 1, 2, \ldots, m$) and the nonlinearity inputs u_j ($j = 1, 2, \ldots, n$) stay within the prescribed bounds $\pm\varepsilon_i$ and $\pm\sigma_j$, respectively, for all time and for all inputs $f \in \mathcal{F}$ where $\mathcal{F}$ is one of the possible sets described by (3). The methodology developed in this work can be seen as an adjunct to Zakian's framework, which is a control design framework comprising the method of inequalities and the principle of matching (Zakian, 1979b, 1986, 1987b, 1996, 2005).

Being obtained by using Schauder's fixed point theorem, Theorem 2.6 provides an essential basis for developing the practical design inequalities (29), which are associated with the nominal linear system subject to the input $f \in \mathcal{F}$ and the additional input vector $\mathbf{d} \in \mathcal{D}$. As a consequence, a solution of the original design problem can be

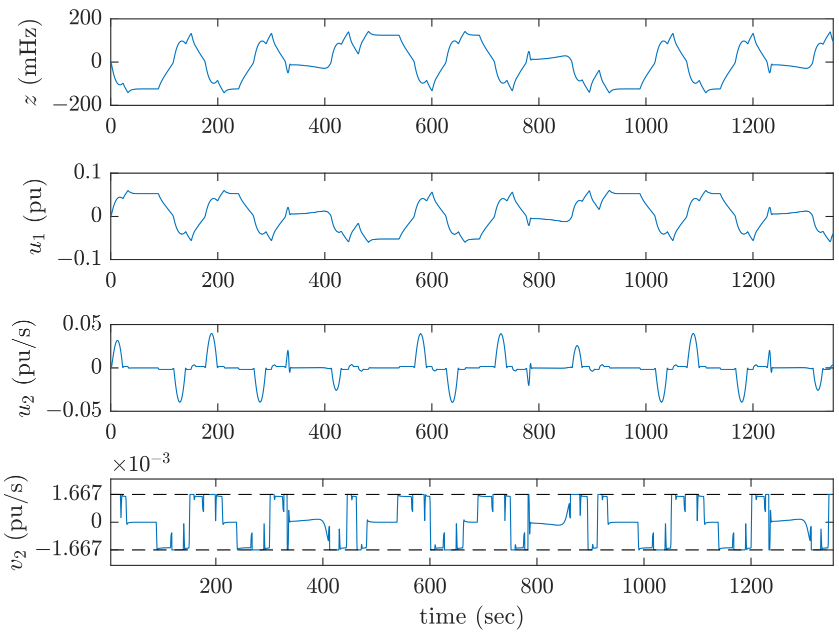


Figure 11. Responses of the nonlinear LFC system due to the test input $\tilde{f}$ with the controller (63)

obtained by solving the surrogate problem with the computational tools developed previously for LTI systems. In addition, because the nonlinear system (8) uses the convolution representation, the methodology developed here is applicable to control systems having rational and/or infinite-dimensional systems as long as Assumption 4 is satisfied.

In the numerical example, we design a controller for the LFC system with time-delay, where deadzone and saturation nonlinearities are considered in the design formulation. With regard to the criteria (52), Section 6.3 shows what can go wrong when such nonlinearities are neglected in the formulation whereas Section 6.4 shows that the developed methodology can be used to obtain the PID controller (63), which provides satisfactory performance for the nonlinear LFC system. Hence, the example clearly demonstrates the usefulness of the contribuition of this work.

For clarity, we assume that the nonlinear system (8) is subjected to only one exogenous input. However, it should be noted that the results obtained here can be extended in a straightforward manner to the case of multiple exogenous inputs.

It may be noted that the linear differential inclusion (LDI) approach (see, e.g., Boyd, El Ghaoui, Feron, & Balakrishnan, 1994) can also be used to handle control systems whose inputs and outputs are constrained. However, there are differences between the problem formulation in this work and the LDI approach. First, the LTI system (8a) can be any LTI system whose input and output are related by convolution integral (see equations (8a)), whereas the LDI approach considers finite-dimensional systems, which are special cases of convolution systems. More importantly, the performance measures used in this work are the infinity norms of individual outputs, while the LDI approach uses a weighted sum of the two norms of outputs. Furthermore, in this work, system inputs are restricted in both magnitude and slope (see the description in (3)), whereas the LDI approach considers system inputs that are restricted in only magnitude.

