Multi-Mode Gaussian State Analysis with Total-Photon Counting

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The continuing improvement in the qualities of photon-number-resolving detectors opens new possibilities for measuring quantum states of light. In this work we consider the question of what properties of an arbitrary multi-mode Gaussian state are determined by a single photon-number-resolving detector that measures totalphoton number. We find an answer to this question in the ideal case where the exact photon-number probabilities are known. We show that the quantities determined by the total-photon-number distribution are the spectrum of the covariance matrix, the absolute displacement in each eigenspace of the covariance matrix, and nothing else. In the case of pure Gaussian states, the spectrum determines the squeezing parameters.

1. INTRODUCTION

Gaussian states in continuous variable systems are relatively easy to prepare experimentally and can be used in quantum communication, quantum cryptography, quantum sensing and other areas [1, 2]. In this context, the problem of experimentally analyzing and characterizing Gaussian states becomes important. The most common methods of characterizing single-mode and multi-mode states involve homodyne tomography and variants thereof [2– 12. Homodyne tomography is based on quadrature measurements requiring photodiodes that record the intensity of absorbed light in a high-amplitude regime. More recently, several groups have proposed schemes for characterizing Gaussian states with click detectors or photon-number-resolving (PNR) detectors. In particular, Ref. [13] proposed a scheme that only uses beam splitters and single-photon detectors to measure the purity, squeezing and entanglement of Gaussian states. Ref. [14] verified this scheme experimentally on singlemode Gaussian states. For single-mode states several studies [15-20] have shown that the statistics of photon counts obtained after displacing the state by different amounts allows one to estimate the Wigner function. Photon counting with single-mode squeezed states have applications in quantum metrology - in particular, they have been shown to enhance the sensitivity of measuring the phase of an interferometer [21] as well as the coherent displacement of a mechanical oscillator [22]. Numerical [23] studies show that comparatively few number of measurements of the photon number of a single-mode non-displaced state allow for accurate determination of its squeezing and temperature parameters. A recent experimental study demonstrated that photon counting allows for a better precision measurement of a weakly-squeezed vacuum state as compared to homodyning [24]. Ref. [25] showed experimentally that the statistics of two-photon counters can be used to reconstruct

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the mode structure of a parametric down-conversion source. Full characterization of multimode Gaussian states has been shown to be theoretically possible using PNR detectors on each mode after passing the state through different linear optical circuits [26, 27]. The proposed schemes either yield only general features of an arbitrary multi-mode state like the mean displacement and the determinant of the covariance matrix [13], or are difficult to realize in practice [26, 27].

In this light, we consider the problem of what can be learned about an arbitrary multimode Gaussian state with a very simple setup - the state is measured by a single PNR detector that returns the total number of photons in all modes. For this study, we make the idealizing assumptions that the detector has no losses nor noise and that we learn the exact probability of observing n photons for every n. These assumptions can be significantly weakened, for example, the PNR detector can be lossy, see Sect. 5. We find that the total-photon-number distribution determines the number of modes that are not in vacuum as well as the spectrum of the covariance matrix in these modes. By spectrum we mean the set of eigenvalues with their multiplicities, if there are degeneracies. The absolute values of the displacement within each eigenspace of the covariance matrix are also determined by the distribution. Conversely, any two states with the same covariance matrix spectrum and absolute eigenspace displacements have the same photon-number distributions. If the state is pure, this implies that the distribution determines the squeezing spectrum, by which we mean the set of squeezing parameters, as well as the absolute displacement along each squeezing axis (or subspace when the given squeezing value is degenerate). We discuss the interpretation of the covariance matrix spectrum for mixed states and identify representatives with diagonal covariance matrices for each equivalence class of Gaussian states with identical photon-number distributions.

The paper is organized as follows. In Sect. 2 we formulate the problem. Our main result is a parametrization theorem that shows that for Gaussian states, the total-photonnumber distributions are bijectively parametrized by the covariance matrix spectrum and the absolute displacements in the covariance matrix eigenspaces. The parametrization theorem is established in Sect. 3. Our main tool is the Husimi representation, from which we compute the expectations of the anti-normally ordered powers of the total number operators. Then, we use generating functions to prove the parametrization theorem. In Sect. 4 we show that for pure Gaussian states, the covariance matrix spectrum determines the squeezing spectrum. For mixed Gaussian states we determine the set of squeezing spectra of pure states from which the state being measured could be obtained by adding Gaussian displacement noise. We also study the set of diagonal covariance matrices with the same spectrum. We discuss and conclude in Sect. 5.

2. PROBLEM FORMULATION

We assume familiarity with quantum optics mode operators and phase space representations. See [28] for a pedagogical treatment.

Consider S modes characterized by annihilation operators \hat{a}_i for $i = 1, \ldots, S$. Our analysis does not depend on the particular physical realization of these modes, but we refer to the excitations of the modes as photons. The canonical quadrature operators \hat{q}_i and \hat{p}_i of mode i are defined so that $\hat{a}_i = \frac{1}{\sqrt{2}}(\hat{q}_i + i\hat{p}_i)$ and $\hat{a}_i^{\dagger} = \frac{1}{\sqrt{2}}(\hat{q}_i - i\hat{p}_i)$. We define the vector of quadrature operators as $\vec{r} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_S, \hat{p}_S)$. We further define the vectors of annihilation and creation operators as $\vec{a} = (\hat{a}_1, \ldots, \hat{a}_S)$ and $\vec{a}^{\dagger} = (\hat{a}_1^{\dagger}, \ldots, \hat{a}_S^{\dagger})$, respectively. We use the convention that operators are denoted with "hats". Density operators are excluded from this convention. The variables without the hats denote scalar values or vectors of values. For example, \vec{r} is a vector of 2S values, which we interpret as values of phase-space coordinates.

