A NOTE ON TANGENTIAL QUADRILATERALS

PRADEEP DAS, ABHISHEK JUYAL AND DUSTIN MOODY

Abstract. A tangential quadrilateral is a convex quadrilateral whose sides are simultaneously tangent to a single circle. In this paper, the primary objective is to construct *rational* tangential quadrilaterals characterized by having rational area, as well as rational side and diagonal lengths. We relate the existence of such tangential quadrilaterals to properties of a certain elliptic curve. Studying the curve, we are able to construct an infinite family of rational tangential quadrilaterals.

1. Introduction

While studying Euclidean geometry in school, we encounter many geometrical objects, like circles, parabolas, triangles, squares, polygons etc. A quadrilateral is a polygon with four sides. In this paper, we will focus on *tangential quadrilaterals*, sometimes known as a tangent or circumscribing quadrilateral. A tangential quadrilateral is a convex quadrilateral whose sides are all tangent to a single circle within the quadrilateral (see Figure 1).

There are many problems related to quadrilaterals which have attracted the interests of mathematicians. In particular, the problem of finding *rational* quadrilaterals has interested mathematicians since antiquity. A polygon is said to be a rational polygon if all of its sides and diagonals have rational lengths. In the seventh century, Brahmagupta gave an elegant method to construct rational quadrilaterals. Brahmagupta's method was elaborated further by Bhaskara in the 12th century and by Chasles in the 19th century (as quoted by Dickson [15, pp.216–217]). In 1848, Kummer [24] gave a method for generating all rational quadrilaterals. Dickson later presented Kummer's construction in a somewhat simplified manner [14].

Since the publication of Kummer's classic paper, various related problems concerning rational quadrilaterals have also been studied (see, for example, [2, 3, 7, 9, 14, 20, 21, 23, 27, 28]). Some authors have adopted a stricter definition of rational quadrilaterals which requires the area to be rational too. For instance, Sastry [28, pp.170–171] gave a parameterization of cyclic quadrilaterals whose sides, diagonals and area are all expressible by rational numbers.

Many of the questions about rational triangles and quadrilaterals can be reduced to solving certain Diophantine equations. For example, this technique is employed to find right-angle triangle and rectangle pairs which have the same area and the same perimeter [6], pairs of incongruent Heron triangles with the same area and perimeter [5], integral right triangle and rhombus pairs with a common area and a common perimeter [8], right triangle and parallelogram pairs with a common area and a common perimeter [35], integral isosceles triangle-parallelogram and Heron triangle-rhombus pairs with a common area and common perimeter [11] (see also [22, 25, 35, 36]). Researchers have also studied other problems related to certain rational quadrilaterals, such as [18, 19, 21]. Typically the results in these works are established by transforming these problems into questions dealing with the existence of rational points on certain families of elliptic curves.

To the best of our knowledge, the study of rational tangential quadrilaterals has never been addressed. The main result of this article is to construct an infinite family of rational tangential quadrilaterals which have rational area. We use the method of transforming the problem into a question on rational points in a family of elliptic curves. The existence of rational points on these

2020 *Mathematics Subject Classification.* 51M04, 14H52, 14G05, 11G05.
Key words and phrases. Tangential Quadrilateral, Rational Quadrilateral, Elliptic Curve.

curves will lead directly to the existence of rational tangential quadrilaterals with rational area. We also study the properties of the elliptic curves used in the construction described above.

This paper is organized as follows. In Section 2, we provide background information on tangential quadrilaterals. We give a proof of the main result in Section 3 by constructing an infinite family of rational tangential quadrilaterals. In Section 4, we consider the curve family (from Section 3) over function fields. We examine the torsion group over $\mathbb{Q}$, before concluding with directions for future work.

2. Construction of Tangential Quadrilaterals

2.1. **Background.** The well-known Pitot's theorem states that for a tangential quadrilateral, the sums of the opposite side lengths are equal. In other words, if the side lengths of a tangential quadrilateral are (respectively) a, b, c, and d then $a + c = b + d$. The converse is also true: any convex quadrilateral whose opposite sides sum to the same value must be a tangential quadrilateral.

