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Internal and external harmonics in bi-cyclide coordinates

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Abstract

The Laplace equation in three-dimensional Euclidean space is $\mathcal{R}$-separable in bi-cyclide coordinates leading to harmonic functions expressed in terms of Lamé–Wangerin functions called internal and external bi-cyclide harmonics. An expansion for a fundamental solution of Laplace's equation in products of internal and external bi-cyclide harmonics is derived. In limiting cases this expansion reduces to known expansions in bi-spherical and prolate spheroidal coordinates.

Keywords: Laplace's equation, fundamental solution, separable curvilinear coordinate system, bi-cyclide coordinates, Lamé–Wangerin functions

(Some figures may appear in colour only in the online journal)

1. Introduction

The Laplace equation $\Delta u = 0$ is separable in various coordinate systems in three-dimensional Euclidean space, among them the bi-cyclide coordinate system. In the theory of separation of variables for Laplace's equation in separable coordinate systems, one may obtain a basis of solutions with a corresponding set of quantum numbers which allows one to expand arbitrary solutions of the homogeneous equation as a sum of separated eigenfunctions over the set of quantum numbers. Once you have obtained this basis, one of the most natural functions to

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expand in terms of this basis is a fundamental solution of Laplace's equation because it satisfies the homogeneous equation almost everywhere. For Laplace's equation in three-dimensions, a fundamental solution of Laplace's equation is proportional to the reciprocal distance between two points. The expansions of the reciprocal distance between two points in terms of the harmonics in separable coordinate systems has a rich full history going back to the 1800s. Such expansions are known for many Laplace separable coordinate systems, but not for all of them. In two previous papers [3, 4], the expansion of the reciprocal distance was given in flat-ring coordinates for the first time. In this paper we derive the expansion of the reciprocal distance in terms of harmonic functions separated in bi-cyclide coordinates.

Bi-cyclide coordinates were originally introduced by Wangerin [25]. They can also be found in Miller [18, p 211, system 14] and Moon and Spencer [19, p 124] (the connection between these forms of bi-cyclide coordinates is made in appendix A). In these references the ordinary differential equations obtained by applying the method of separation of variables to $\Delta u = 0$ in bi-cyclide coordinates are given. However, formulas for the internal and external harmonics as well as the corresponding expansion of the reciprocal distance using these harmonic functions are missing. It is the purpose of this paper to supply these missing results.

In section 2 we define bi-cyclide coordinates in the form given by Miller and carry out the process of separation of variables to the Laplace equation. The form of the bi-cyclide coordinates used by Miller has the advantage that two of the separated ordinary differential equations appear in the standard form of the Lamé equation. This is not the case if the coordinates are given as in Wangerin or Moon and Spencer. In section 3 we review Lamé–Wangerin functions that appear in the definitions of internal and external bi-cyclide harmonics. Lamé–Wangerin functions are particular solutions of Lamé's differential equation that have recessive behavior at two neighboring regular singularities. Moreover, an estimate for Lamé–Wangerin functions is given that is needed to prove convergence of various series expansions. In section 4 internal and external bi-cyclide harmonics are introduced and their main properties are established. In section 5 various results involving internal and external bi-cyclide harmonics are proved. These results include the solution of a Dirichlet problem and an integral representation of external harmonics in terms of internal harmonics. Finally, as the main result of this paper, the expansion of the reciprocal distance between two points in a series of internal and external bi-cyclide harmonics is given. As corollaries we find an addition theorem and integral relations for Lamé–Wangerin functions in terms of zero order toroidal harmonics of the second kind. In section 6 we introduce a second kind of internal and external bi-cyclide harmonics. In contrast to corresponding results in flat-ring coordinates, these internal and external bi-cyclide harmonics of the second kind can be reduced to the ones of the first kind by a Kelvin transformation. In the final two sections 7.1 and 7.2 we show that limiting cases of bi-cyclide coordinates include bi-spherical and prolate spheroidal coordinates. We connect the expansion of the reciprocal distance in bi-cyclide coordinates to the known expansions in bi-spherical and prolate spheroidal coordinates.

2. Bi-cyclide coordinates

Bi-cyclide coordinates are orthogonal curvilinear coordinates on $\mathbb{R}^3$ which are one of the seventeen conformally inequivalent coordinate systems which allow for separation of variables of Laplace's equation in three-dimensions (see [18, tables 14 and 17]). All of these coordinate systems are triply-orthogonal coordinate systems in that all two-dimensional coordinate equal constant surfaces in $\mathbb{R}^3$ intersect at right angles. This coordinate system is also rotationally-invariant about the z-axis, of which there are a total of nine out of these seventeen coordinate systems. Another interesting property of these seventeen conformally inequivalent coordinate systems is that eleven of these have coordinate equal constant surfaces which are second order quadrics (three-dimensional generalizations of the conic sections) and the remaining six coordinate systems have coordinate equal constant surfaces which are fourth-order surfaces referred to as cyclides. Bi-cyclide coordinates are one of the rotationally-invariant coordinate systems which are cyclidic with two fourth-order coordinate equal constant surfaces (not including the half-planes which correspond to a fixed angle about the z-axis) referred to as bi-cyclides and apple-shaped cyclides respectively.

Miller [18, p 211, (6.28)] introduces bi-cyclide coordinates α, β, ϕ in $\mathbb{R}^3$ by

$$x = R\cos\phi, \quad y = R\sin\phi, \quad z = ikR\,\mathrm{sn}(\alpha,k)\,\mathrm{sn}(\beta,k), \tag{2.1}$$

where

$$\frac{1}{R} = \frac{i}{k'}\left(\mathrm{dn}(\alpha,k)\,\mathrm{dn}(\beta,k) - k\,\mathrm{cn}(\alpha,k)\,\mathrm{cn}(\beta,k)\right).$$

Note that we corrected a typo in the definition of R. These coordinates depend on a given modulus $k \in (0,1)$, and involve the Jacobian elliptic functions $\mathrm{cn}, \mathrm{sn}, \mathrm{dn}$ [21, chapter 22]. We also use the complementary modulus $k' = \sqrt{1-k^2}$ and the complete elliptic integrals of the first kind $K = K(k)$ and $K' = K'(k) = K(k')$. The complex coordinates α and β vary in the segments $\alpha \in (iK', 2K + iK')$, $\beta \in (2K - iK', 2K + iK')$ and $\phi \in (-\pi, \pi]$.

Bi-cyclide coordinates can also be seen as a coordinate system in the (R,z)-plane, where $R = (x^2+y^2)^{1/2}$ denotes the distance of a point (x,y,z) in $\mathbb{R}^3$ to the z-axis. Three-dimensional bi-cyclide coordinates are then obtained by adding the rotation angle ϕ about the z-axis.

We prefer a real version of bi-cyclide coordinates. Setting $\alpha = s + K + iK'$, $\beta = 2K + it$ with $s \in (-K,K)$, $t \in (-K',K')$, we obtain

$$R = \frac{\mathrm{cn}(s,k)\,\mathrm{cn}(t,k')}{1 - \mathrm{sn}(s,k)\,\mathrm{dn}(t,k')}, \tag{2.2}$$

$$z = \frac{\mathrm{dn}(s,k)\,\mathrm{sn}(t,k')}{1 - \mathrm{sn}(s,k)\,\mathrm{dn}(t,k')}. \tag{2.3}$$

In the derivation of (2.2) and (2.3), standard identities for Jacobian elliptic functions are used [21, sections 22.4 and 22.6].

The mapping $(s,t) \in (-K,K) \times (-K',K') \mapsto (R,z) \in (0,\infty) \times \mathbb{R}$ is bijective, (real) analytic and its inverse is also analytic. The proof that the coordinate mapping $(s,t) \mapsto (R,z)$ is bijective is similar to the corresponding proof in flat-ring coordinates [3]. In fact, planar bi-cyclide and planar flat-ring coordinates are closely related as can be seen as follows.

Letting $s = \sigma - K$, $\sigma \in (0,2K)$ and $t = K' - \tau$, $\tau \in (0,K')$, we obtain

$$z = T^{-1}, \quad R = (x^2+y^2)^{1/2} = -\frac{ik}{T}\,\mathrm{sn}(\sigma,k)\,\mathrm{sn}(i\tau,k),$$

where

$$T = \frac{1}{k'}\,\mathrm{dn}(\sigma,k)\,\mathrm{dn}(i\tau,k) + \frac{k}{k'}\,\mathrm{cn}(\sigma,k)\,\mathrm{cn}(i\tau,k).$$

Thus σ, τ are exactly the planar flat-ring coordinates (coordinates on $\mathbb{R}^2$, see for instance [19, figure 2.16]) treated in [3, section 2.2] except that $(x,y) = (R,z)$ are interchanged. Therefore, we can say that planar flat-ring and planar bi-cyclide coordinates are the same (in the first quadrant) but their three-dimensional versions become different because we rotate about different coordinate axes.

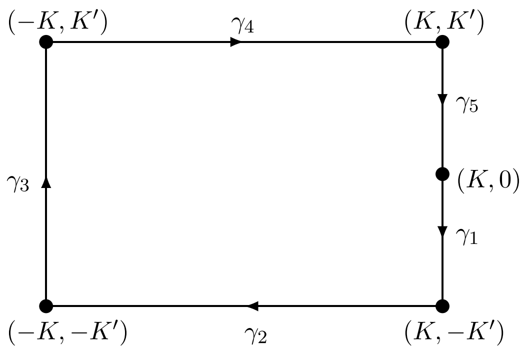

Figure 1. The rectangle $[-K,K]\times[-K',K']$ of coordinates s,t.

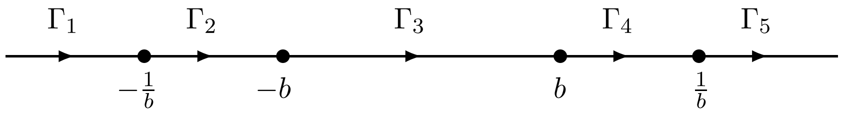

Figure 2. Bi-cyclide coordinates on the z-axis.