We assume that the state ρ being measured is Gaussian, characterized by a displacement \vec{d} with entries $d_i = \langle \hat{r}_i \rangle$, and a covariance matrix Γ with entries $\Gamma_{ij} = \langle \hat{r}_i \hat{r}_j + \hat{r}_j \hat{r}_i \rangle - 2\langle \hat{r}_i \rangle \langle \hat{r}_j \rangle$. The notation $\langle \dots \rangle$ denotes expectation with respect to ρ . With these conventions the covariance matrix of the vacuum state corresponds to the identity. The Wigner function of ρ is given by

$$W(\vec{r}) = \frac{1}{\pi^S} \frac{1}{\sqrt{\det(\Gamma)}} e^{-(\vec{r} - \vec{d})^T \Gamma^{-1}(\vec{r} - \vec{d})}.$$
(2.1)

This corresponds to Eq. 20 in Sect. IIA of [2], where this expression is derived. We note that we use a different convention than Ref. [2]. The annihilation operators are defined as $\hat{a}_i = \frac{1}{2}(\hat{q}_i + i\hat{p}_i)$ there, and the covariance matrix is equal to $\frac{1}{2}\Gamma$, which results in a different form of the Wigner function.

We use the Husimi representation [28][Ch. 3] of Gaussian states. The Husimi representation can be obtained from the Wigner function by convolution with the Gaussian $\frac{1}{\pi^S}e^{-|\vec{r}|^2}$ and is therefore also Gaussian. With our conventions, the Husimi representation of ρ is

$$Q(\vec{r}) = \frac{1}{\pi^S} \frac{1}{\sqrt{\det(I+\Gamma)}} e^{-(\vec{r}-\vec{d})^T (\Gamma+I)^{-1} (\vec{r}-\vec{d})},$$
(2.2)

where I is the $2S \times 2S$ identity matrix.

We analyze the situation where the S modes are measured by an ideal PNR detector that does not distinguish the modes. The detector's output is the number n of photons observed, and the associated operator is the projector $\hat{\Pi}_n$ onto the subspace of states with n photons. Let \hat{n} denote the total-photon-number operator on the S modes so that $\hat{n} = \sum_{i=1}^{S} \hat{a}_i^{\dagger} \hat{a}_i$. For this work we assume that we have learned the exact probabilities $\langle \hat{\Pi}_n \rangle$ of having n photons for every n. This means that we can assume as given the photon-number distribution. In the next section, we solve the following problem:

Problem. What features of S-mode Gaussian states are determined by their total-photonnumber distribution?

The total-photon-number distribution is not affected by a passive linear optical transformation defined by a unitary transformation of the mode annihilation operators. As a result, the features that can be determined must be invariant under these transformations. Such features include the spectrum of the covariance matrix of the state and the absolute displacements in the eigenspaces of the covariance matrix, see Sect. 3. Other invariants include squeezing and thermal spectra, see Sect. 4. We find that of these invariants, the covariance matrix spectrum and the absolute displacements can always be determined, while the squeezing and thermal spectra are not fully determined in general. However, the latter are determined for states occupying a single mode, and for pure states.

For special states of one mode the relationship between the photon-number distribution and the Gaussian parameters of interest is readily identified. For coherent states, which are displaced vacuum states, the photon-number distribution is Poissonian with the probability of n photons given by $e^{-d^2} d^{2n}/n!$, where d is the absolute value of the displacement. For single-mode squeezed states, the probability is zero for n odd, and $\frac{n!}{2^n(n/2)!}(\tanh r^n/\cosh r)$ for *n* even, where *r* is the squeezing parameter. For single-mode thermal states, it is $2(\nu - 1)^n/(\nu + 1)^{n+1}$, where ν is the thermal parameter. In each case the relevant parameter - the displacement, the squeezing or the thermal, is readily determined from the distribution. At the same time, the three distributions are very different from each other, suggesting that all three parameters can be simultaneously identified for general Gaussian one-mode states. General expressions for photon-number distributions of single-mode Gaussian states have been obtained and analysed, for example see [29–32]. In Sect. 4, we show that absolute displacement, squeezing and temperature parameters can all be inferred from the photon-number distribution for single-mode Gaussian states.

One way to tackle the multi-mode problem is to compute the expectations of the operators $\hat{\Pi}_n$ directly by summing the known expressions for the joint probabilities of detecting n_i photons in mode *i*. Examples of such expressions are in Ref. [33]. A review of related work and further expressions can be found in Ref. [34]. These expressions depend non-linearly on Γ and \vec{d} . Ref. [35] notes that their evaluation requires computing Hafnians, and that computing Hafnians is in general $\#\mathbf{P}$ hard, suggesting that working directly with the joint photon-number probabilities could be difficult. Thus we proceed differently.