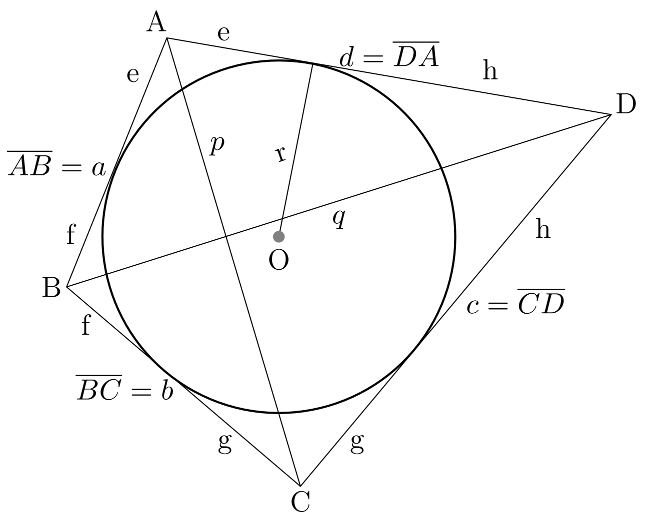

Figure 1. A tangential quadrilateral.

The tangent lengths of the quadrilateral are the lengths of the line segments from the quadrilateral vertices to the points of contact with the tangential circle (see Figure 1). If the quadrilateral is $\overline{ABCD}$, we will refer to the tangent lengths as e, f, g, and h, where $\overline{AB} = a = e + f, \overline{BC} = b = f + g, \overline{CD} = c = g + h$, and $\overline{DA} = d = h + e$.

It can be worked out that the diagonals have lengths p, q where

$$p = \sqrt{\frac{e+g}{f+h}((e+g)(f+h) + 4fh)},$$

$$q = \sqrt{\frac{f+h}{e+g}((e+g)(f+h) + 4eg)}.$$

The radius of the tangential circle is given by

$$r = \sqrt{\frac{efg + fgh + ghe + hef}{e+f+g+h}},$$

and the semiperimeter of ABCD is $s = (a+b+c+d)/2 = e+f+g+h$. With these quantities defined, there are several equivalent formulas for the area K:

$$K = rs,$$
$$K = \frac{1}{2}\sqrt{p^2q^2 - (ac-bd)^2},$$
$$K = \sqrt{(e+f+g+h)(efg+fgh+ghe+hef)},$$
$$K = \sqrt{abcd - (ef-gh)^2}.$$

We also note that the tangential quadrilateral is cyclic if and only if $eg = fh$, which shows the area is maximal ($K = \sqrt{abcd}$) when the quadrilateral is cyclic.

2.2. **A Rational Family.** In this section we seek to construct rational tangential quadrilaterals. We use the same notation as indicated above (and illustrated in Figure 1). We begin by finding rational e, f, g and h so that we obtain a rational tangential quadrilateral with rational area. For simplicity, we will assume $h = 1$, as if $h \neq 1$ we may simply rescale the other lengths e, f, g by h. Note that the side lengths a, b, c, d will be rational, as will the semiperimeter (assuming e, f, g are rational). It follows that the area K will be rational if and only if the radius r is rational, since $K = rs$. From above, this requires

$$r^2s^2 = (e+f+g+1)(efg+fg+ge+ef),$$

or equivalently

$$r^2(e+f+g+1) = efg+fg+ge+ef.$$

We solve for e to obtain

$$e = -\frac{fg - r^2(f+g+1)}{fg+f+g-r^2}.$$

Substituting this value back in, the area will then be

$$K = \frac{(g+1)(f+1)(f+g)r}{fg+f+g-r^2},$$

which is rational provided that e, f, g, and r are rational.

If we look at the equation for the diagonal q to be rational, we will need that

$$\frac{f+1}{e+g}((e+g)(f+1)+4eg)$$

is a rational square. We parameterize this expression by setting

$$f = \frac{4g^2r^2 - g^2w^2 - r^2w^2 + 4r^2g + g^2 + r^2}{2(g^2w - 2r^2g + r^2w + g^2 - r^2)},$$

so that then

$$q = \frac{4g^2r^2 + g^2w^2 - 4gr^2w + r^2w^2 + 2g^2w + 4r^2g - 2r^2w + g^2 + r^2}{2(g^2w - 2r^2g + r^2w + g^2 - r^2)}.$$

The diagonal length q will be rational as long as w, r, and g are rational.