We extend planar bi-cyclide coordinates to the z-axis as follows. We note that the denominator on the right-hand sides of (2.2) and (2.3) is positive on the rectangle $(s,t)\in[-K,K]\times[-K',K']$ with the exception of the point $s=K, t=0$. Therefore, R,z are continuous functions on this rectangle with the point $(K,0)$ removed. The points on the boundary of the rectangle are mapped to $R=0$. As we go around the boundary of this rectangle in a clockwise direction as shown in figure 1, z transverses the z-axis from $-\infty$ to $+\infty$. The segments γ_j are mapped to Γ_j for each $j=1,2,3,4,5$ as shown in figure 2 using the notation

$$b=\frac{1-k}{k'}=\frac{k'}{1+k}\in(0,1). \tag{2.4}$$

For $s_0\in(-K,K)$ and $t_0\in(-K',K')$ we introduce the polynomials

$$P_1(x,y,z)=\operatorname{sn}^2(t_0,k')(R^2+z^2+1)^2-k^2\operatorname{sd}(t_0,k')^2(R^2+z^2-1)^2-4z^2, \tag{2.5}$$

$$P_2(x,y,z)=\operatorname{sn}^2(s_0,k)(R^2+z^2+1)^2-(R^2+z^2-1)^2-4k'^2\operatorname{sd}^2(s_0,k)z^2, \tag{2.6}$$

where we used Glaisher's notation for the Jacobi elliptic functions [21, (22.2.10)]. If s,t are bi-cyclide coordinates of (x,y,z) then

$$R^2+z^2+1=\frac{2}{1-\operatorname{sn}(s,k)\operatorname{dn}(t,k')},\quad R^2+z^2-1=\frac{2\operatorname{sn}(s,k)\operatorname{dn}(t,k')}{1-\operatorname{sn}(s,k)\operatorname{dn}(t,k')}, \tag{2.7}$$

so a computation gives

$$P_1(x,y,z)=\frac{4(\operatorname{sn}^2(t_0,k')-\operatorname{sn}^2(t,k'))(\operatorname{dn}^2(t_0,k')-k^2\operatorname{sn}^2(s,k))}{\operatorname{dn}^2(t_0,k')(1-\operatorname{sn}(s,k)\operatorname{dn}(t,k'))^2}, \tag{2.8}$$

$$P_2(x,y,z)=\frac{4(\operatorname{sn}^2(s_0,k)-\operatorname{sn}^2(s,k))(\operatorname{dn}^2(t,k')-k^2\operatorname{sn}^2(s_0,k))}{\operatorname{dn}^2(s_0,k)(1-\operatorname{sn}(s,k)\operatorname{dn}(t,k'))^2}. \tag{2.9}$$

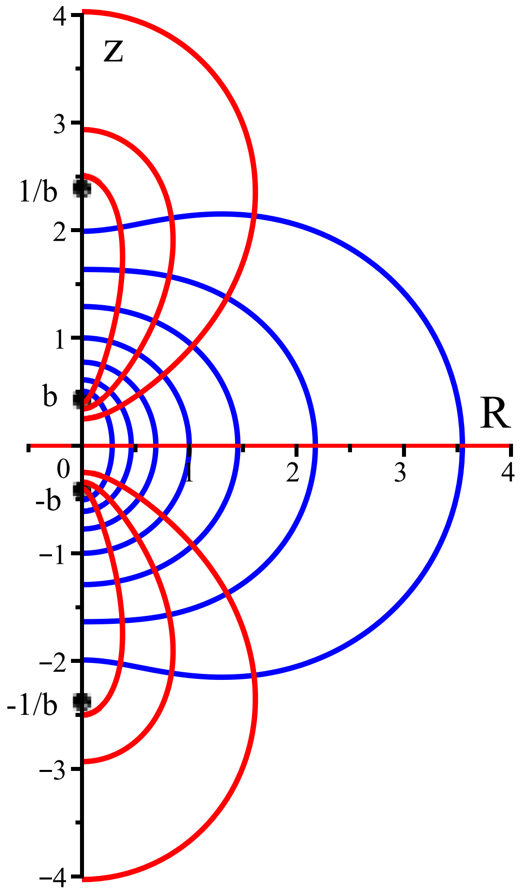

Figure 3. Coordinate lines $s=\pm\frac{3}{5}K,\pm\frac{2}{5}K,\pm\frac{1}{5}K,0$ in blue and $t=\pm\frac{4}{5}K',\pm\frac{3}{5}K',\frac{2}{5}K',0$ in red for bi-cyclide coordinates with $k=0.7$.

Therefore, $P_1(x,y,z)=0$ if and only if $t=t_0$ or $t=-t_0$, and $P_2(x,y,z)=0$ if and only if $s=s_0$ or $s=-s_0$.

Figure 3 depicts coordinate lines of planar bi-cyclide coordinates. The coordinate lines $s=s_0$ and $t=t_0$ are shown in blue and red, respectively. The coordinate line $t=0$ is the positive R-axis, and the coordinate line $s=0$ is half the unit circle. The mapping $(s,t)\to(s,-t)$ corresponds to $(R,z)\to(R,-z)$, and $(s,t)\to(-s,t)$ corresponds to $(R,z)\to(R^2+z^2)^{-1}(R,z)$, the inversion at the unit circle. The rectangle $(s,t)\in(0,K)\times(0,K')$ corresponds to the region $\{(R,z):R,z>0,R^2+z^2>1\}$. Figure 3 also shows the position of the four points $(0,\pm b)$, $(0,\pm b^{-1})$ on the z-axis.

2.1. *The inverse mapping*

Let $R\geqslant 0$, $z\in\mathbb{R}$ with corresponding bi-cyclide coordinates s,t. Then $P_1=P_2=0$ with $s=s_0$, $t=t_0$ imply that $\eta_1=-\frac{k'^2}{k^2}\,\mathrm{cn}^2(s,k)$ and $\eta_2=\mathrm{cn}^2(t,k')$ are solutions of the quadratic equation

$$E\eta^2+F\eta+G=0,$$

where

$$E=\frac{k'^2}{k^2}(R^2+z^2+1)^2,\quad F=\frac{k'^2}{k^2}\left(4z^2-(R^2+z^2+1)^2\right)+4(R^2+z^2),\quad G=-4R^2.$$

Therefore, if $\eta_1\leqslant\eta_2$ denote the solutions of this quadratic equation, then we find

$$s=\begin{cases}s_1 & \text{if } R^2+z^2\geqslant 1,\\ -s_1 & \text{if } R^2+z^2<1,\end{cases}\qquad t=\begin{cases}t_1 & \text{if } z\geqslant 0,\\ -t_1 & \text{if } z<0,\end{cases}$$

where

$$s_1=\operatorname{arccn}\left(\frac{k'}{k}\sqrt{-\eta_1},k\right)\in[0,K],\quad t_1=\operatorname{arccn}(\sqrt{\eta_2},k')\in[0,K'].$$

These formulas apply to every point $(x,y,z)\in\mathbb{R}^3$.

2.2. $\mathcal{R}$-separation of variables for Laplace's equation in bi-cyclide coordinates

Wangerin showed [25] that in rotationally-invariant coordinate systems, the Laplace equation $\Delta u=0$ can be solved by functions

$$u(x,y,z)=\mathcal{R}(x,y,z)u_1(s)u_2(t)u_3(\phi),\tag{2.10}$$

where $\mathcal{R}$ is a known elementary function called the modulation factor, following Morse and Feshbach [20], and u_1, u_2, u_3 solve ordinary differential equations. Here $\mathcal{R}=R^{-1/2}$, where R is the distance to the axis of rotation. In the literature, this is referred to as $\mathcal{R}$-separation of variables, and we say that Laplace's equation is $\mathcal{R}$-separable. See [18, section 3.6] for a detailed description of $\mathcal{R}$-separation of variables for the three-variable Laplace equation. In bi-cyclide coordinates one has $\mathcal{R}$-separated solutions

$$u(x,y,z)=R^{-1/2}u_1(\alpha)u_2(\beta)\mathrm{e}^{im\phi},\quad R=(x^2+y^2)^{1/2},$$

where u_1 and u_2 satisfy the ordinary differential equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}\zeta^2}+\left(\lambda-(m^2-\tfrac14)k^2\operatorname{sn}^2(\zeta,k)\right)w=0.\tag{2.11}$$

This is stated in [18, p 211, (6.28)] and will be confirmed in theorem 2.1 below.

The Lamé equation is

$$\frac{\mathrm{d}^2w}{\mathrm{d}\zeta^2}+(\lambda-\nu(\nu+1)k^2\operatorname{sn}^2(\zeta,k))w=0.\tag{2.12}$$

So (2.11) is the Lamé equation with $\nu=|m|-\frac12$.

If we write $v_1(s)=u_1(s+K+iK')$ and $v_2(t)=u_2(2K+it)$, we obtain the differential equations

$$\frac{\mathrm{d}^2v_1}{\mathrm{d}s^2}+(\lambda-(m^2-\tfrac14)\operatorname{dc}^2(s,k))v_1=0,\tag{2.13}$$

$$\frac{\mathrm{d}^2v_2}{\mathrm{d}t^2}-(\lambda+(m^2-\tfrac14)k^2\operatorname{sc}^2(t,k'))v_2=0.\tag{2.14}$$

Using $\operatorname{dc}^2(s,k)=1+k'^2\operatorname{sc}^2(s,k)$ we see that equation (2.13) is the same as (2.14) with k replaced by k' and λ replaced by $\nu(\nu+1)-\lambda$. We summarize the result in the following theorem.

Theorem 2.1. *If $m \in \mathbb{Z}$, $\lambda \in \mathbb{R}$, v_1 solves (2.13) on $(-K,K)$ and v_2 solves (2.14) on $(-K',K')$. Then*

$$u(x,y,z) = R^{-1/2} v_1(s) v_2(t) \mathrm{e}^{im\phi} \tag{2.15}$$

is a harmonic function in $\mathbb{R}^3 \setminus \{(0,0,z) : z \in \mathbb{R}\}$.

Proof. The metric coefficients of bi-cyclide coordinates are given by $h_\phi = R$ and

$$h_s = h_t = R\left(\mathrm{dc}^2(s,k) + k^2\,\mathrm{sc}^2(t,k')\right)^{1/2}. \tag{2.16}$$

In circular cylindrical coordinates R, z, ϕ, the Laplace equation $\Delta u = 0$ takes the form

$$\frac{\partial^2 v}{\partial R^2} + \frac{\partial^2 v}{\partial z^2} + R^{-2}\left(\frac{\partial^2 v}{\partial \phi^2} + \frac{1}{4}v\right) = 0,$$

where $u = R^{-1/2} v$. Using $h_s = h_t$ this equation transforms to

$$\frac{\partial^2 v}{\partial s^2} + \frac{\partial^2 v}{\partial t^2} + R^{-2} h_s^2 \left(\frac{\partial^2 v}{\partial \phi^2} + \frac{1}{4}v\right) = 0.$$

We now easily confirm that $v_1(s) v_2(t) \mathrm{e}^{im\phi}$ satisfies this equation. □

3. Lamé–Wangerin functions

We recall the Lamé–Wangerin eigenvalue problem. The Lamé equation (2.12) has regular singular points at $\zeta = iK'$ and $\zeta = 2K + iK'$ with exponents $-\nu$ and $\nu + 1$ at both points. The eigenvalue problem asks for solutions of (2.12) (with $\nu \geqslant -\frac{1}{2}$) on the segment $\zeta \in (iK', 2K + iK')$ which belong to the exponent $\nu + 1$ at both end points $iK^{'}$ and $2K + iK'$. In [13, section 15.6] the eigenfunctions of this eigenvalue problem are denoted by $F_\nu^n(\zeta, k^2)$. Alternatively, in [4] we used the notation $W_\nu^n(s,k) = F_\nu^n(s + K + iK', k^2)$.

We list the most important properties of the Lamé–Wangerin functions $W_\nu^n(s,k)$, $\nu \geqslant -\frac{1}{2}$, $n \in \mathbb{N}_0$. See [24] for further details.

(a) The function $W_\nu^n(s,k)$ is real-valued on the interval $s \in (-K,K)$ and has exactly n zeros in this open interval. In addition, $W_\nu^n(s,k) \to 0$ as $s \to \pm K$.

(b) For $s < K$ close to K we have the expansion

$$W_\nu^n(s,k) = \sum_{\ell=0}^{\infty} c_\ell (K - s)^{\nu + 1 + 2\ell}, \tag{3.1}$$

with real coefficients c_ℓ and $c_0 \neq 0$.

(c) The function $W_\nu^n(s,k)$ is even/odd with n:

$$W_\nu^n(-s,k) = (-1)^n W_\nu^n(s,k). \tag{3.2}$$

(d) For every fixed $\nu \geqslant -\frac{1}{2}$ and $k \in (0,1)$, the sequence of functions $\{W_\nu^n(s,k)\}_{n\in\mathbb{N}_0}$ forms an orthonormal basis of the Hilbert space $L^2(-K,K)$. See also appendix B for some of the details relating to this fact. Note that one may also find relevant information about this in the appendix of the Bi thesis [2, p 43] and also in [26, chapter 14].