We first make a few observations about relationships between the photon-number distribution, its moments, and the expectations of the anti-normally ordered powers of the total-photon-number operator. The moments of the photon-number distribution are given by $\langle \hat{n}^k \rangle$. Our calculations are simplified by considering instead the anti-normally ordered moments given by $\langle : \hat{n}^l : \rangle$. Here, the vertical triple dots denote anti-normal ordering of all mode operators in the formal expression between the triple dots - that is, all creation operators are moved to the right of the annihilation operators.

The Husimi representation of states satisfies the optical equivalence theorem for antinormal order. A general treatment of this theorem is in Refs. [36, 37]. These references explain orderings for one mode. To generalize the treatment to multiple modes, it suffices to apply the fact that operators from different modes commute. For the Husimi representation, it implies that expectations of expressions $f(\vec{a}, \vec{a}^{\dagger})$ whose terms are already in anti-normal order can be evaluated as the expectation of $f(\vec{\alpha}, \vec{\alpha})$ with respect to the Husimi representation of the state. That is, if $Q(\vec{\alpha}, \vec{\alpha})$ is the Husimi representation of ρ , then

$$\langle f(\vec{\hat{a}}, \vec{\hat{a}}^{\dagger}) \rangle_{\rho} = \langle f(\vec{\alpha}, \vec{\alpha}) \rangle_{H} = \int \prod_{i} d\alpha_{i} d\bar{\alpha}_{i} f(\vec{\alpha}, \vec{\alpha}) Q(\vec{\alpha}, \vec{\alpha}).$$
(2.3)

It follows that anti-normal ordering, like other such orderings, has the property that for a general expression $g(\vec{a}, \vec{a}^{\dagger})$ that may include terms that are not in anti-normal order, we have $\langle :g(\vec{a}, \vec{a}^{\dagger}): \rangle_{\rho} = \langle g(\vec{\alpha}, \vec{\alpha}) \rangle_{H}$ - see Thm. I of Ref. [36]. We utilize the expectations of the anti-normally ordered powers of the number operator. The *l*'th power of the total number operator is expressed as $\hat{n}^{l} = (\sum_{i=1}^{S} \hat{a}_{i} \hat{a}_{i}^{\dagger})^{l}$. In phase space, $\alpha_{i} \bar{\alpha}_{i} = (p_{i}^{2} + q_{i}^{2})/2$, and as a

result we obtain

$$\langle \hat{:}\hat{n}^{l} \hat{:} \rangle = \langle \hat{:} (\sum_{i=1}^{S} \hat{a}_{i}^{\dagger} \hat{a}_{i})^{l} \hat{:} \rangle$$

$$= \int \prod_{i} d\alpha_{i} d\bar{\alpha}_{i} Q(\vec{\alpha}, \vec{\alpha}) (\sum_{i} \alpha_{i} \bar{\alpha}_{i})^{l}$$

$$= \frac{1}{2^{l}} \int d^{2S} \vec{r} Q(\vec{r}) |\vec{r}|^{2l}.$$

$$(2.4)$$

Notice that for Gaussian states these integrals converge. Below we show that the antinormally ordered moments determine and are determined by the $\langle \hat{n}^k \rangle$. This implies that the latter are defined as well.

Our result relies on analyzing the generating function for anti-normally ordered moments:

$$G(z) = \langle :e^{-z\hat{n}}: \rangle$$

= $\sum_{j} (-1)^{j} \frac{1}{j!} \langle :\hat{n}^{j}: \rangle z^{j}.$ (2.5)

For Gaussian states, we find that G(z) is analytic in a neighborhood of z = 0, see Eqs. 3.3 and 3.4 in the next section. The next paragraph shows that the anti-normally ordered moments carry the same information as the usual moments $\langle \hat{n}^k \rangle$. The relationship between the two types of moments implies that the generating function for the $\langle \hat{n}^k \rangle$ is also analytic in a neighborhood of z = 0, where it is determined by G(z). Crucially, this means that for Gaussian states G(z) determines the photon-number distribution [38, Ch. 4].

According to Ref. [39], for one mode (S = 1), the operator-valued generating functions $e^{-x\hat{n}}$ and $e^{-x\hat{n}}$ are related by

$$e^{-x\hat{n}} = e^x : e^{(1-e^x)\hat{n}} : .$$
 (2.6)

This identity extends to the total-photon number in an arbitrary numbers of modes as follows.

$$e^{-x\sum_{i=1}^{S}\hat{n}_{i}} = \prod_{i=1}^{S} e^{-x\hat{n}_{i}} = \prod_{i=1}^{S} \left[e^{x} e^{(1-e^{x})\hat{n}_{i}} \right]$$
$$= \left[\prod_{i=1}^{S} e^{x}\right] e^{(1-e^{x})\sum_{i=1}^{S}\hat{n}_{i}}$$
$$= e^{xS} e^{(1-e^{x})\sum_{i=1}^{S}\hat{n}_{i}}, \qquad (2.7)$$

where the second line is obtained from the first by applying the observation that antinormal ordering and products over distinct, commuting modes can be interchanged without changing the resulting operator. Introduce the new variable z satisfying $x = \ln(1+z)$ and substitute in Eq. 2.7 to obtain

$$e^{-\ln(1+z)\hat{n}} = (1+z)^S : e^{-z\hat{n}} :$$
(2.8)

From this we can write

$$G(z) = \langle :e^{-z\hat{n}} : \rangle = \langle e^{-\ln(1+z)\hat{n}} \rangle / (1+z)^S.$$
(2.9)