We note that by further substitution, we obtain

$$e = \frac{(w-1)(2r^2g - r^2w + gw + r^2 + g)}{4r^2g - gw^2 - 2r^2w + 2gw + 2r^2 - w^2 + 3g + 1}.$$

The condition for the diagonal p to be rational then becomes:

$$\begin{aligned}(2g-w+1)^2(2g+w+3)^2r^4 - 2(w+1)(2g-w+1)(2g^3w + g^2w^2 - 6g^3 + 4g^2w \\ +2gw^2 - 13g^2 + 2w^2 - 10g - 2)r^2 + g^2(w+1)^2(gw-3g+2w-2)^2 = z^2,\end{aligned} \tag{1}$$

for some rational z. This is an elliptic curve in (r, z), and for specific values of g and w can be an elliptic curve with positive rank. For example, when $(g, w) = (1, -3)$ then the equation (1) becomes

$$C : 144r^4 - 96r^2 + 784 = z^2,$$

which is birationally equivalent to the Weierstrass curve

$$E : y^2 = x^3 - 454656x + 28835840.$$

This particular curve E has rank 1 with generator $(x, y) = (-128, 9216)$. This generator corresponds to $(r, z) = (7/6, 91/3)$. The corresponding tangential quadrilateral has lengths

$$(e, f, g, h) = (74/11, 24/61, 1, 1).$$

As can be checked, the area is $7140/671$ and the diagonal lengths are $(p, q) = (91/11, 159/61)$. Thus, we have an example of a rational tangential quadrilateral. See Figure 2 for an illustration.

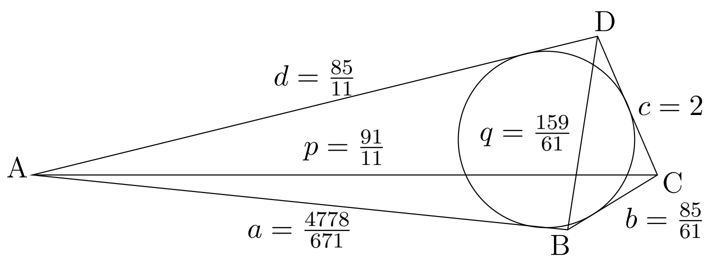

FIGURE 2. Tangential quadrilateral with $e = 74/11, f = 24/61, g = h = 1, p = 91/11, q = 159/61$.

3. The Main Result

We now utilize the construction above to prove the main result of this work, showing not just a single example of a rational tangential quadrilateral, but infinitely many.

Theorem 1. *There are infinitely many rational tangential quadrilaterals whose area, side lengths, and diagonal lengths are all rational.*

Proof. We use the construction in Section 2.2. If we set $(g, w) = (1, -3)$ in equation (1), then we previously noted we have a positive rank elliptic curve E, which has an infinite number of rational points. By the construction, each such rational point will lead to values of (e, f, g, h), where we have $g = h = 1$. To obtain the formulas for e and f, we use the inverse transformation $r = 56y/(x^2 + 64x - 450560)$ to map from E to C. Some calculation yields that

$$e = -2\frac{3r^2 - 1}{3r^2 - 5} = \frac{-2(x^2 - 9344x + 141552)}{(5x + 832)(x - 1984)},$$

and also that

$$f = \frac{2}{3r^2 + 1} = \frac{2(x + 704)(x - 640)}{x^2 + 9472x - 1052672}.$$

As e and f are rational (since x is), then these values will yield a constructible rational quadrilateral with tangent lengths e, f, g, h as long as they are positive. Clearly $f = 2/(3r^2 + 1)$ is always positive, and it is easy to see that e will be positive precisely when $\frac{1}{\sqrt{3}} < |r| < \sqrt{\frac{5}{3}}$. As the elliptic curve C has only one connected component, by a theorem of Hurwitz and Poincaré on the density of rational points on curves (see [34, p. 78]), we have that there will be an infinite number of rational points (r, z) which satisfy the condition for e to be positive. We can therefore conclude that there are an infinite number of rational tangential quadrilaterals. □