(e) The function $W(s) = W_\nu^n(s)$ satisfies differential equation

$$\frac{\mathrm{d}^2 W}{\mathrm{d}s^2} + \left(\Lambda_\nu^n(k) - \nu(\nu+1)\,\mathrm{dc}^2(s,k)\right) W = 0, \tag{3.3}$$

where $\Lambda_\nu^n(k)$ denotes an eigenvalue. Some properties of these eigenvalues are given in [4, section 2].

(f) The function $W_\nu^n(s)$ can be continued analytically to an analytic function in the strip $|\Im s| < 2K'$ with branch cuts $(-\infty, -K]$ and $[K, +\infty)$ removed.

It should be mentioned that there are no explicit formulas for the Lamé–Wangerin functions W_ν^n nor for the eigenvalues Λ_ν^n. However, efficient methods for their numerical computation are available.

In the application to bi-cyclide coordinates we use Lamé–Wangerin functions $W_\nu^n(s,k)$ not only for $s \in (-K, K)$ but also for complex s with real part $-K$ or K. This is an important difference to the application of Lamé–Wangerin functions to flat-ring coordinates [4]. In that case, $W_\nu^n(s,k)$ was used for purely imaginary s. The following two lemmas state a property of Lamé–Wangerin functions with complex argument that is required in the subsequent analysis.

Lemma 3.1. *Let $\nu \geqslant 0$, $n \in \mathbb{N}_0$, $k \in (0,1)$.*

(i) $W_\nu^n(K+ir,k) \neq 0$ *for all* $r \in (0, 2K')$.

(ii) If $0 < r_1 < r_2 < 2K'$ *then*

$$0 < \frac{W_\nu^n(K+ir_1,k)}{W_\nu^n(K+ir_2,k)} \leqslant 2\mathrm{e}^{-\omega(n+\nu+1)(r_2-r_1)}, \quad \textit{where } \omega := \frac{\pi}{2K}.$$

Proof. The function $W_\nu^n(K+ir,k)$, $r \in (0,2K')$, is usually not real-valued. However, it follows from (3.1) that we can write $W_\nu^n(K+ir,k) = Cw(r)$ with a suitable complex constant C such that w is real-valued and has the expansion

$$w(r) = \sum_{\ell=0}^{\infty} \mathrm{d}_\ell r^{\nu+1+2\ell} \quad \text{with } d_\ell \in \mathbb{R},\ d_0 = 1, \tag{3.4}$$

for small $r > 0$. We will replace $W_\nu^n(K+ir,k)$ by $w(r)$ in the proof. Now (3.3) gives

$$w'' = q(r)w, \quad q(r) = \Lambda_\nu^n(k) + \nu(\nu+1)\,\mathrm{cs}^2(r,k'). \tag{3.5}$$

Using $\nu \geqslant 0$ and by Sturm's comparison theorem [15, section 10.4] (see also [4, lemma 2.3]), we find

$$q(r) \geqslant \Lambda_\nu^n(k) \geqslant \gamma^2, \quad \gamma := \omega(n+\nu+1) > 0.$$

It follows from (3.4) that $w(r) > 0$ and $w'(r) > 0$ for small $r > 0$ so (3.3) and $q(r) > 0$ imply $w(r) > 0$ and $w'(r) > 0$ for all $r \in (0, 2K')$. Now $u = w'/w$ satisfies the Riccati equation $u' + u^2 = q(r)$, so by comparison with the equation $v' + v^2 = \gamma^2$,

$$u(r) \geqslant \gamma \tanh(\gamma(r - r_1)) \quad \text{for } r \geqslant r_1.$$

Integrating from $r = r_1$ to $r = r_2$ gives

$$\ln \frac{w(r_2)}{w(r_1)} \geqslant \ln\cosh(\gamma(r_2 - r_1)) \geqslant \ln\left(\tfrac{1}{2}\mathrm{e}^{\gamma(r_2-r_1)}\right),$$

as desired. □

The proof of lemma 3.1 does not work for negative ν. When working with bi-cyclide coordinates we only use ν of the form $\nu = m - \frac{1}{2}$ with $m \in \mathbb{N}_0$, so it is sufficient to treat the case $\nu = -\frac{1}{2}$ in the following lemma.

Lemma 3.2. *Let* $k \in (0,1)$.

(i) For each $n \in \mathbb{N}_0$, $r \in (0,2K')$, $W^n_{-1/2}(K+ir,k) \neq 0$.
(ii) Let $0 < r_1 < r_2 < 2K'$. *Then there exist positive constants N such that*

$$0 < \frac{W^n_{-1/2}(K+ir_1,k)}{W^n_{-1/2}(K+ir_2,k)} \leqslant 2\mathrm{e}^{-\frac{\sqrt{3}}{2}\omega(n+\frac{1}{2})(r_2-r_1)} \quad \textit{for } n \geqslant N.$$

Proof. As in the proof of lemma 3.1 we replace $W^n_{-1/2}(K+ir,k)$ by the function $w_n(r)$ which satisfies differential equation (3.3) with $\nu = -\frac{1}{2}$ and admits the expansion (3.4) with $\nu = -\frac{1}{2}$. We abbreviate $\lambda_n = \Lambda^n_{-1/2}(k)$.

(i) We set $w_n(r) = \mathrm{sn}^{1/2}(r,k')u_n(r)$, $r \in (0,2K')$. Then equation (3.3) transforms to

$$u_n'' + \mathrm{ds}(r,k')\,\mathrm{cn}(r,k')u_n' - p(r)u_n = 0, \tag{3.6}$$

where

$$p(r) = \lambda_n - \tfrac{3}{4}k'^2\,\mathrm{sn}^2(r,k') + \tfrac{1}{4}k'^2 + \tfrac{1}{2}.$$

By [4, lemma 2.3], $\lambda_n \geqslant \frac{1}{2}\omega^2 - \frac{1}{4}$. Since $\omega^2 > k' > k'^2$, this gives

$$p(r) \geqslant \tfrac{1}{2}k'^2 - \tfrac{1}{4} - \tfrac{3}{4}k'^2 + \tfrac{1}{4}k'^2 + \tfrac{1}{2} = \tfrac{1}{4} > 0 \quad \text{for } r \in (0,2K').$$

Now (3.6) yields $u_n(r) = 1 + cr^2 + \dots$ for r close to 0 with $c = \frac{1}{16}(2+4\lambda_n+k'^2) > 0$, so $u_n(r) > 0$, $u_n'(r) > 0$ for small positive r. Since $p(r) > 0$, equation (3.6) shows that $u_n(r) > 0$, $u_n'(r) > 0$ for all $r \in (0,2K')$. Therefore, $w_n(r) > 0$ for $r \in (0,2K')$ and $w_n'(r) > 0$ for $r \in (0,K')$. Note that we cannot show that $w_n'(r) > 0$ for all $r \in (0,2K')$ because $w_n(r) \to 0$ as $r \to 2K'$ (the regular singularity $r = 2K'$ of (3.5) has two negative exponents $-\frac{1}{2}, -\frac{1}{2}$). This proves (i).

(ii) We are using equation (3.5). Let N be so large that $\lambda_n > 1$ for $n \geqslant N$. For $n \geqslant N$, we consider the interval

$$I_n := \left[\lambda_n^{-1/2}K', 2K' - \lambda_n^{-1/2}K'\right].$$

Since $\mathrm{sn}(r,k')$ is a concave function of $r \in [0,2K']$, we have $\mathrm{sn}(r,k') \geqslant \frac{r}{K'}$ for $r \in [0,K']$. Therefore,

$$q(r) \geqslant \lambda_n + \tfrac{1}{4} - \frac{K'^2}{4r^2} \quad \text{for } r \in (0,K'].$$

This implies that

$$q(r) \geqslant \tfrac{3}{4}\lambda_n + \tfrac{1}{4} > 0 \quad \text{for } r \in I_n. \tag{3.7}$$

In (i) we proved that w_n and w_n' are positive on the interval $(0,K']$, so (3.3) and (3.7) show that w_n and w_n' are positive on $(0,K') \cup I_n$. Now choose N so large that $r_1, r_2 \in I_n$ for $n \geqslant N$. Arguing as in the proof of lemma 3.2, we obtain from (3.3) and (3.7) that

$$0 < \frac{w_n(r_1)}{w_n(r_2)} \leqslant 2\exp\left(-\tfrac{\sqrt{3}}{2}(\lambda_n + \tfrac{1}{4})^{1/2}(r_2 - r_1)\right).$$

This together with [4, lemma 2.3] yields the desired estimate. □

4. Harmonics of the first kind

The coordinate surface $t = 0$ is the plane $z = 0$. If $t_0 \in (0, K')$ then the closed coordinate surface $t = t_0$ is the part of the cyclidic surface $P_1(x,y,z) = 0$ with P_1 defined in (2.5) which lies in the half-space $z > 0$. Similarly, if $t_0 \in (-K', 0)$ then the coordinate surface $t = t_0$ is given by the part of the surface $P_1(x,y,z) = 0$ which lies in the half-space $z < 0$. These surfaces are shown in red in figure 4 (see also figure 5).

Let $t_0 \in (0, K')$. Then the bounded domain D_1 interior to the surface $t = t_0$ is given by $t \in (t_0, K']$ in bi-cyclide coordinates and by

$$D_1 = \{(x,y,z) : P_1(x,y,z) < 0, z > 0\}$$

in Cartesian coordinates. Its boundary is the coordinate surface $t = t_0$.

We now introduce harmonic functions $u(x,y,z)$ of the separated form (2.15) which are harmonic in the union of all D_1 with $t_0 \in (0, K')$. In particular, these functions must be harmonic on the positive z-axis. For $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$, we define internal bi-cyclide harmonics of the first kind by

$$\mathrm{G}_{m,n}(x,y,z) = R^{-1/2} W^n_{|m|-\frac{1}{2}}(s,k) W^n_{|m|-\frac{1}{2}}(it - K - iK', k) \mathrm{e}^{im\phi}. \tag{4.1}$$

Theorem 4.1. *The internal bi-cyclide harmonic $\mathrm{G}_{m,n}(x,y,z)$ is harmonic on all of $\mathbb{R}^3$ with the exception of the segment $\{(0,0,z) : -b^{-1} \leqslant z \leqslant -b\}$, where b is given by (2.4).*

Proof. Using (3.3) we see that $v_1(s) = W_{|m|-\frac{1}{2}}(s,k)$, $v_2(t) = W^n_{|m|-\frac{1}{2}}(it - K - iK', k)$ satisfy (2.13), (2.14) with $\lambda = \Lambda^n_{|m|-\frac{1}{2}}(k)$ in both equations. It follows from theorem 2.1 that $\mathrm{G}_{m,n}(x,y,z)$ is harmonic on all of $\mathbb{R}^3$ minus the z-axis. Using (2.2), (3.1) and (3.2), we see that the function

$$(s,t) \mapsto R^{-1/2} W^n_{|m|-\frac{1}{2}}(s,k) W^n_{|m|-\frac{1}{2}}(it - K - iK', k)$$

is locally bounded at every point on the boundary of the rectangle $(s,t) \in (-K, K) \times (-K', K')$ with the exception of the closed segment γ_2 (defined in figure 1) and the point $(K, 0)$. Since the map $(R,z) \mapsto (s,t)$ is continuous, we obtain that $\mathrm{G}_{m,n}(x,y,z)$ is locally bounded at every point of the z-axis with the exception of the closed segment Γ_2 (defined in figure 2). Note that we cannot claim that $\mathrm{G}_{m,n}(x,y,z)$ is locally bounded at the points of the closed segment Γ_2 because the function W^n_ν is a solution of (3.3) which belongs to the exponent $\nu + 1$ at the regular singular points $-K$ and K but possibly not at $K - 2iK'$ (actually, it cannot belong to the exponent $\nu + 1$ there). The local boundedness of $\mathrm{G}_{m,n}$ at a point on the z-axis implies that the function $\mathrm{G}_{m,n}$ can be continued to an harmonic function in a neighborhood of this point according to the following lemma. This completes the proof. □