Both expressions for G(z) are well-defined as generating functions. This identity implies that the anti-normally ordered moments of degree j are a linear combination of the usual moments of degree at most j and vice-versa. This can be verified as follows: Eq. 2.5 shows that the coefficient of j'th power of z in $\langle :e^{-z\hat{n}}: \rangle$ is proportional to $\langle :\hat{n}^{j}: \rangle$. Expanding the rightmost expression in Eq. 2.9 one can see that the coefficient of z^{j} is a linear combination of the $\langle \hat{n}^{k} \rangle$ for $k \leq j$. To see the reverse, multiply both expressions for G(z) by $(1 + z)^{S}$ and substitute $z = e^{x} - 1$. Then, the rightmost expression has the coefficients proportional to the powers of $\langle \hat{n}^{j} \rangle$, and for each j the transformed expression on the left can be seen to be a linear combination of the $\langle :\hat{n}^{k}: \rangle$ for $k \leq j$.

3. THE PARAMETRIZATION THEOREM

In this section, we prove the parameterization theorem. We first show that the antinormally ordered generating function G(z) defined in Eq. 2.5 may be expressed as a Gaussian integral by means of the expression for the anti-normally ordered moments in terms of the Husimi representation in Eq. 2.4. Further, we show that G(z) is analytic in a neighborhood of z = 0. As explained in Sect. 2, this implies that G(z) determines the photon-number distribution. We find that G(z) only depends on the spectrum of the state's covariance matrix Γ and the absolute displacement of the state within the eigenspaces of Γ . According to the parametrization theorem the reverse also holds, that is, these parameters are determined by G(z). Let $\{\lambda_i\}_{i=1}^N$ be the distinct eigenvalues of Γ in decreasing order. Let V_i be the eigenspace of Γ for eigenvalue λ_i and k_i the dimension of V_i . The displacement \vec{d} can be written uniquely as a sum $\sum_{i=1}^{N} \vec{d_i}$ with $\vec{d_i} \in V_i$. Let $d_i = |\vec{d_i}|$. We refer to the family $\{(\lambda_i, k_i, d_i)\}_{i=1}^N$ as the "normal parameters" of the Gaussian state.

Theorem 3.1 (Parametrization Theorem). The total-photon-number distribution of a Gaussian state determines and is determined by the normal parameters of the state.

Proof. As explained in Sect. 2, the photon-number distribution determines the antinormally ordered generating function G(z). We show that G(z) is analytic in a neighborhood of the origin, which implies that G(z) determines the photon-number distribution. This implies an equivalence between the photon-number distribution and G(z). We further show that G(z) determines and is determined by the normal parameters. The theorem statement then follows from these two equivalences.

In terms of the Husimi representation $Q(\vec{r})$ and in consideration of Eq. 2.4, G(z) is

$$G(z) = \sum_{l=0}^{\infty} \frac{(-z)^{l}}{l!} \langle :\hat{n}^{l} : \rangle$$

= $\sum_{l=0}^{\infty} \frac{(-z)^{l}}{2^{l} l!} \int dr^{2S} Q(\vec{r}) r^{2l}$
= $\int dr^{2S} Q(\vec{r}) e^{-\frac{1}{2}zr^{2}},$ (3.1)

where we used the convention that $r = |\vec{r}|$. According to Eq. 2.2 $Q(\vec{r})$ is a Gaussian with covariance matrix $(\Gamma + I)/2$ and displacement \vec{d} . Thus, the integral in Eq. 3.1 is a Gaussian integral that can be evaluated to obtain a closed form expression for G(z). See, for example, Ref. [40, Ch. 4]. To simplify the expressions, we define $\Gamma' = (\Gamma + I)^{-1}$ and $\vec{y} = \vec{r} - \vec{d}$. The evaluation goes as follows:

$$\begin{aligned} G(z) &= \int dr^{2S} Q(\vec{r}) e^{-\frac{1}{2}zr^{2}} \\ &= \frac{\sqrt{\det(\Gamma')}}{\pi^{S}} \int dr^{2S} e^{-(\vec{r}-\vec{d})^{T} \Gamma'(\vec{r}-\vec{d})} e^{-\frac{1}{2}zr^{2}} \\ &= \frac{\sqrt{\det(\Gamma')}}{\pi^{S}} \int dr^{2S} e^{-(\vec{r}-\vec{d})^{T} \Gamma'(\vec{r}-\vec{d})} e^{-\frac{1}{2}z(\vec{r}-\vec{d})^{T}(\vec{r}-\vec{d})-z(\vec{r}-\vec{d})^{T}\vec{d}-\frac{1}{2}z\vec{d}^{T}\vec{d}} \\ &= \frac{\sqrt{\det(\Gamma')}}{\pi^{S}} \int dy^{2S} e^{-\vec{y}^{T} \Gamma' \vec{y}} e^{-\frac{1}{2}z\vec{y}^{T} \vec{y}-z\vec{y}^{T}\vec{d}-\frac{1}{2}z\vec{d}^{T}\vec{d}} \\ &= \frac{\sqrt{\det(\Gamma')}}{\pi^{S}} e^{-\frac{1}{2}z\vec{d}^{T}\vec{d}} \int dy^{2S} e^{-\vec{y}^{T}(\Gamma'+\frac{1}{2}zI)\vec{y}-z\vec{y}^{T}\vec{d}}. \end{aligned}$$
(3.2)