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Appendix A. Proof of Lemma 2.4 and Proposition 2.5

Theorem A.1. [McShane (1944), p. 227 and Desoer and Vidyasagar (1975), p. 223]. *If* $h \in \mathbb{L}^1$*, then*

$$\lim_{\Delta t \to 0} \int_{-\infty}^{\infty} |h(\tau + \Delta t) - h(\tau)| \mathrm{d}\tau = 0.$$

Definition A.2. [McShane (1944)]. Let (E, ρ) be a metric space. Let $\mathcal{G}$ denote a set of functions that are defined and finite-valued on E. The set $\mathcal{G}$ is said to be equicontinuous if, for every $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that for all $f \in \mathcal{G}$, $|f(x_1) - f(x_2)| < \epsilon$ whenever $x_1, x_2 \in E$ and $\rho(x_1, x_2) < \delta$. The set $\mathcal{G}$ is said to be uniformly bounded if there is an $M < \infty$ such that $|f(x)| \le M$, $\forall\, x \in E$, $\forall\, f \in \mathcal{G}$.

Theorem A.3. [Ascoli's Theorem (McShane, 1944)]. *Let* $\mathcal{G}$ *denote a set of functions that are defined on a bounded and closed set. If* $\mathcal{G}$ *is equicontinuous and uniformly bounded, then it is possible to select a uniformly convergent subsequence from every sequence* $\{f_n\}$ *of functions of* $\mathcal{G}$*.*

Proposition A.4. [Zeidler (1986)]. *In a Banach space, a subset* K *is relatively compact if and only if every sequence in* K *contains a subsequence.*

Definition A.5. [Zeidler (1986)]. Let X and Y be Banach spaces and $\mathcal{H} : D \subset X \to Y$ be an operator. Then $\mathcal{H}$ is called compact if and only if (i) $\mathcal{H}$ is continuous; and (ii) $\mathcal{H}$ maps bounded sets into relatively compact sets.

Proof of Lemma 2.4. Since $h_k^j \in \mathbb{L}^1$ for all j, k, there exists C_0 such that $\|h_k^j\|_1 \le C_0 < \infty$ for all j, k. Then one can verify from (7) and (16) that

$$\|\mathcal{H}\mathbf{x} - \mathcal{H}\mathbf{y}\| \le \sum_{j=1}^{n} \sum_{k=1}^{n} \|h_k^j\|_1 \|x_k - y_k\|_\infty \le nC_0 \|\mathbf{x} - \mathbf{y}\|.$$

Thus, we conclude that $\mathcal{H}$ is continuous.

Let $\{\mathbf{x}^{(l)}\}$ be any sequence in X_T and let $\mathbf{y}^{(l)}$ be defined as

$$\mathbf{y}^{(l)}(t) = \big(\mathcal{H}\mathbf{x}^{(l)}\big)(t), \quad \forall\, t \in [0, T].$$

Then we can write

$$y_j^{(l)}(t) = \big(\mathcal{H}_j\mathbf{x}^{(l)}\big)(t), \quad j = 1, 2, \ldots, n. \tag{A1}$$

Since $X_T \subset \mathbb{L}^\infty_{n,T}$, $\|r_j\| \leq M_r$ for all j and $h_k^j \in \mathbb{L}^1$ for all j, k, it follows from (16) that $\{y_j^{(l)}\}$ $(j = 1, 2, \ldots, n)$ are uniformly bounded on $[0, T]$ for any fixed $T > 0$.