Lemma 4.2. *Consider the ball*

$$B_r = \{(x,y,z) : x^2 + y^2 + z^2 < r^2\},$$

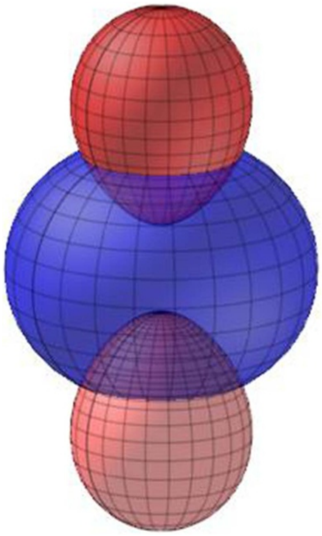

Figure 4. Coordinate surfaces $s = 0.2K$ in blue and $t = \pm 0.5K'$ in red of system (2.1) with $k = 0.5$.

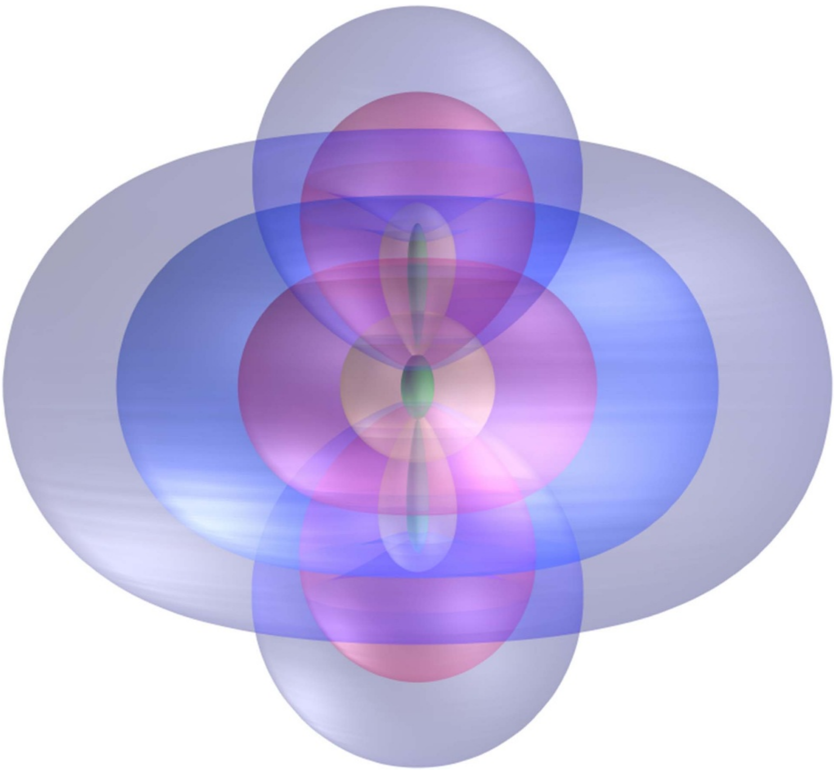

Figure 5. For $k = 0.7$ this figure depicts a three-dimensional visualization of rotationally-invariant bi-cyclides for $t \in \{\pm 0.29K', \pm 0.38K', \pm 0.70K', \pm 0.90K'\}$ (respectively green, yellow, red, blue) and orthogonal bi-concave disk cyclides $s \in \{-0.6K, 0, 0.48K, 0.66K, 0.74K\}$ (respectively green, yellow, red, blue and dark blue). Note that biconcave disk at $s = 0$ (rendered in yellow) corresponds to the unit sphere.

and a bounded continuous function

$$u : B_r^* := \overline{B}_r \setminus \{(0,0,z) : z \in \mathbb{R}\} \to \mathbb{R}$$

such that u is harmonic on $B_r \setminus \{(0,0,z) : z \in \mathbb{R}\}$. *Then u has a harmonic extension to* B_r.

Proof. Using the Poisson integral [17, p 241] we solve the Dirichlet problem on B_r with boundary values u. We obtain a solution U which is harmonic on B_r, continuous on $\overline{B}_r \setminus \{(0,0,r),(0,0,-r)\}$ and agrees with u on $\partial B_r \setminus \{(0,0,r),(0,0,-r)\}$ [17, p 243, remark]. Define a function v by

$$v(x,y,z) = -\ln \frac{\sqrt{x^2+y^2}}{r}.$$

This function is harmonic on $\mathbb{R}^3$ minus the z-axis. Let $\epsilon > 0$. Consider the function

$$w = \epsilon v - (u - U).$$

Then $w \geqslant 0$ on $\partial B_r \setminus \{(0,0,r),(0,0,-r)\}$. There is a constant M such that $|u| \leqslant M$ on B_r^*. Then also $|U| \leqslant M$ on B_r^*, so $|u-U| \leqslant 2M$ on B_r^*. Choose $\delta > 0$ so small that $\epsilon v \geqslant 2M$ if $x^2+y^2 \leqslant \delta^2$. Then $w \geqslant 0$ on the boundary of the set $A := B_r \setminus \{(x,y,z) : x^2+y^2 \leqslant \delta^2\}$. By the maximum principle for harmonic functions, $w \geqslant 0$ on $\overline{A}$. We can choose $\delta > 0$ as small as we want, so $w \geqslant 0$ on B_r^*. Since $\epsilon > 0$ is arbitrary, we get $U - u \geqslant 0$ on B_r^*. In a similar way, we get $U - u \leqslant 0$ on B_r^*. Therefore, $u = U$ on B_r^*, so U is the desired extension of u. □

Let

$$\sigma(\mathbf{r}) = \|\mathbf{r}\|^{-2}\mathbf{r}$$

denote the inversion at the unit sphere in $\mathbb{R}^3$. Then the corresponding Kelvin transform of a harmonic function $u(\mathbf{r})$ is

$$\hat{u}(\mathbf{r}) = \|\mathbf{r}\|^{-1} u(\sigma(\mathbf{r})) \tag{4.2}$$

and this function is also harmonic. The inversion at the unit sphere is expressed by $s \mapsto -s$ in bi-cyclide coordinates. It follows from (3.2) and

$$x^2+y^2+z^2 = \frac{1+\mathrm{sn}(s,k)\,\mathrm{dn}(t,k')}{1-\mathrm{sn}(s,k)\,\mathrm{dn}(t,k')},$$

so that the Kelvin transformation (4.2) of $\mathrm{G}_{m,n}$ satisfies

$$\widehat{\mathrm{G}}_{m,n}(\mathbf{r}) = (-1)^n \mathrm{G}_{m,n}(\mathbf{r}).$$

For $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$, we define external bi-cyclide harmonics of the first kind by

$$\mathrm{H}_{m,n}(x,y,z) = R^{-1/2} W^n_{|m|-\frac{1}{2}}(s,k) W^n_{|m|-\frac{1}{2}}(-it-K-iK',k)\mathrm{e}^{im\phi}. \tag{4.3}$$

The definition of $\mathrm{H}_{m,n}$ is the same as that of $\mathrm{G}_{m,n}$ except that we replaced t by $-t$. Therefore,

$$\mathrm{H}_{m,n}(x,y,z) = \mathrm{G}_{m,n}(x,y,-z). \tag{4.4}$$

By theorem 4.1, $\mathrm{H}(x,y,z)$ is harmonic on all of $\mathbb{R}^3$ except the segment $\{(0,0,z) : b \leqslant z \leqslant b^{-1}\}$. Note that the notions 'internal' and 'external' refer to the surfaces $t = t_0$ with $t_0 \in (0,K')$.

5. Applications of bi-cyclide harmonics of the first kind

We solve the Dirichlet problem for the region D_1 given by $t\in(t_0,K']$, where $t_0\in(0,K')$. We say that a harmonic function u defined in D_1 attains the boundary values f on ∂D_1 in the weak sense if $R^{1/2}u$ (expressed in terms of bi-cyclide coordinates s,t,ϕ) evaluated at $t_1\in(t_0,K')$ converges to $R^{1/2}f$ in the Hilbert space

$$H_1 := L^2((-K,K)\times(-\pi,\pi))$$

as $t_1\to t_0$. As in [3, section 5.2], one can show that the solution of the Dirichlet problem is unique.

Theorem 5.1. *Let f be a function defined on the boundary ∂D_1 of the region D_1 given by $t\in(t_0,K']$ for some $t_0\in(0,K')$. Suppose that f is represented in bi-cyclide coordinates as*

$$f(\mathbf{r}) = R^{-1/2}g(s,\phi),\quad s\in(-K,K),\quad \phi\in(-\pi,\pi],$$

such that $g\in H_1$. For all $m\in\mathbb{Z}$ and $n\in\mathbb{N}_0$ define

$$\begin{aligned} c_{m,n} &:= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{-im\phi}\int_{-K}^{K}g(s,\phi)W^n_{|m|-\frac{1}{2}}(s,k)\,\mathrm{d}s\,\mathrm{d}\phi \\ &= \frac{1}{2\pi\,W^n_{|m|-\frac{1}{2}}(it_0-K-iK',k)}\int_{\partial D_1}\frac{1}{h_s(\mathbf{r})}f(\mathbf{r})\,\mathrm{G}_{-m,n}(\mathbf{r})\,\mathrm{d}S(\mathbf{r}), \end{aligned}$$

where h_s is given in (2.16). Then the function

$$u(\mathbf{r}) = \sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty}\mathrm{d}_{m,n}G_{m,n}(\mathbf{r}),\quad \mathrm{d}_{m,n} := c_{m,n}\{W^n_{|m|-\frac{1}{2}}(it_0-K-iK',k)\}^{-1}, \tag{5.1}$$

is harmonic in D_1 and it attains the boundary values f on ∂D_1 in the weak sense. The infinite series in (5.1) converges absolutely and uniformly in compact subsets of D_1.