For further concrete examples of rational tangential quadrilaterals, we could use other rational curve points of E. The point $(x, y) = (2368, 110592)$ leads to this same quadrilateral already illustrated in Figure 2. The point $(x, y) = (3904/361, -33546240/6859)$ yields $[e, f, g, h] = [1826/31535, 16224/17137, 1, 1]$ with

$$[r, K, p, q] = \left[\frac{95}{156}, \frac{197764008}{108083059}, \frac{56209}{31535}, \frac{35187}{17137}\right].$$

The proof could have used other choices for (g, w) in equation (1) to similarly produce infinitely many rational tangential quadrilaterals. For example, some other values for (g, w) which lead to positive rank curves are

$$(1, -6), (1, -7), (1, -8), (1, -9), (2, -8), (3, -6), (3, -8), (4, -7), (4, -8).$$

Experiments seem to suggest that if $w = -8$ the rank is always positive, though we have been unable to prove this. Using $w = -8$, the curve E is then $y^2 = x^3 + a_4 x + a_6$, where

$$a_4 = -\frac{784}{3}(2g+9)^2(484g^6 - 836g^5 - 3907g^4 + 2300g^3 + 10753g^2 + 7434g + 3969),$$

$$a_6 = \frac{21952}{27}(2g+9)^3(22g^3 - 19g^2 - 76g + 63)(44g^3 - 38g^2 - 194g - 63)(22g^3 - 19g^2 - 118g - 126).$$

If we could find rational points on this curve (in terms of g), we would obtain equations parameterizing an infinite family of rational tangential quadrilaterals. We leave this as an open question.

4. Study of curve over function field

The aim of this section is to study some of the elliptic curves of Section 3 over $\mathbb{Q}(t)$. First we recall some basic notions about elliptic surfaces.

Definition. Let C be a smooth, irreducible projective curve over an algebraically closed field k. An elliptic surface over C is a pair (S, f), where S is a smooth, irreducible, projective surface over k, and $f : S \longrightarrow C$ is a relatively minimal elliptic fibration having a singular fiber and a zero section. We often write $f : S \longrightarrow C$ to denote the elliptic surface (S, f) over C.

Let $k(C)$ denote the function field of the curve C. Given an elliptic curve E over $k(C)$, one can associate an elliptic surface $f : \mathcal{E} \longrightarrow C$ with generic fiber E, the existence and uniqueness of which is guaranteed by the work of Kodaira and Néron. This elliptic surface is known as the Kodaira-Néron model of the elliptic curve E over $k(C)$.

Given that all the relevant results needed to prove our main theorem are well known, we just give their statements and omit the proofs.

Theorem 2 ([29, Corollary 2.2])**.** *Let (S, f) be an elliptic surface over C. The Néron-Severi group, denoted $NS(S)$, is finitely generated and torsion-free.*

Theorem 3 ([29, Corollary 5.3])**.** *Let (S, f) be an elliptic surface over C. For each point v of C having singular fiber, let m_v denote the number of components of the singular fiber above v. Let E denote the generic fiber of S. The rank of the Néron-Severi group of S, denoted $\rho(S)$, can be obtained from the equality*

$$\rho(S) = \text{rank } E(k(C)) + 2 + \sum_v (m_v - 1),$$

where the summation ranges over the the points of C under singular fibers.

We will also need the following lemma.

Lemma 1 ([32, Theorem IV.8.2] and [30, Corollary 7.5])**.** *Let E be an elliptic curve over $\overline{\mathbb{Q}}(t)$. Let $\Sigma \subset \mathbb{P}^1(\overline{\mathbb{Q}}(t))$ be the set of points of bad reduction of E. Let $G(F_v)$ denote the group generated by simple components of the fiber F_v at $v \in \Sigma$. Then there exists an injective homomorphism*

$$\phi : E(\overline{\mathbb{Q}}(t))_{tors} \longrightarrow \prod_{v\in\Sigma} G(F_v).$$

If F_v is of multiplicative type I_n in Kodaira notation, the corresponding group is $\mathbb{Z}/n\mathbb{Z}$. If F_v is of additive type I^*_{2n}, the group is $(\mathbb{Z}/2\mathbb{Z})^2$.