Next we show that $\{y_j^{(l)}\}$ $(j = 1, 2, \ldots, n)$ are equicontinuous. Since $X_T \subset \mathbb{L}^\infty_{n,T}$, there exists $C < \infty$ such that $\|x_j^{(l)}\|_\infty \leq C$ for all j. Consequently, one can verify from (16) and (A1) that

$$|y_j^{(l)}(t_1) - y_j^{(l)}(t_2)| \leq C \sum_{k=1}^{n} \int_0^T |h_k^j(t_1 - \tau) - h_k^j(t_2 - \tau)| \mathrm{d}\tau + |r_j(t_1) - r_j(t_2)|. \tag{A2}$$

Define $\Delta t \triangleq t_1 - t_2$. Since $h_k^j \in \mathbb{L}^1$ for all j, k, it follows from Theorem A.1 that

$$\lim_{\Delta t \to 0} \int_0^T |h_k^j(t_1 - \tau) - h_k^j(t_1 - \Delta t - \tau)| \mathrm{d}\tau = 0, \ \forall\, j, k. \tag{A3}$$

Then statement (A3) implies that, for any $\epsilon > 0$, there exists $\delta_1 > 0$ such that

$$\int_0^T |h_k^j(t_1 - \tau) - h_k^j(t_1 - \Delta t - \tau)| \mathrm{d}\tau \leq \frac{\epsilon}{2nC}, \ \forall\, j, k \quad \text{whenever } \Delta t \leq \delta_1.$$

Therefore, if $\Delta t \leq \delta_1$, then for all j,

$$C \sum_{k=1}^{n} \int_0^T |h_k^j(t_1 - \tau) - h_k^j(t_2 - \tau)| \mathrm{d}\tau \leq nC \cdot \frac{\epsilon}{2nC} = \frac{\epsilon}{2}. \tag{A4}$$

Furthermore, since r_j is continuous on $[0, T]$, it is also uniformly continuous on $[0, T]$. Consequently, there exists $\delta_2 > 0$ such that

$$|r_j(t_1) - r_j(t_2)| \leq \frac{\epsilon}{2}, \ \forall\, j \quad \text{whenever } \Delta t \leq \delta_2. \tag{A5}$$

From (A2)–(A5), it follows that, for any $t_1, t_2 \in [0, T]$ and for any $l > 0$, there exists $\delta = \min\{\delta_1, \delta_2\}$ such that

$$|y_j^{(l)}(t_1) - y_j^{(l)}(t_2)| \leq \epsilon, \ \forall\, j \quad \text{whenever } \Delta t \leq \delta.$$

Consequently, we conclude by Definition A.2 that $\{y_j^{(l)}\}$ $(j = 1, 2, \ldots, n)$ are equicontinuous. Hence, in view of Theorem A.3, each $\{y_j^{(l)}\}$ contains a convergent subsequence.

Since $\mathcal{H}_j(\mathbf{x}^{(l)}) = y_j^{(l)}$, it readily follows from Proposition A.4 that the sets $\mathcal{H}_j(X_T)$ $(j = 1, 2, \ldots, n)$ are relatively compact. Thus, by virtue of Tychonoff's theorem (see, e.g., Zeidler, 1986, p.756), the set $\mathcal{H}(X_T) = \mathcal{H}_1(X_T) \times \mathcal{H}_2(X_T) \times \cdots \times \mathcal{H}_n(X_T)$ is relatively compact and so we conclude that the operator $\mathcal{H}$ maps bounded sets into relatively compact sets. Because of this and the continuity of $\mathcal{H}$, the compactness of $\mathcal{H}$ readily follows from Definition A.5. □

Proof of Proposition 2.5. We prove the proposition by using Definition A.5. Since $\mathcal{Q}(D_1) \subset D_2$, the operator $\mathcal{H}$ is continuous over $\mathcal{Q}(D_1)$. Consequently, because a composite of continuous functions is continuous, we conclude that the operator $\mathcal{HQ}$ is continuous over D_1. Since $\mathcal{Q}$ is continuous, the boundedness of $\mathcal{Q}(D_1)$ readily follows from the boundedness of D_1. Hence, by Definition A.5, the compactness of $\mathcal{H}$ implies that the set $\mathcal{HQ}(D_1)$ is relatively compact. This means that $\mathcal{HQ}$ maps bounded sets into relatively compact sets. Because of this and the continuity of $\mathcal{HQ}$, we conclude by Definition A.5 that $\mathcal{HQ}$ is compact over D_1. □