Proof. Since $\mathrm{d}S(\mathbf{r}) = Rh_s(\mathbf{r})\,\mathrm{d}s\,\mathrm{d}\phi$, the two formulas for $c_{m,n}$ agree. The system of functions $W^n_{|m|-\frac{1}{2}}(s,k)\mathrm{e}^{im\phi}$, $m\in\mathbb{Z}$, $n\in\mathbb{N}_0$, are orthogonal with respect to the scalar product

$$\langle f,g\rangle_{H_1} = \int_{-K}^{K}\int_{-\pi}^{\pi}f(s,\phi)\,\overline{g(s,\phi)}\,\mathrm{d}\phi\,\mathrm{d}s,$$

where $f,g\in H_1$, with corresponding norm

$$\|f\|_{H_1} = \left(\int_{-K}^{K}\int_{-\pi}^{\pi}|f(s,\phi)|^2\,\mathrm{d}\phi\,\mathrm{d}s\right)^{1/2}.$$

and complete in the Hilbert space H_1. Therefore one has the corresponding Fourier expansion

$$g(s,\phi)\sim\sum_{m\in\mathbb{Z}}\sum_{n=0}c_{m,n}W^n_{|m|-\frac{1}{2}}(s,\phi)\,\mathrm{e}^{-im\phi}.$$

In particular, the sequence $\{c_{m,n}\}$ is bounded: $|c_{m,n}|\leqslant C_1$. We use the Weierstrass M-test to show uniform convergence of the series in (5.1) on the compact set $t\geqslant t_1>t_0$. Using the maximum principle for harmonic functions it is sufficient to find bounds $M_{m,n}$ such that $|d_{m,n}G_{m,n}(\mathbf{r})| \leqslant M_{m,n}$ for $t=t_1$ and $\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{N}_0} M_{m,n} < \infty$. Using (2.2) we find for $s\in(-K,K)$ and $t=t_1$,

$$\frac{1}{R} = \frac{1-\mathrm{sn}(s,k)\,\mathrm{dn}(t_1,k')}{\mathrm{cn}(s,k)\,\mathrm{cn}(t_1,k')} \leqslant \frac{2}{\mathrm{cn}(s,k)\,\mathrm{cn}(t_1,k')} \leqslant \frac{2}{k'}\,\mathrm{dc}(s,k)\,\mathrm{nc}(t_1,k').$$

Using [4, lemmas 2.4 and 2.5] and lemmas 3.1, 3.2, we estimate

$$\left|\frac{W^n_{|m|-\frac{1}{2}}(it_1-K-iK',k)}{W^n_{|m|-\frac{1}{2}}(it_0-K-iK',k)} R^{-1/2} W^n_{|m|-\frac{1}{2}}(s,k)\mathrm{e}^{im\phi}\right| \leqslant C_2 p^{|m|+n}(1+|m|+n),$$

where the constants C_2 and $p\in(0,1)$ are independent of m,n,s,ϕ. Therefore, we can take $M_{m,n}=C_1C_2p^{|m|+n}(1+|m|+n)$ and the proof of convergence is complete. Hence $u(\mathbf{r})$ defined by (5.1) is a harmonic function on D_1. We show that u attains the boundary values f on ∂D_1 in the weak sense by the method based on Parseval's equality as used in the proof of [3, theorem 5.3]. □

Define the Wronskian $\mathcal{W}(U,V)$ by

$$\mathcal{W}(U(t),V(t)) := w_{m,n} := U(t)V'(t) - U'(t)V(t), \tag{5.2}$$

where $U(t) := W^n_{|m|-\frac{1}{2}}(it-K-iK',k)$, $V(t) := U(-t)$. External harmonics admit an integral representation in terms of internal harmonics.

Theorem 5.2. *Let $t_0\in(0,K')$, $m\in\mathbb{Z}$, $n\in\mathbb{N}_0$, and let $\mathbf{r}^*$ be a point outside $\overline{D}_1$, where D_1 is the region given by $t\in(t_0,K']$. Then*

$$\mathrm{H}_{m,n}(\mathbf{r}^*) = \frac{w_{m,n}}{4\pi\{W^n_{|m|-\frac{1}{2}}(it_0-K-iK',k)\}^2}\int_{\partial D_1}\frac{\mathrm{G}_{m,n}(\mathbf{r})}{h_s(\mathbf{r})\|\mathbf{r}-\mathbf{r}^*\|}\,\mathrm{d}S(\mathbf{r}). \tag{5.3}$$

We omit the proof of this theorem which is very similar to the proof of [3, theorem 5.5]. It follows from (5.3) that $w_{m,n}\neq 0$.

We obtain the expansion of the reciprocal distance of two points in internal and external bi-cyclide harmonics by combining theorems 5.1 and 5.2.

Theorem 5.3. *Let $\mathbf{r},\mathbf{r}^*\in\mathbb{R}^3$ have bi-cyclide coordinates (s,t,ϕ) and (s^*,t^*,ϕ^*), respectively. If $-K'<t^*<t<K'$ then*

$$\frac{1}{\|\mathbf{r}-\mathbf{r}^*\|} = 2\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty}\frac{1}{w_{m,n}}\mathrm{G}_{m,n}(\mathbf{r})\,\mathrm{H}_{-m,n}(\mathbf{r}^*). \tag{5.4}$$

Proof. If $t>0$ we choose $t_0\in(0,K')$ such that $t^*<t_0<t$, and consider the region D_1 interior to the surface $t=t_0$. Then we apply theorem 5.1 to the function $f(\mathbf{u}) = \|\mathbf{u}-\mathbf{r}^*\|^{-1}$ which is harmonic on an open set containing the closure of D_1 (because $\mathbf{r}^*$ lies outside the closure of D_1). Using theorem 5.2 to evaluate the Fourier coefficients, we obtain (5.4).

If $t\leqslant 0$, we replace $\mathbf{r}$ and $\mathbf{r}^*$ by their reflections at the plane $z=0$. Then t,t^* are replaced by $-t,-t^*$. Now we apply the result from the first part of the proof to the reflected points (in reversed order) and obtain again (5.3) observing (4.4). □

As a corollary we obtain the following addition formula for Lamé–Wangerin functions in terms of zero order toroidal harmonics of the second kind. The key insight which provides meaning for the following addition theorem for Lamé–Wangerin functions is the azimuthal

Fourier expansion of the reciprocal distance between two points expressed in circular cylindrical coordinates $\mathbf{r} = (R, \phi, z)$, $\mathbf{r}^* = (R^*, \phi^*, z^*)$, $R, R^* \in [0, \infty)$, $\phi, \phi^*, z, z^* \in \mathbb{R}$ [8, (15)]

$$\frac{1}{\|\mathbf{r}-\mathbf{r}^*\|} = \frac{1}{\pi\sqrt{RR^*}} \sum_{m=-\infty}^{\infty} Q_{m-\frac{1}{2}}(\chi) \mathrm{e}^{im(\phi-\phi^*)}, \tag{5.5}$$

where

$$\chi = \frac{R^2 + R^{*2} + (z-z^*)^2}{2RR^*}.$$

Expansions of the reciprocal distance between two points on $\mathbb{R}^3$ expressed in rotationally-invariant coordinate systems which separates Laplace's equation can all be represented as separated eigenfunction expansion with azimuthal Fourier basis $\mathrm{e}^{im(\phi-\phi')}$. These include circular cylindrical, spherical, oblate & prolate spheroidal coordinates, parabolic coordinates, toroidal coordinates and all the remaining rotationally invariant coordinate systems which separate Laplace's equation in three-dimensions. Corresponding addition theorems in terms of toroidal harmonics of the second kind therefore exist in all these coordinate systems and we have presented many of these addition theorems in previous publications. The following addition theorem in terms of the toroidal harmonic of the second kind with vanishing order is its representation in the rotationally-invariant bicyclide coordinate system.

Toroidal harmonics represent solutions to Laplace's equation in $\mathbb{R}^3$ expressed in toroidal coordinates [1, p 461]. The solutions are given in terms of associated Legendre functions of the first and second kind with odd-half-integer degree and integer order $P^n_{m-\frac{1}{2}}(z)$, $Q^n_{m-\frac{1}{2}}(z)$, where $n, m \in \mathbb{Z}$ and $z \in (1, \infty)$. These are often referred to as toroidal harmonics of the first and second kind. For properties of the associated Legendre functions, see [21, chapter 14]. The definition of a vanishing order toroidal harmonic of the second kind in terms of the Gauss hypergeometric function is given by [21, (14.3.7)]

$$Q_{m-\frac{1}{2}}(z) = \frac{\sqrt{\pi}\,\Gamma(m+\frac{1}{2})}{m!(2z)^{m+\frac{1}{2}}}\, {}_2F_1\left(\begin{array}{c} \frac{1}{2}m+\frac{1}{4}, \frac{1}{2}m+\frac{3}{4} \\ m+1 \end{array} ; \frac{1}{z^2} \right), \tag{5.6}$$

where $m \in \mathbb{Z}$. Now we present this addition formula and a corresponding integral formula for Lamé–Wangerin functions.

Theorem 5.4. *Let $m \in \mathbb{N}_0$, $s, s^* \in (-K, K)$, $-K' < t^* < t < K'$. Then*

$$\begin{aligned} Q_{m-\frac{1}{2}}(\chi) = 2\pi \sum_{n=0}^{\infty} \frac{1}{w_{m,n}} W^n_{m-\frac{1}{2}}(s,k) W^n_{m-\frac{1}{2}}(it - K - iK', k) \\ \times W^n_{m-\frac{1}{2}}(s^*,k) W^n_{m-\frac{1}{2}}(-it^* - K - iK', k), \end{aligned} \tag{5.7}$$

where $\chi : ((-K,K) \times (-K',K'))^2 \times (0,1) \to (1,\infty)$ is given by

$$\begin{aligned} \chi(s,t,s^*,t^*,k) = {} & \mathrm{nc}(s,k)\,\mathrm{nc}(t,k')\,\mathrm{nc}(s^*,k)\,\mathrm{nc}(t^*,k') \\ & - \mathrm{dc}(s,k)\,\mathrm{sc}(t,k')\,\mathrm{dc}(s^*,k)\,\mathrm{sc}(t^*,k') \\ & - \mathrm{sc}(s,k)\,\mathrm{dc}(t,k')\,\mathrm{sc}(s^*,k)\,\mathrm{dc}(t^*,k'), \end{aligned} \tag{5.8}$$

and $w_{m,n}$ is the Wronskian (5.2).

Proof. This follows from comparison of (5.4) with the azimuthal Fourier expansion (5.5) with R, R^*, z, z^* given in terms of bi-cyclide coordinates s, t and s^*, t^* respectively. The identity (5.8) can be verified by a direct computation. □

Theorem 5.4 leads to an integral relation for Lamé–Wangerin functions.

Theorem 5.5. *Let $m,n \in \mathbb{N}_0$, $s^* \in (-K,K)$, $-K' < t^* < t < K'$. Then*

$$w_{m,n} \int_{-K}^{K} Q_{m-\frac{1}{2}}(\chi) W^n_{m-\frac{1}{2}}(s,k)\,\mathrm{d}s$$
$$= 2\pi\, W^n_{m-\frac{1}{2}}(it - K - iK', k) W^n_{m-\frac{1}{2}}(s^*, k) W^n_{m-\frac{1}{2}}(-it^* - K - iK', k).$$

Using the method employed in [23], one can show that theorem 5.5 remains true if we replace $m - \frac{1}{2}$ everywhere by ν. See [4] for more details.

6. Harmonics of the second kind

For $s_0 \in (-K,K)$ the coordinate surface $s = s_0$ is a closed surface. An instance of this surface is shown in blue in figure 4. If $s_0 = 0$ this surface is the unit sphere $x^2 + y^2 + z^2 = 1$. If $s_0 \in (-K,0)$ the surface $s = s_0$ is given by the part of the cyclidic surface $P_2(x,y,z) = 0$ with P_2 defined in (2.6) which lies in the unit ball $B = \{(x,y,z) : x^2 + y^2 + z^2 < 1\}$. If $s_0 \in (0,K)$ the surface $s = s_0$ is given by the part of the surface $P_2(x,y,z) = 0$ which lies outside $\bar{B}$. The surface $s = s_0$ encloses the bounded domain D_2 given by $s \in [-K, s_0)$ in bi-cyclide coordinates.