That G(z) is determined by the normal parameters can be deduced from the last expression. First, det(Γ') depends only on the eigenvalues of Γ' , and these eigenvalues are derived from the normal parameters as $1/(\lambda_i + 1)$ with multiplicity k_i . Second, we can change variables in the integral to diagonalize Γ' and standardize \vec{d} . Let O be an orthogonal matrix for which $O\Gamma'O^T$ is diagonal with the eigenvalues in non-ascending order on the diagonal and $O^T \vec{d}$ has the property that its nonzero entries are non-negative and associated with the first coordinate of each eigenspace block of $O\Gamma'O^T$ with the same eigenvalue. To achieve the latter property, it suffices to choose appropriate orthogonal transformations within each eigenspace block. Then $O\Gamma'O^T$ is determined by the λ_i and k_i , and the normal parameter d_i is the nonzero entry of $O^T \vec{d}$ associated with the eigenspace block for eigenvalue $\lambda'_i = 1/(\lambda_i + 1)$ of $O\Gamma'O^T$. Changing variables according to $\tilde{\vec{y}} = O\vec{y}$ is equivalent to replacing Γ' with $O\Gamma'O^T$ and \vec{d} by $O^T \vec{d}$. This equivalence hinges on the rotational invariance of the measure of integration dy^{2S} . After this transformation, the integral factors as a product over each coordinate separately and we find that the value of the integral is determined by the normal parameters. Here is the explicit calculation. To express this transformation in the integral we index the new variable of integration \vec{y} according to the eigenspace blocks as \tilde{y}_{ij} , where i indicates the i'th of N blocks and j indicates the j'th of k_i coordinates in the block. Then

$$G(z) = \frac{\sqrt{\prod_{i=1}^{N} (\lambda_i')^{k_i}}}{\pi^S} e^{-\frac{1}{2}z\sum_{i=1}^{N} d_i^2} \int d\tilde{y}^{2S} e^{-\sum_{i=1}^{N} \left[(\lambda_i' + \frac{1}{2}z) \sum_{j=1}^{k_i} \tilde{y}_{ij}^2 \right] - \sum_{i=1}^{N} z \tilde{y}_{i1} d_i}$$
(3.3)

The integrand in Eq. 3.3 factors as a product of exponentials, each a function of one of the coordinates \tilde{y}_{ij} . Therefore, the integral can be expressed as a product of one-dimensional Gaussian integrals. Provided $\frac{1}{2}z > -\min\{\lambda'_i\}$, every integral in the product is finite, so G(z) evaluates to

$$G(z) = \frac{\sqrt{\prod_{i=1}^{N} (\lambda'_{i})^{k_{i}}}}{\pi^{S}} e^{-\frac{1}{2}z \sum_{i=1}^{N} d_{i}^{2}} \prod_{i=1}^{N} \left[\sqrt{\frac{\pi}{\lambda'_{i} + \frac{1}{2}z}} \right]^{k_{i}} e^{\frac{z^{2} d_{i}^{2}}{4(\lambda'_{i} + \frac{1}{2}z)}}$$
$$= \prod_{i=1}^{N} \left[\frac{\lambda'_{i}}{\lambda'_{i} + \frac{1}{2}z} \right]^{k_{i}/2} e^{-\frac{1}{2}z d_{i}^{2} + \frac{z^{2} d_{i}^{2}}{4(\lambda'_{i} + \frac{1}{2}z)}}$$
$$= \prod_{i=1}^{N} \left[\frac{\lambda'_{i}}{\lambda'_{i} + \frac{1}{2}z} \right]^{k_{i}/2} e^{-\frac{-z d_{i}^{2} \lambda'_{i}}{2(\lambda'_{i} + \frac{1}{2}z)}}.$$
(3.4)

Since $\lambda_i > 0$ the minimum of the λ'_i is a strictly positive number. Therefore, G(z) is analytic in a neighborhood of the origin.

One way to obtain the normal parameters from G(z) is to look at the first derivative of its natural logarithm. $\ln(G(z))$ is a multi-valued function, where the different "branches" differ by an additive constant. Thus, the derivative of $\ln(G(z))$ is a well-defined, single valued function for $z > -2\min\{\lambda'_i\}$:

$$L(z) \doteq \frac{d \ln(G(z))}{dz}$$

$$= -\frac{1}{2} \sum_{i} \frac{d}{dz} k_{i} \ln\left(\lambda_{i}' + \frac{1}{2}z\right) - \frac{1}{2} \sum_{i} \frac{d}{dz} z d_{i}^{2} \lambda_{i}' (\lambda_{i}' + \frac{1}{2}z)^{-1}$$

$$= -\frac{1}{4} \sum_{i} k_{i} (\lambda_{i}' + \frac{1}{2}z)^{-1} - \frac{1}{2} \sum_{i} d_{i}^{2} \lambda_{i}' (\lambda_{i}' + \frac{1}{2}z)^{-1} + \frac{1}{4} \sum_{i} z d_{i}^{2} \lambda_{i}' (\lambda_{i}' + \frac{1}{2}z)^{-2}$$

$$= -\sum_{i} \left(\frac{k_{i}}{2} (2\lambda_{i}' + z)^{-1} + 2d_{i}^{2} (\lambda_{i}')^{2} (2\lambda_{i}' + z)^{-2}\right).$$
(3.5)