Using the above theory, we study the curve $E_g : y^2 = x^3 + a_4x + a_6$, where
(2)

$$\begin{aligned} a_4 &= -\frac{784}{3}(2g+9)^2(484g^6 - 836g^5 - 3907g^4 + 2300g^3 + 10753g^2 + 7434g + 3969),\\ a_6 &= \frac{21952}{27}(2g+9)^3(22g^3 - 19g^2 - 76g + 63)(44g^3 - 38g^2 - 194g - 63)(22g^3 - 19g^2 - 118g - 126). \end{aligned}$$

Recall that if we could find nontrivial rational points on E_g then we would have a parameterization of infinitely many rational tangential quadrilaterals.

Theorem 4. *Let E_g denote the elliptic curve over $\mathbb{Q}(g)$ given by*

$$E_g : y^2 = x^3 + a_4x + a_6,$$

where a_4 and a_6 are given by the equations (2). Then

(1) The associated elliptic surface (denoted $\mathcal{E}$) is a $K3$ surface.
(2) The rank of $E_g(\overline{\mathbb{Q}}(g))$ is not more than 2.
(3) The group $E_g(\mathbb{Q}(g))_{\text{tors}}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Proof. The discriminant of this elliptic curve E_g is

$$\Delta(g) = (614656)^2g^2(22g - 63)^2(2g - 5)^2(g + 1)^4(11g + 18)^2(2g + 9)^8.$$

From the above expression for the discriminant, we see that the associated elliptic surface has singular fibers at the values $g = 0, 63/22, 5/2, -1, -18/11, -9/2$ and ∞. We determine the numbers m_v, of irreducible components of the fiber over v, from Kodaira types of singular fibers.

v	coefficients				
	$ord_{g=v}(A)$	$ord_{g=v}(B)$	$ord_{g=v}(\Delta)$	Kodaira type	$m_v - 1$
0	0	0	2	I_2	1
63/22	0	0	2	I_2	1
5/2	0	0	2	I_2	1
-1	0	0	4	I_4	3
$-18/11$	0	0	2	I_2	1
$-9/2$	2	3	8	I_2^*	6
∞	0	0	4	I_4	3

The Euler number $e(S)$ of the associated elliptic surface S equals

$$e(S) = \Sigma_v\, e(F_v),$$

where the sum varies over the set of points having singular fibers. The local Euler number $e(F_v)$ is equal to the number of components m_v if the fiber has multiplicative reduction, or to $m_v + 1$ if the reduction is additive (cf. [10, Proposition 5.1.6]). Using the above table we find that $e(S) = 24$. It follows that S has Kodaira dimension 0, and hence it is a $K3$ surface. For $K3$ surfaces we have the Picard rank $\rho(S) \le 20$. Therefore, Theorem 2 gives

$$20 \ge \rho(S) = 2 + 16 + \operatorname{rank} E_g(\overline{\mathbb{Q}}(g)),$$

which yields $\operatorname{rank} E_g(\overline{\mathbb{Q}}(g)) \le 2$.

By Lemma 1 and Mazur's theorem, the torsion subgroup of $E_g(\mathbb{Q}(g))$ is embedded in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, where $n = 2$ or 4. We have three points of order 2, namely

$$Q_1 = \frac{28}{3}(2g+9)\left(22g^3 - 19g^2 - 118g - 126, 0\right),$$
$$Q_2 = \frac{28}{3}(2g+9)(22g^3 - 19g^2 - 76g - 63, 0),$$
$$Q_3 = -\frac{28}{3}(2g+9)(44g^3 - 38g^2 - 194g - 63, 0).$$

If $P = (x, y)$ is a point of order 4, then $2P$ must be one of these points. Setting $2P = Q_i$ ($i = 1, 2, 3$), we find there are no solutions in $= \mathbb{Q}(g)$ for x and y. We can conclude that there are no points of order 4, and hence $E_g(\mathbb{Q}(g))_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