The coordinate surfaces $s = s_0$ and $t = t_0$ are connected through the inversion M at the sphere with center $(0,0,1)$ and radius $\sqrt{2}$. This inversion is given by

$$M(x,y,z) = (x^2 + y^2 + (z-1)^2)^{-1}(2x, 2y, x^2 + y^2 + z^2 - 1).$$

If (x,y,z) has bi-cyclide coordinates (s,t,ϕ) then (2.7) gives

$$M(x,y,z) = (u\cos\phi, u\sin\phi, v),$$

where

$$u = \frac{\mathrm{cn}(s,k)\,\mathrm{cn}(t,k')}{1 - \mathrm{dn}(s,k)\,\mathrm{sn}(t,k')}, \qquad v = \frac{\mathrm{sn}(s,k)\,\mathrm{dn}(t,k')}{1 - \mathrm{dn}(s,k)\,\mathrm{sn}(t,k')}.$$

Therefore, the point $M(x,y,z)$ has bi-cyclide coordinates t, s, ϕ (with s,t exchanged) with bi-cyclide coordinates taken with respect to the complementary modulus k'. This means that the coordinate surface $s = s_0$ (with respect to k) is mapped to the coordinate surface $t = s_0$ (with respect to k'). If $t_0 \in (0, K')$ then M maps the domain D_1 given by $t > t_0$ with respect to k to the exterior of the domain D_2 given by $s > t_0$ with respect to k'.

Because of this connection between the coordinate surfaces, the results on bi-cyclide harmonics of the second kind adapted to the domains D_2 will be very similar to the ones for bi-cyclide harmonic of the first kind. Therefore, we will keep the following treatment of bi-cyclide harmonics of the second kind short.

We are looking for harmonic functions $u(x,y,z)$ of the $\mathcal{R}$-separated form (2.15) which are harmonic in the union of all D_2 with $s_0 \in (-K,K)$. This requires that u must be harmonic on the interval $\{(0,0,z) : -b^{-1} < z < b^{-1}\}$ on the z-axis. For $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$, we define internal bi-cyclide harmonics of the second kind by

$$\mathrm{G}_{m,n}(x,y,z) = R^{-1/2} W^n_{|m|-\frac{1}{2}}(-is - K' - iK, k') W^n_{|m|-\frac{1}{2}}(t, k') \mathrm{e}^{im\phi}.$$

Theorem 6.1. *The internal harmonic $\mathrm{G}_{m,n}(x,y,z)$ is a harmonic function on all of $\mathbb{R}^3$ with the exception of set $\{(0,0,z) : |z| \geqslant b^{-1}\}$.*

For $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$, we define external bi-cyclide harmonics of the second kind by

$$\mathrm{H}_{m,n}(x,y,z) = R^{-1/2} W^n_{|m|-\frac{1}{2}}(is - K' - iK), k') W^n_{|m|-\frac{1}{2}}(t, k') \mathrm{e}^{im\phi}.$$

Then $\mathrm{H}_{m,n}$ is the Kelvin transform of $\mathrm{G}_{m,n}$ with respect to the unit sphere:

$$\mathrm{H}_{m,n}(x,y,z) = \widehat{\mathrm{G}}_{m,n}(x,y,z).$$

The function $\mathrm{H}_{m,n}(x,y,z)$ is harmonic on $\mathbb{R}^3$ with the exception of the segment $\{(0,0,z) : |z| \leqslant b\}$.

Arguing as in section 5 we prove the expansion of the reciprocal distance of two points in bi-cyclide harmonics of the second kind. Alternatively, employing the inversion M, the result can be derived directly from theorem 5.3.

Theorem 6.2. *Let $\mathbf{r}, \mathbf{r}^* \in \mathbb{R}^3$ with bi-cyclide coordinates (s,t,ϕ) and (s^*,t^*,ϕ^*), respectively. If $-K < s < s^* < K$ then*

$$\frac{1}{\|\mathbf{r}-\mathbf{r}^*\|} = 2\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty}\frac{1}{w_{m,n}}G_{m,n}(\mathbf{r})H_{-m,n}(\mathbf{r}^*), \tag{6.1}$$

where $\mathcal{W}(U(s),V(s)) := w_{m,n} = U(s)V'(s) - U'(s)V(s)$ is the Wronskian of the functions $U(s) = W^n_{|m|-\frac{1}{2}}(is - K' - iK, k')$ and $V(s) = U(-s)$.

As in section 5, theorem 6.2 yields an addition theorem and integral relations for Lamé–Wangerin functions. We omit these results because they can be obtained from theorems 5.4 and 5.5 by exchanging $k \leftrightarrow k'$, $K \leftrightarrow K'$ and $s \leftrightarrow t$.

7. Analysis of the limiting behavior for bi-cyclide coordinates

In figure 6, we graphically depict the behavior of bi-cyclide coordinates for three different values of $k \in \{\frac{1}{10}, \frac{1}{2}, \frac{4}{5}\}$ in order to illustrate the behavior that changing the value of k has upon the coordinates themselves. From figure 6, one qualitatively sees that as $k \to 0, 1$, bi-cyclide coordinates approaches bi-spherical and spherical coordinates respectively. For instance, using (2.2) and (2.3), one can study this limiting process precisely. In section 7.1 we treat the $k \to 0$ limit of bi-cyclide coordinates to bi-spherical coordinates. A careful analysis of the $k \to 1$ limit of bi-cyclide coordinates to spherical coordinates is left to the reader. However, in section 7.2 we study the $k \to 1$ limit of a modified bi-cyclide coordinates to prolate spheroidal coordinates.

7.1. The $k \to 0$ bi-spherical limit of bi-cyclidic coordinates

In this section we show that bi-cyclide coordinates approach bi-spherical coordinates as $k \to 0$, and our expansion of the reciprocal distance between two points (5.4) approaches term-by-term the corresponding known expansion in bi-spherical coordinates.

Bi-spherical coordinates θ, t, ϕ [19, p 110] are given by

$$x = \frac{\sin\theta\cos\phi}{\cosh t - \cos\theta}, \quad y = \frac{\sin\theta\sin\phi}{\cosh t - \cos\theta}, \quad z = \frac{\sinh t}{\cosh t - \cos\theta},$$

where $t \in \mathbb{R}$, $\theta \in (0,\pi)$, $\phi \in (-\pi,\pi]$. According to [20, (10.3.74)] we have the expansion

$$\frac{1}{\|\mathbf{r}-\mathbf{r}^*\|} = (\cosh t - \cos\theta)^{1/2}(\cosh t^* - \cos\theta^*)^{1/2}$$
$$\times\sum_{\ell=0}^{\infty}\mathrm{e}^{-(\ell+\frac{1}{2})(t-t^*)}\sum_{m=-\ell}^{\ell}\frac{(\ell-m)!}{(\ell+m)!}\mathsf{P}^m_\ell(\cos\theta)\mathsf{P}^m_\ell(\cos\theta^*)\mathrm{e}^{im(\phi-\phi^*)}, \tag{7.1}$$

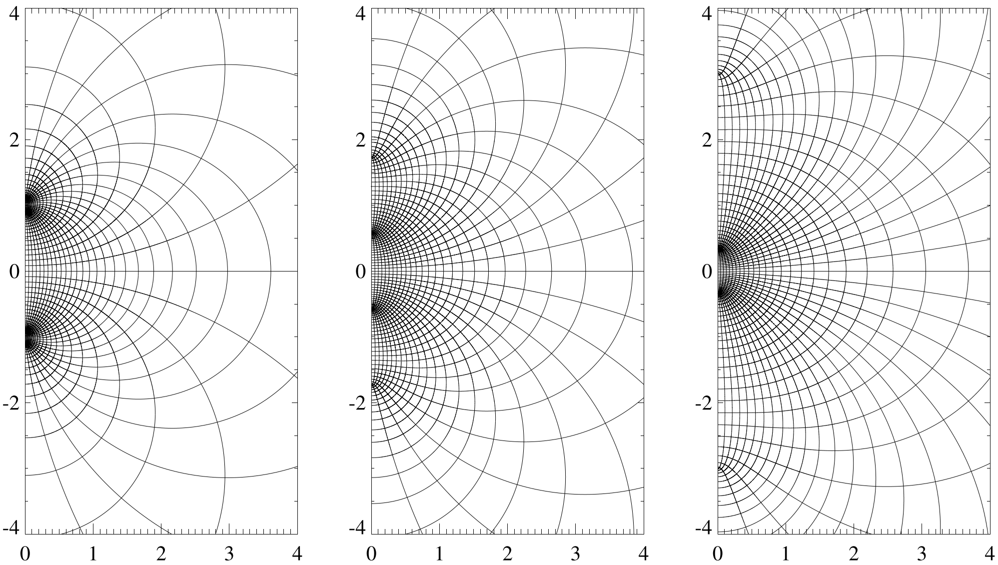

Figure 6. In bi-cyclide coordinates, the figures depict coordinate lines for constant values of $s \in (-K, K)$ and $t \in (-K', K')$ with uniform spacing for $k = \frac{1}{10}, \frac{1}{2}, \frac{4}{5}$ respectively from left to right. The abscissa represents the radial coordinate $R = (x^2 + y^2)^{1/2}$ and the ordinate represents the z-axis. One can see that as k approaches zero, the bi-cyclidic coordinate system approaches bi-spherical coordinates. Similarly, as k approaches unity, the bi-cyclidic coordinate system approaches spherical coordinates.

where P_ℓ^m denotes the Ferrers function of the first kind, (θ, t, ϕ), (θ^*, t^*, ϕ^*) are bi-spherical coordinates of $\mathbf{r}$, $\mathbf{r}^*$, respectively, and it is assumed that $t^* < t$.

In our analysis it is more convenient to interchange the summation in (7.1) so

$$\frac{1}{\|\mathbf{r} - \mathbf{r}^*\|} = \sum_{m \in \mathbb{Z}} c_m \mathrm{e}^{im(\phi - \phi^*)}.$$

If $m \in \mathbb{N}_0$ then by setting $\ell = m + n$, we have

$$c_m = \sum_{n=0}^{\infty} B_{m,n}(\theta, \theta^*, t, t^*),$$

where, for $m, n \in \mathbb{N}_0$,

$$B_{m,n} = (\cosh t - \cos\theta)^{1/2} (\cosh t^* - \cos\theta^*)^{1/2} \\ \times \mathrm{e}^{-(m+n+\frac{1}{2})(t-t^*)} \frac{n!}{(2m+n)!} \mathsf{P}_{m+n}^m(\cos\theta) \mathsf{P}_{m+n}^m(\cos\theta^*).$$

If a real-valued function $f(\phi)$ with period 2π is expanded in a complex Fourier series $f(\phi) = \sum_{m \in \mathbb{Z}} c_m \mathrm{e}^{im\phi}$, then we must have $c_{-m} = \overline{c}_m$. In our case, the coefficients c_m are real, so we have $c_{-m} = c_m$. Therefore, we obtain

$$\frac{1}{\|\mathbf{r} - \mathbf{r}^*\|} = \sum_{m \in \mathbb{Z}} \mathrm{e}^{im(\phi - \phi^*)} \sum_{n=0}^{\infty} B_{m,n}(\theta, \theta^*, t, t^*),$$

where we define $B_{-m,n} := B_{m,n}$ for $m < 0$.