Because the G(z) is defined in a neighborhood of the origin, L(z) is also defined in a neighborhood of the origin, but it can be extended to a function with a maximal domain of definition on the complex plane. This is true for any function defined on a non-empty, open subset of the complex plane, and the procedure is known as analytic continuation [41, Ch. 16]. In this case the analytic continuation of L(z) is the extension of the domain to all z where the expression on the right side of Eq. 3.5 is defined. We refer to the analytic continuation of L(z) as $L_{a.c.}(z)$. According to Eq. 3.5, $L_{a.c.}(z)$ is analytic except at poles of at most second order at $z_i = -2\lambda'_i$ for each *i*. By uniqueness of analytic continuations, the locations of the poles and their coefficients are determined by G(z). The positions z_i of the poles and the coefficients of the corresponding orders $1/(z - z_i)$ and $1/(z - z_i)^2$ can in principle be extracted by contour integration. The position of each pole determines a λ'_i and

therefore a λ_i . From the coefficient of the pole at z_i we obtain the multiplicity parameter k_i and the displacement d_i . It follows that G(z) determines the normal parameters.

4. INTERPRETATION OF NORMAL PARAMETERS FOR PURE AND MIXED STATES

Thm. 3.1 shows that the photon-number distribution determines the spectrum of the covariance matrix and the absolute displacement in each eigenspace, and nothing else. But what does this tell us about the physical properties of the Gaussian state such as the amount of squeezing along different directions of phase-space or the temperature parameters (defined in the paragraph below) of the modes? The set of all Gaussian states can be divided into equivalence classes, such that the states in a given equivalence class have the same normal parameters. We are interested in characterizing the physical properties of the states that belong to the same equivalence class. We show that for pure Gaussian states, the squeezing parameters are determined by the normal parameters. We characterize the set of normal parameters of Gaussian states, and we show that for such normal parameters, there is always a Gaussian state with diagonal covariance matrix in a fixed mode basis with these normal parameters. We further investigate sets of states that have the same normal parameters and whose covariances are all diagonal in the same mode basis. The background material for this section can be found in reviews and textbooks such as Ref. [42].

Let Γ be the $2S \times 2S$ covariance matrix of the observed state in some mode basis. The mode basis determines an antisymmetric matrix J that is preserved by the action of Gaussian unitaries on the mode operators. We order the coordinates so that the antisymmetric matrix J is block diagonal with S blocks of the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Gaussian unitaries that involve no displacement are characterized by transformation matrices A that satisfy $A^T J A = J$. Such matrices are called symplectic. There exists a symplectic matrix A such that $\Gamma = A^T T A$ [42][Secs. 3.2.3 and 3.2.4] with T diagonal and consisting of S blocks of the form $\begin{pmatrix} \nu_i & 0 \\ 0 & \nu_i \end{pmatrix}$ with $\nu_i \geq 1$, where we normalized mode operators so that the vacuum covariance matrix is the identity. We refer to T as the symplectic diagonalization of Γ , and to the family consisting of the ν_i as the symplectic spectrum of Γ . The Gaussian state with covariance matrix T is thermal in each mode, and the modes are uncorrelated. We call such states "independently thermal states", where ν_i is the temperature parameter for the *i*'th mode. In terms of the expected number of quanta in mode i, the temperature parameter ν_i is given by $\nu_i = 2\langle \hat{n}_i \rangle + 1$. Symplectic transformations can be physically realized by a combination of squeezing and linear optical transformations. Passive linear optical transformations are represented by symplectic matrices O that are also orthogonal, that is $O^T O = I$. Every symplectic matrix has a representation A = KQL where K and L are symplectic and orthogonal, and Q squeezes each mode by different amounts [42][Sect. 5.1.2]. Such a Q is diagonal with diagonal blocks of the form $\begin{pmatrix} e^{r_i} & 0\\ 0 & e^{-r_i} \end{pmatrix}$ where r_i is the squeezing parameter for mode *i*. The squeezing parameters of Γ are determined by Q. For one mode, S = 1, the symplectic diagonalization T is proportional to the identity and commutes with K. Consequently $\Gamma = L^T Q T Q L$, where Q T Q has spectrum $(\nu_1 e^{2r_1}, \nu_1 e^{-2r_1})$, and therefore, so does Γ . In this case, the thermal and squeezing parameters are determined by the spectrum of Γ .

For a multi-mode state where Γ is diagonal, the previous paragraph implies that Γ is composed of S consecutive blocks of the form $\begin{pmatrix} \nu_i e^{2r_i} & 0\\ 0 & \nu_i e^{-2r_i} \end{pmatrix}$, where ν_i and r_i are the temperature and squeezing parameters of mode i, respectively. The product of the diagonal elements of these 2×2 matrices are ν_i^2 , which satisfy $\nu_i^2 \geq 1$. Conversely, consider any diagonal positive matrix M with S diagonal 2×2 blocks, where the block for mode i is of the form $\begin{pmatrix} \gamma_i & 0\\ 0 & \gamma_i' \end{pmatrix}$ with $\gamma_i \gamma_i' \geq 1$. Then M is the covariance matrix of a Gaussian state. To see this it suffices to transform for each i, the i'th mode's block with the symplectic diagonalization 2×2 matrix $B_i = \begin{pmatrix} (\gamma_i'/\gamma_i)^{1/4} & 0\\ 0 & (\gamma_i/\gamma_i')^{1/4} \end{pmatrix}$. This gives a covariance matrix that is independently thermal in each mode as described above. We say that M is the covariance matrix of a Gaussian state where the i'th mode is a squeezed thermal state. The i'th mode has temperature parameter $\nu_i = \sqrt{\gamma_i \gamma_i'}$ and squeezing parameter $r_i = \ln(\gamma_i/\nu_i)/2$.