□

We note that for certain values of g, the torsion group can be larger than $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ For example, if $g = 27/8$, there is a point of order 3, $(23402547/256, -19297377225/1024)$ and the torsion group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. If $g = 45/4$, there is a point of order 4, $(167318487/16 : 189114296805/16)$, and the torsion group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. In fact, there are an infinite number of values of g for which there will be a point of order 4. This can be seen be using the doubling formula to check for possible points R such that $2R = Q_1$. Some easy algebra shows for a rational solution to exist the following expression needs to have rational solutions

$$7g(11g + 18)(2g - 5)(2g + 9) = z^2.$$

This curve equation is birationally equivalent to the elliptic curve $y^2 = x^3 - 8362683x + 3602177082$, which has rank 1. For each rational point on this curve, we could substitute back to find a value of g that will produce a point of order 4 on E_g. We performed some experiments to determine if other torsion groups were possible, but did not find any besides $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, for $N = 2, 4$, or 6.

We also did some computations to find high rank curves in the family E_g. We found several values of g for which the rank is 6: namely when $g = -11/134, -107/207, 383/6, -686/271$, or $-883/449$.

5. Conclusion

In this work we proved that there are infinitely many rational tangential quadrilaterals which have rational side lengths, diagonal lengths, and rational area. The proof technique used a specific elliptic curve, but it would be nice if a more general approach could be found. Specifically, we believe that a parameterized family of rational tangential quadrilaterals can be constructed, although we were unable to do so. One way to produce such a parameterization could be to use specific values of g or w in (1) that lead to a curve family with rational points. As noted in Section 3, we think it likely that the curve family with $w = -8$ (studied in Section 4) has generic rational points, but we were unable to establish this. It would be interesting to find such rational points, or to produce other values of g or w where such rational points can be found. This would yield a constructive proof, rather than our proof which only establishes the existence of infinitely many rational tangential quadrilaterals. Alternatively, it would be interesting to find other ways of constructing rational tangential quadrilaterals, or to do a similar study for other geometric shapes yet investigated.

References

[1] J. H. J. Almering, *Rationaliteitseigenschappen*, in de vlakke meetkunde. Diss. Amsterdam, (1950).
[2] J. H. J. Almering, *Rational quadrilaterals*, Indag. Math. 25 (1963), 192–199.
[3] J. H. J. Almering, *Rational quadrilaterals. II*, Indag. Math. 27 (1965), 290–304.
[4] A. S. Besicovitch, *Rational Polygons*, Mathematika, 6:98, 1959.