If we let $k\to 0$ in (2.2), (2.3), and observe [21, tables 22.5.3 and 22.5.4],

$$\mathrm{sn}(s,k)\to\sin s,\quad \mathrm{cn}(s,k)\to\cos s,\quad \mathrm{dn}(s,k)\to 1,$$

$$\mathrm{sn}(t,k')\to\tanh t,\quad \mathrm{cn}(t,k')\to\operatorname{sech} t,\quad \mathrm{dn}(t,k')\to\operatorname{sech} t,$$

we find that bi-cyclide coordinates approach bi-spherical coordinates with $s=\frac{\pi}{2}-\theta$.

Let us write the expansion (5.4) in the form

$$\frac{1}{\|\mathbf{r}-\mathbf{r}^*\|}=\sum_{m\in\mathbb{Z}}\mathrm{e}^{im(\phi-\phi^*)}\sum_{n=0}^{\infty}A_{m,n}(s,s^*,t,t^*,k),$$

where $A_{-m,n}=A_{m,n}$, and, for $m,n\in\mathbb{N}_0$,

$$\begin{aligned}A_{m,n}=\frac{2}{w_m^n}(RR^*)^{-1/2}&W^n_{m-\frac12}(s,k)W^n_{m-\frac12}(s^*,k)\\ &\times W^n_{m-\frac12}(it-K-iK',k)W^n_{m-\frac12}(-it^*-K-iK',k).\end{aligned}$$

The following theorem states the main result of this section.

Theorem 7.1. *Let $m\in\mathbb{Z}$, $n\in\mathbb{N}_0$, $s,s^*\in(-\frac12\pi,\frac12\pi)$, $t,t^*\in\mathbb{R}$. Then*

$$A_{m,n}(s,s^*,t,t^*,k)\to B_{m,n}(\tfrac12\pi-s,\tfrac12\pi-s^*,t,t^*)\quad \textit{as } k\to 0.$$

Proof. It is sufficient to consider $m\geqslant 0$. All limits in this proof are taken as $k\to 0$. We first note that

$$R^{-1/2}R^{*-1/2}\to\left(\frac{\cos s}{\cosh t-\sin s}\right)^{-1/2}\left(\frac{\cos s^*}{\cosh t^*-\sin s^*}\right)^{-1/2}. \tag{7.2}$$

Using [4, corollary 4.4], we have

$$\begin{aligned}&W^n_{m-\frac12}(s,k)W^n_{m-\frac12}(s^*,k)\\ &\quad\to(m+n+\tfrac12)\frac{n!}{(2m+n)!}(\cos s)^{1/2}\mathsf{P}^n_{m+n}(\sin s)(\cos s^*)^{1/2}\mathsf{P}^m_{m+n}(\sin s^*)\end{aligned} \tag{7.3}$$

By [4, theorem 4.7], we have

$$\frac{W^n_{m-\frac12}(i(\sigma-K'),k)}{W^n_{m-\frac12}(-iK',k)}\to\mathrm{e}^{-(m+n+\frac12)\sigma}$$

locally uniformly for $\sigma\in\mathbb{C}$. Actually, it was assumed there that $|\Im\sigma|<\frac12\pi$ but the proof shows that this restriction is superfluous. If we set $\sigma=t+iK$ and note that $K(k)\to\frac12\pi$, it follows that

$$\begin{aligned}&\frac{2}{w^m_n}W^n_{m-\frac12}(it-K-iK',k)W^n_{m-\frac12}(-it^*-K-iK',k)\\ &\qquad\to\frac{\mathrm{e}^{-(m+n+\frac12)(t+i\frac12\pi)}\mathrm{e}^{(m+n+\frac12)(t^*+i\frac12\pi)}}{m+n+\frac12}.\end{aligned} \tag{7.4}$$

After multiplying out (7.2)–(7.4) and minor simplification, we obtain the desired statement. □

7.2. The $k\to 1$ prolate spheroidal limit of modified bi-cyclidic coordinates

If we let $k\to 1$ in (2.2) and (2.3) we find that bi-cyclide coordinates approach spherical coordinates. However, by modifying the limiting process we show that bi-cyclide coordinates can also approach prolate spheroidal coordinates as $k\to 1$.

Prolate spheroidal coordinates [19, p 28] are given by

$$x=a\sinh\sigma\sin\theta\cos\phi,\quad y=a\sinh\sigma\sin\theta\sin\phi,\quad z=a\cosh\sigma\cos\theta,$$

where $\sigma\in(0,\infty)$, $\theta\in(0,\pi)$, $\phi\in(-\pi,\pi]$. According to [14, section 245] (see also [9, section 5.1]) we have the expansion

$$\frac{1}{\|\mathbf{r}-\mathbf{r}^*\|}=\frac{1}{a}\sum_{\ell=0}^{\infty}(2\ell+1)\sum_{m=-\ell}^{\ell}(-1)^m\left[\frac{(\ell-m)!}{(\ell+m)!}\right]^2 \times\mathsf{P}_\ell^m(\cos\theta)\mathsf{P}_\ell^m(\cos\theta^*)P_\ell^m(\cosh\sigma)Q_\ell^m(\cosh\sigma^*)\mathrm{e}^{im(\phi-\phi^*)},$$

where P_ℓ^m denotes the Ferrers function of the first kind, P_ℓ^m, Q_ℓ^m are associated Legendre functions of the first and second kind, respectively, (σ,θ,ϕ), $(\sigma^*,\theta^*,\phi^*)$ are prolate spheroidal coordinates of $\mathbf{r}$, $\mathbf{r}^*$, respectively, and it is assumed that $\sigma<\sigma^*$. We may write the expansion in the equivalent form

$$\frac{1}{\|\mathbf{r}-\mathbf{r}^*\|}=\sum_{m\in\mathbb{Z}}\mathrm{e}^{im(\phi-\phi^*)}\sum_{n=0}^{\infty}B_{m,n}(\sigma,\sigma^*,\theta,\theta^*),$$

where $B_{-m,n}=B_{m,n}$, and for $m,n\in\mathbb{N}_0$,

$$B_{m,n}=\frac{1}{a}(-1)^m(2m+2n+1)\left[\frac{n!}{(2m+n)!}\right]^2 \times\mathsf{P}_{m+n}^m(\cos\theta)\mathsf{P}_{m+n}^m(\cos\theta^*)P_{m+n}^m(\cosh\sigma)Q_{m+n}^m(\cosh\sigma^*).$$

We modify bi-cyclide coordinates by setting $\sigma=s+K$, $(x,y,z)=\frac{1}{2a}k'(X,Y,Z)$. The modified bi-cyclide coordinates (σ,t,ϕ) of (X,Y,Z) are defined as those of (x,y,z) with $\sigma=s+K$. If we let $k\to 1$, we obtain

$$\lim_{k\to 1}(X,Y,Z)=(a\sinh\sigma\cos t\cos\phi,\ a\sinh\sigma\cos t\sin\phi,\ a\cosh\sigma\sin t),$$

so we approach prolate spheroidal coordinates with $\theta=\frac{1}{2}\pi-t$.

Let us write the expansion (6.1) in the form

$$\frac{1}{\|\mathbf{r}-\mathbf{r}^*\|}=\sum_{m\in\mathbb{Z}}\mathrm{e}^{im(\phi-\phi^*)}\sum_{n=0}^{\infty}A_{m,n}(\sigma,\sigma^*,t,t^*,k),\tag{7.5}$$

where $A_{-m,n}=A_{m,n}$, and, for $m,n\in\mathbb{N}_0$,

$$A_{m,n}=(X^2+Y^2)^{-1/4}(X^{*2}+Y^{*2})^{-1/4} \times\frac{2}{w_m^n}W_{m-\frac12}^n(t,k')W_{m-\frac12}^n(t^*,k')W_{m-\frac12}^n(K'-i\sigma,k')W_{m-\frac12}^n(K'-i(2K-\sigma^*)),k').$$

In (7.5) we take $\mathbf{r}=(X,Y,Z)$, $\mathbf{r}^*=(X^*,Y^*,Z^*)$.

Lemma 7.2. *Let* $\sigma_0 > 0$, $\nu \geqslant -\frac{1}{2}$, $n \in \mathbb{N}_0$. *Then we have*

$$\frac{W_\nu^n(K' - i(2K-\sigma), k')}{W_\nu^n(K' - i(2K-\sigma_0), k')} \to \frac{(\sinh\sigma)^{1/2} Q_{n+\nu+\frac{1}{2}}^{\nu+\frac{1}{2}}(\cosh\sigma)}{(\sinh\sigma_0)^{1/2} Q_{n+\nu+\frac{1}{2}}^{\nu+\frac{1}{2}}(\cosh\sigma_0)}$$

as $k \to 1$ *locally uniformly for* $\Re\sigma > 0$.

Proof. The function $w(\sigma) = W_\nu^n(K' - i(2K-\sigma), k')$, $0 < \sigma < 2K$, satisfies the differential equation

$$w'' + \left(\nu(\nu+1) - \Lambda_\nu^n(k') - \nu(\nu+1)\frac{1}{\operatorname{sn}^2(\sigma,k)}\right) w = 0. \tag{7.6}$$

By [4, lemma 2.3], $\Lambda_\nu^k(k') \to (n+\nu+1)^2$ as $k \to 1$. The differential equation (7.6) appeared in the proof of [3, theorem 7.2] with k' in place of k. The sequence $\Lambda_\nu^n(k')$ was replaced by another sequence that converged to n^2. We can now follow the proof of [3, theorem 7.2] to complete the proof the lemma. □

The main result of this section follows.

Theorem 7.3. *Let* $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$, $\sigma, \sigma^* \in (0,\infty)$, $t, t^* \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ *and set* $\theta = \frac{1}{2}\pi - t$, $\theta^* = \frac{1}{2}\pi - t^*$. *Then*

$$A_{m,n}(\sigma,\sigma^*,t,t^*,k) \to B_{m,n}(\sigma,\sigma^*,\theta,\theta^*) \quad \text{as } k \to 1.$$

Proof. It is sufficient to consider $m \geqslant 0$. All limits in this proof are taken as $k \to 1$. We first note that

$$(X^2+Y^2)^{-1/4}(X^{*2}+Y^{*2})^{-1/4} \to (\sinh\sigma\sin\theta)^{-1/2}(\sinh\sigma^*\sin\theta^*)^{-1/2}. \tag{7.7}$$

Using [4, corollary 4.4], we have

$$\begin{aligned} &2W_{m-\frac{1}{2}}^n(t,k')W_{m-\frac{1}{2}}^n(t^*,k') \\ &\quad\to (2m+2n+1)\frac{n!}{a(2m+n)!}(\sin\theta)^{1/2}\mathsf{P}_{m+n}^n(\cos\theta)(\sin\theta^*)^{1/2}\mathsf{P}_{m+n}^m(\cos\theta^*). \end{aligned} \tag{7.8}$$

We define two functions

$$\begin{aligned} f(u,k) &:= W_{m-\frac{1}{2}}^n(u,k'), \\ g(u^*,k) &:= W_{m-\frac{1}{2}}^n(K' - i(2K-u^*),k'). \end{aligned}$$