We call covariance matrices of Gaussian pure states "pure covariance matrices". To determine these states' parameters, we need the following characterizations of pure covariance matrices:

Lemma 4.1. Let Γ be a covariance matrix of a Gaussian state on S modes. The following are equivalent: 1. The matrix Γ is pure. 2. $\det(\Gamma) = 1$. 3. The eigenvalues $(\gamma_j)_{j=1}^{2S}$ in non-ascending order of Γ satisfy the tight pairing condition $\gamma_j \gamma_{2S+1-j} = 1$. Furthermore, in case 3. the quantities $\ln(\gamma_j)/2$ for $j \leq S$ are the squeezing parameters of the state.

Proof. The equivalence of 1. and 2. can be found in [42, Ch. 3, Sect. 5], but we provide a proof for completeness. Since displacements are realized unitarily and do not affect the covariance matrix, we may assume that the state is undisplaced so that the quadrature operators have zero mean. Write $\Gamma = A^T T A$ with A simplectic and T diagonal with thermal blocks. Since A is realized by a Gaussian unitary transformation, Γ is pure iff T is. The covariance matrix T is pure iff T = I, or equivalently, iff the temperature parameters of all modes are 0. The identity $A^T J A = J$ implies that $\det(A) = \pm 1$. Thus $\det(\Gamma) = \det(T)$. The form of T implies that $\det(T) \ge 1$ with $\det(T) = 1$ iff all temperature parameters are zero, that is, iff T is the covariance matrix of vacuum. This proves the equivalence of 1. and 2.

Write A = KQL with K and L simplectic orthogonal and Q diagonal with blocks of the form $\begin{pmatrix} e^{r_j} & 0\\ 0 & e^{-r_j} \end{pmatrix}$. We may assume without loss of generality that $r_j \ge 0$. According to the previous paragraph, if Γ is pure, then T = I. Since $K^T K = I$, we have $\Gamma = A^T T A = L^T Q^T K^T T K Q L = L^T Q^2 L$. Since L is orthogonal, the spectrum of Γ is that of Q^2 , and the pairing condition is satisfied by Q^2 . The relationship of the eigenvalues to the squeezing parameters is implied by this form. Conversely, suppose that the pairing condition is satisfied by Γ . Then det $(\Gamma) = 1$ so Γ is pure.

The next theorem establishes the relationship between normal parameters and squeezing parameters of pure Gaussian states.

Theorem 4.2. Let $\mathcal{F} = \{(\lambda_i, k_i, d_i)\}_{i=1}^N$ be the family of normal parameters of a Gaussian state. The state is pure iff $\prod_i \lambda_i^{k_i} = 1$. For pure states, the squeezing parameters are determined as follows: Let $(\gamma_j)_{j=1}^{2S}$ be the non-ascending sequence of length $2S = \sum_i k_i$ in which λ_i occurs k_i times. The S squeezing parameters of the state are given by $\ln(\gamma_j)/2$ for $j = 1, \ldots, S$.

Proof. Let Γ be the covariance matrix for the Gaussian state. For the first statement, it suffices to observe that $\prod_i \lambda_i^{k_i}$ is the determinant of Γ and apply Lem. 4.1. If the state is pure, write $\Gamma = L^T Q K^T I K Q L$ as discussed at the beginning of this section. Since K is orthogonal, $K^T I K = I$ and $\Gamma = L^T Q^2 L$. Since L is orthogonal, the spectrum of Γ is the spectrum of Q^2 , whose entries are $e^{\pm 2r_j}$, where the r_j are the squeezing parameters. This proves the second statement.

As noted at the beginning of the section, for one mode, the squeezing and the thermal parameters of a Gaussian state are determined by the spectrum. Therefore, in this case the normal parameters determine the temperature and squeezing parameter of the state. If the state is unsqueezed, one can determine the absolute displacement. Otherwise, one can determine the absolute displacements in the squeezed and in the antisqueezed directions. For mixed Gaussian states on two or more modes, it is in general not possible to determine the squeezing and thermal parameters from the normal parameters, but we can determine diagonal representatives of the set of Gaussian states with the same normal parameters and characterize the set of normal parameters.

Lemma 4.3. Let Γ be the covariance matrix of a Gaussian state. Then there exists a diagonal covariance matrix D of a Gaussian state with the same spectrum as Γ .

Proof. To prove the lemma we use the fact that there exists a pure state covariance matrix Γ_p such that $\Gamma_p \leq \Gamma$ and then apply Weyl's monotonicity principle [43, Ch. 3] to compare the spectra. For the first step, we write $\Gamma = A^T T A$ with A simplectic and T independently thermal in each mode. Then $T \geq I$, and I is the covariance matrix of vacuum, which is a pure Gaussian state. Therefore, $\Gamma_p = A^T I A$ is the covariance matrix of a pure state, and $\Gamma = A^T T A \geq A^T I A = \Gamma_p$. Let $(\gamma_j)_{j=1}^{2S}$ and $(\gamma'_j)_{j=1}^{2S}$ be the eigenvalues of Γ and Γ_p in non-ascending order. By Weyl's monotonicity principle, $\gamma_j \geq \gamma'_j$. By Lem. 4.1, for $j \leq S$ we have $\gamma'_j \gamma'_{2S+1-j} = 1$, which implies that $\gamma_j \gamma_{2S+1-j} \geq 1$. Let D be the $2S \times 2S$ diagonal matrix where the j'th mode's 2×2 block has diagonal $(\gamma_j, \gamma_{2S+1-j})$. Then D has the same spectrum as Γ and as observed at the beginning of this section, D is the covariance matrix of a Gaussian state.