[5] A. Bremner, On Heron triangles, Ann. Math. Inform. 33 (2006) 15–21.
[6] A. Bremner, R. K. Guy, Triangle-rectangle pairs with a common area and a common perimeter, Int. J. Number Theory (2006), no.2, 217–223.
[7] R. H. Buchholzand, J. A. MacDougall, *Cyclic polygons with rational sides and area*, J. Number Theory 128 (2008), 17–48.
[8] S. Chern, Integral right triangle and rhombus pairs with a common area and a common perimeter, Forum Geom. 16 (2016) 25–27.
[9] A. Choudhry, *Rational quadrilaterals*, to appear, Rocky Mt. J. Math.
[10] F. R. Cossec and I. V. Dolgachev, *Enriques Surfaces. I*, Progr. Math. 76, Birkhäuser Boston, Boston, MA, 1989.
[11] P. Das, A. Juyal and D. Moody, Integral isosceles triangle-parallelogram and Heron triangle-rhombus pairs with a common area and common perimeter, J. Number Theory 180(2017), 208–218.
[12] D. E. Daykin, *Rational Polygons*, Mathematika, 10:125–131, 1963.
[13] D. E. Daykin, *Rational triangles and parallelograms*, Mathematics Magazine, 38 (1965), 46–47.
[14] L. E. Dickson, *Rational triangles and quadrilaterals*, Am. Math. Month. 28 (1921), 244–250.
[15] L. E. Dickson, *History of the theory of numbers* Vol. 2, reprint, Chelsea Publishing Company, 1992.
[16] A. Dujella, *High rank elliptic curves with prescribed torsion*, `http://web.math.hr/~duje/tors/tors.html` (accessed December 2020).
[17] A. Dujella, *Infinite families of elliptic curves with high rank and prescribed torsion*, `https://web.math.pmf.unizg.hr/~duje/tors/generic.html` (accessed December 2020).
[18] F. Izadi, F. Khoshnam, and D. Moody, *Heron quadrilaterals via elliptic curves* Rocky Mountain Journal of Mathematics, 47 (2017).
[19] F. Izadi, F. Khoshnam, D. Moody, and A.S. Zargar, *Elliptic curves arising from Brahmagupta quadrilaterals*, Bulletin of the Australian Mathematical Society, 90 (1), (2014) 47-56.
[20] M. Hajja and J. Sondow, *Newton quadrilaterals, the associated cubic equations, and their rational solutions*, Am. Math. Mon. 126(2019),135–150.
[21] F. Izadi, F. Khoshnamand and D. Moody, *Heron quadrilaterals via elliptic curves*, Rocky Mt. J. Math. 47 (2017), 1227–1258.
[22] A. Juyal and D. Moody, Pairs of Heron and right triangles with a common area and a common perimeter, Publ. Math. Debrecen 100 (2022), no. 3–4, 449–460.
[23] C. P. A. Kumar, Approximation of quadrilaterals by rational quadrilaterals in the plane, Proc. Indian Acad. Sci (Math.Sci.) Vol.127, No.1, February 2017, pp.69–90.
[24] E. E. Kummer, *Collected papers, Vol. 1*, Springer-Verlag 1975, ¨Uber die Vierecke, deren Seiten und Diagonalen rational sind, pp. 253–273,(published originally in Journal f¨ur die reine und angewandte Mathematik 37(1848), 1–20).
[25] M. Laĺin, X. Ma, θ-triangle and ω-parallelogram pairs with common area and common perimeter, J. Number Theory 202 (2019), 1–26.
[26] R. Miranda, *An overview of algebraic surfaces, in: Algebraic Geometry (Ankara, 1995)*, Lecture Notes in Pure and Appl. Math 193, Dekker, New York, 1997, 197–217.
[27] L. J. Mordell, *Rational quadrilaterals*, J. London Math. Soc. 35 (1960), 277–282.
[28] K. R. S. Sastry, *Brahmagupta quadrilaterals*, Forum Geom. 2 (2002), 167–173.
[29] T. Shioda, *On the Mordell-Weil lattices*, *Comment. Math. Univ. St. Pauli*, **39** (1990), 211–240.
[30] T. Shioda and M. Schütt, *Elliptic surfaces.* Algebraic geometry in East Asia-Seoul 2008, 51–160, *Adv. Stud. Pure Math.*, **60**, *Math. Soc. Japan, Tokyo*, 2010.
[31] J. H. Silverman, *The Arithmetic of Elliptic Curves*, Springer-Verlag, New York-Berlin-Heidelberg-tokyo, 1986.
[32] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics, **151** Springer-Verlag, New York (1994).
[33] T. Skolem, Diophantische gleichungen, Berlin, 1938.
[34] T. Skolem, Diphantische Gleichungen, Chelsea, 1950.
[35] Y. Zhang, Right triangle and parallelogram pairs with a common area and a common perimeter, J. Number Theory 164 (2016) 179–190.
[36] Y. Zhang and A. S. Zargar, Integral triangles and perpendicular quadrilateral pairs with a common area and a common perimeter, Funct. Approx. Comment. Math. 63 (2020), no. 2, 165–180.

(Pradeep Das) Department of Mathematical Sciences, Rajiv Gandhi Institute of Petroleum Technology, Jais, Amethi, 229304, India
Email address: `pradeepdas0411@gmail.com`

(Abhishek Juyal) Hemvati Nandan Bahuguna Garhwal University, Srinagar Garhwal, India, 246174.
Email address: `abhinfo1402@gmail.com`

(Dustin Moody) Computer Security Division, National Institute of Standards & Technology, Gaithersburg, Maryland, USA.
Email address: dustin.moody@nist.gov