These functions are well-defined for $u, u^* \in (0,2K)$. Then we consider the expression

$$h(u,u^*,k) = \frac{f(u,k)g(u^*,k)}{\mathcal{W}(g,f)}, \tag{7.9}$$

where $\mathcal{W}(g,f) := w_m^n$ denotes the Wronskian of $g(\cdot,k)$ and $f(\cdot,k)$. We notice that (7.9) remains unchanged when we multiply f and/or g by real or complex constants. Therefore, using [4, corollary 4.4] and lemma 7.2 one obtains

$$h(\sigma,\sigma^*,k) \to \frac{F(\sigma)G(\sigma^*)}{\mathcal{W}(G,F)},$$

where

$$F(\sigma) = (\sinh\sigma)^{1/2} P^m_{m+n}(\cosh\sigma), \quad G(\sigma^*) = (\sinh\sigma^*)^{1/2} Q^m_{m+n}(\cosh\sigma^*).$$

The known Wronskian [21, (14.2.10)]

$$[P^\mu_\nu(x), Q^\mu_\nu(x)] = \mathrm{e}^{i\mu\pi} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} \frac{1}{1-x^2},$$

implies

$$\mathcal{W}(G,F) = (-1)^m \frac{(2m+n)!}{n!}.$$

Thus we have shown

$$\frac{1}{w^n_m} W^n_{m-\frac{1}{2}}(K' - i\sigma, k') W^n_{m-\frac{1}{2}}(K' - i(2K - \sigma^*), k')$$
$$\to (-1)^m \frac{n!}{a(2m+n)!} (\sinh\sigma)^{1/2} P^n_{m+n}(\cosh\sigma)(\sinh\sigma^*)^{1/2} Q^n_{m+n}(\cosh\sigma^*). \tag{7.10}$$

After multiplying out (7.7), (7.8), (7.10) and minor simplification, we obtain the desired statement. □

8. Conclusions and future work

This paper concludes the analysis of the internal and external harmonics and an expansion of a fundamental solution of Laplace's equation associated with bi-cyclide coordinates, a rotationally invariant coordinate system which separates Laplace's equation. The existence of addition theorems in terms of toroidal harmonics of the second kind for rotationally-invariant coordinate systems which separate Laplace's equation in three-dimensions which was introduced in [5, 7–9] is nearly complete. The only such coordinate systems of this type whose analysis has not been completed and rotationally-invariant addition theorems derived until now is flat-disk cyclide coordinates [18, system 16, p 211]. The completion of these efforts for flat-disk cyclide coordinates, hopefully in the coming years, will be an exciting conclusion to these activities.

Concerning the three-variable Laplace equation in Euclidean space, and the study of the internal and external harmonics and an expansion of its fundamental solution in the conformally inequivalent coordinate systems which separate Laplace's equation, one should consider any other remaining coordinate systems which have not received adequate attention. These coordinate systems include, as just mentioned, flat-disk cyclide coordinates [18, system 16, p 211], but as well, asymmetric cyclidic coordinates of the second kind [18, system 13, p 210]. The asymmetric cyclidic coordinates of the first kind was considered in [10–12]).

Other topics of related interest include the fact that there should also exist integral formulas for a fundamental solution of Laplace's equation which will arise when the coordinate surfaces are noncompact, such as those which occur in the asymmetric cyclidic coordinates of the second kind, confocal ellipsoidal coordinates, paraboloidal coordinates, sphero-conical coordinates, parabolic coordinates, oblate spheroidal coordinates, prolate spheroidal coordinates, spherical coordinates, parabolic cylinder coordinates, elliptic-cylinder coordinates, circular cylindrical coordinates, and Cartesian coordinates. Additional work could be done on limits of one coordinate system to another with corresponding limit relations for special functions.

Of course an analysis of eigenfunction expansions in conformally inequivalent separable coordinate systems for a fundamental solution of Laplace's equation on $\mathbb{R}^d$ see [6, (2.3)]

$$\mathcal{G}^d(\mathbf{x},\mathbf{x}') = \frac{\Gamma(\frac{1}{2}d-1)}{4\pi^{\frac{1}{2}d}\|\mathbf{x}-\mathbf{x}'\|^{d-2}},$$

which satisfies

$$-\Delta\mathcal{G}^d(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}'),$$

in dimensions greater than three, is almost entirely uninvestigated. In fact, as far as the authors are aware, the total number and classification of such conformally inequivalent separable coordinate systems for Laplace's equation in dimensions greater than three remains outstanding. However, the general theory of $\mathcal{R}$-separability on an n-dimensional Riemannian manifold was derived in [22] (see also [16, 18]).

Data availability statement

No new data were created or analysed in this study.

Appendix A. The bi-cyclide coordinates of Moon and Spencer

Moon and Spencer [19, p 124] define bi-cyclide coordinates μ,ν,ϕ by

$$x = \frac{a}{\Lambda}\operatorname{cn}(\mu,\kappa)\operatorname{dn}(\mu,\kappa)\operatorname{sn}(\nu,\kappa')\operatorname{cn}(\nu,\kappa')\cos\phi,$$

$$y = \frac{a}{\Lambda}\operatorname{cn}(\mu,\kappa)\operatorname{dn}(\mu,\kappa)\operatorname{sn}(\nu,\kappa')\operatorname{cn}(\nu,\kappa')\sin\phi,$$

$$z = \frac{a}{\Lambda}\operatorname{sn}(\mu,\kappa)\operatorname{dn}(\nu,\kappa'),$$

where

$$\Lambda = 1-\operatorname{dn}^2(\mu,\kappa)\operatorname{sn}^2(\nu,\kappa'),$$

a is a positive constant, and $\kappa\in(0,1)$, $\kappa'=(1-\kappa^2)^{1/2}$. Setting $R=(x^2+y^2)^{1/2}$ and using the addition theorem for the Jacobi elliptic function sn [21, (22.8.1)], we can write these coordinates in the complex form

$$z+iR = a\operatorname{sn}(\mu+i\nu,\kappa). \tag{A.1}$$

The function $v=\operatorname{sn}(u,\kappa)$ maps the rectangle

$$-K(\kappa) < \Re u < K(\kappa), \quad 0 < \Im u < K'(\kappa),$$

conformally to the half-plane $\Im v > 0$. Therefore, we choose

$$-K(\kappa) < \mu < K(\kappa), \quad 0 < \nu < K'(\kappa). \tag{A.2}$$

Wangerin [25] introduced coordinates μ,ν in the (R,z)-plane by setting $z+iR=af(\mu+i\nu)$ for $f=\operatorname{cn}$, $f=\operatorname{sn}$ and $f=\operatorname{dn}$. Actually, he considers only $f=\operatorname{cn}$ and $f=\operatorname{dn}$ because the coordinates generated by sn and dn are essentially the same. This follows from the identity

$$\kappa\operatorname{sn}(u,\kappa) = \operatorname{dn}(K'(\kappa)+iK(\kappa)-iu,\kappa').$$

In this paper we used bi-cyclide coordinates $s \in (-K,K)$, $t \in (-K',K')$ defined by (2.2) and (2.3). They can be written in complex form as

$$z + iR = i(\mathrm{sc}(s-it,k) + \mathrm{nc}(s-it,k)). \tag{A.3}$$

The connection between μ,ν and s,t is given by the following theorem.

Theorem A.1. *Take $a = \kappa^{1/2}$ and $\kappa = \frac{1-k}{1+k}$. Then the coordinates μ,ν and s,t of a point (R,z) with $R>0$ are connected by*

$$t = (1+\kappa)\mu, \quad s + K(k) = (1+\kappa)\nu.$$

Proof. The modulus κ is the descending Landen transformation of k' so [21, (19.8.12)] gives

$$K'(k) = (1+\kappa)K(\kappa), \quad 2K(k) = (1+\kappa)K'(\kappa).$$

It follows that $\tilde{s} := (1+\kappa)\nu - K(k)$, $\tilde{t} := (1+\kappa)\mu$ satisfy $-K(k) < \tilde{s} < K(k)$, $-K'(k) < \tilde{t} < K'(k)$. Therefore, using (A.1), (A.3) and setting $u = (1+\kappa)(\mu + i\nu)$, the theorem will follow from the identity

$$\kappa^{1/2}\,\mathrm{sn}\left(\frac{u}{1+\kappa},\kappa\right) = i\left(\mathrm{nc}(iu+K(k),k) - \mathrm{sc}(iu+K(k),k)\right). \tag{A.4}$$

To prove (A.4) we note that

$$\begin{aligned} &i(\mathrm{nc}(iu+K(k),k) - \mathrm{sc}(iu+K(k),k)) \\ &\quad = \frac{i}{k'}\left(\mathrm{cs}(iu,k) - \mathrm{ds}(iu,k)\right) = \frac{1}{k'}\left(\mathrm{ns}(u,k') - \mathrm{ds}(u,k')\right). \end{aligned}$$

Now (A.4) follows from [21, (22.7.2) and (22.7.4)]. □

Appendix B. Orthogonality of Lamé–Wangerin functions

For fixed $k \in (0,1)$ and $\nu \geqslant -\frac{1}{2}$ we consider the differential equation (3.3) for $t \in (-K,K)$. A Lamé–Wangerin function $w(t)$ is a real-valued solution of this equation for some real eigenvalue h with the property that it is a Fuchs-Frobenius solution corresponding to the exponents $\nu+1$ at both endpoints $t=-K$ and $t=K$ (the other exponent is $-\nu$). One could also say that $w(t)$ is a recessive solution at both endpoints.

Lemma B.1. *Let $w_1(t)$ and $w_2(t)$ be two Lamé–Wangerin functions belonging to different eigenvalues h_1,h_2. Then*

$$\int_{-K}^{K} w_1(t)w_2(t)\,\mathrm{d}t = 0.$$

Proof. We have the equations

$$w_1''(t) + (h_1 - \nu(\nu+1)\,\mathrm{dc}^2(t,k))w_1(t) = 0,$$

$$w_2''(t) + (h_2 - \nu(\nu+1)\,\mathrm{dc}^2(t,k))w_2(t) = 0.$$

Multiplying the first equation by $w_2(t)$, the second by $w_1(t)$, and subtracting we find

$$w_1''(t)w_2(t) - w_2''(t)w_1(t) + (h_1-h_2)w_1(t)w_2(t) = 0.$$

Integrating from a to b where $-K<a<b<K$, we get

$$(h_2-h_1)\int_a^b w_1(t)w_2(t)\,dt=\int_a^b (w_1''(t)w_2(t)-w_2''(t)w_1(t))\,dt=(w_1'(t)w_2(t)-w_1(t)w_2'(t))|_a^b.$$

If we let $a\to -K$ and $b\to K$ then the right-hand side tends to 0. This is because for $t>-K$ close to $-K$

$$w_1(t)=(t+K)^{\nu+1}v_1(t),\quad w_2(t)=(t+K)^{\nu+1}v_2(t)$$

with convergent power series $v_1(t)$ and $v_2(t)$ in powers of $t+K$. Then

$$w_1'(t)w_2(t)-w_1(t)w_2'(t)=(t+K)^{2\nu+2}(v_1'(t)v_2(t)-v_1(t)v_2'(t))$$

and this tends to zero as $t\to -K$ because $\nu\geqslant -\frac{1}{2}$. Similarly, we do this analysis at the right endpoint $t=K$. Since $h_1\neq h_2$ we get the desired orthogonality. □

Remark. The proof works for $\nu>-1$. Since $v_1(t)$ and $v_2(t)$ contain only even powers of $t+K$ one could also prove this lemma for $\nu>-\frac{3}{2}$. But if $\nu\leqslant -\frac{3}{2}$ then the orthogonality makes no sense anymore because the function $w_1(t)w_2(t)$ behaves like $(t+K)^{2\nu+2}$ close to $t=-K$ so $w_1(t)w_2(t)$ is not integrable.

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