Corollary 4.4. Consider the family of normal parameters $\mathcal{F} = \{(\lambda_i, k_i, d_i)\}_{i=1}^N$ for S modes. Let $(\gamma_j)_{j=1}^{2S}$ be the non-ascending sequence of length $2S = \sum_i k_i$ in which λ_i occurs k_i times. There exists a displaced Gaussian state with diagonal covariance matrix whose family of normal parameters is \mathcal{F} .

Proof. By Lem. 4.3 there exists a diagonal covariance matrix of a Gaussian state ρ' , whose non-zero entries are composed of the γ_i . We may assume that ρ' is undisplaced, so that it has zero-mean quadratures. To obtain the desired Gaussian state, it suffices to displace the quadratures associated with the first coordinate of each set of coordinates with identical eigenvalues by d_i .

Theorem 4.5. Let $\mathcal{F} = \{(\lambda_i, k_i, d_i)\}_{i=1}^N$ be a general family of triples with λ_i and d_i real and k_i positive integers. Let $(\gamma_j)_{j=1}^{S'}$ be the non-ascending sequence in which λ_i occurs k_i times. \mathcal{F} is the family of normal parameters of a Gaussian state on S modes iff the following conditions hold:

 θ . $\lambda_i > 0$ and $d_i \ge 0$.

- 1. S' is even, S' = 2S.
- 2. $\gamma_j \gamma_{2S-j+1} \ge 1$.

Proof. Suppose first that the conditions hold. Let Γ be the diagonal matrix with diagonal entries determined by $\Gamma_{2j-1} = \gamma_j$ and $\Gamma_{2j} = \gamma_{2S-j+1}$ for $j = 1, \ldots, S$. As explained at the beginning of this section, Γ is the covariance matrix of a Gaussian state where the j'th mode is a squeezed thermal state with temperature parameter $\nu_j = \sqrt{\gamma_j \gamma_{2S-j+1}}$ and squeezing parameter $r_j = \ln(\gamma_j/\nu_j)/2$. For the state to have the given normal parameters, it suffices to displace the quadratures associated with the first coordinate of each set of coordinates with identical eigenvalues by d_i .

Let Γ be the covariance matrix of a Gaussian state. Conditions 0. and 1. hold by the definition of the normal parameters. We prove that condition 2. holds. By Lem. 4.3, there exists a diagonal covariance matrix T with the same spectrum as Γ . The diagonal block of T corresponding to mode j has diagonal entries τ_j, τ_{2S-j+1} and is the covariance matrix of a squeezed thermal state. This implies that $\tau_j \tau_{2S-j+1} \ge 1$. By permuting the blocks and swapping the pair of quadrature coordinates in a block if necessary, we can assume that $\tau_j \ge \tau_{2S-j+1}$ and τ_j is non-ascending, which implies that the entire sequence $(\tau_j)_{j=1}^{2S}$ is non-ascending. Since this sequence is the spectrum of T and Γ , it follows that $\gamma_j = \tau_j$ and condition 2. is satisfied.

We end this section with a brief discussion of the general problem of characterizing the set of covariance matrices of Gaussian states with a given family of normal parameters. We focus on the case of no displacement, in which case the problem is to characterize the set \mathcal{G} of covariance matrices Γ of Gaussian states such that Γ has a given spectrum, namely the spectrum entailed by the family of normal parameters. Let Γ_0 be the diagonal covariance matrix of a Gaussian state with the same spectrum as Γ , constructed as in the proof of Cor. 4.4. Then \mathcal{G} is the intersection of the orbit \mathcal{O} of Γ_0 under the orthogonal group O(2S) and the set \mathcal{C} of covariance matrices of Gaussian states. In general, \mathcal{G} is a strict subset of \mathcal{O} . For example, with S = 2, the two diagonal matrices Diag(2, 1/2, 4, 1) and Diag(1, 1/2, 4, 2) are in the same orbit of O(2S), but the second one is not the covariance matrix of a Gaussian state, because the first mode, associated with the first two coordinates, violates the uncertainty principle, which requires the product of the two diagonal entries to be at least 1.

The set \mathcal{G} is a disjoint union of orbits under the group of orthogonal and symplectic (OS) matrices. Each such orbit is identified by its squeezing and its thermal spectrum. The results of this section imply that for S = 1 or for a pure state (det $\Gamma_0 = 1$) \mathcal{G} consists of a single such orbit. In general, there are more orbits. For example, if the spectrum of Γ_0 is (4,3,1,1), then the diagonal matrices Diag(4,3,1,1) and Diag(4,1,3,1) are both in \mathcal{G} and have different squeezing spectra and thermal parameters. Not all OS orbits have representatives that are diagonal. Examples of such orbits exist for $S \ge 2$. Consider S = 2. It suffices to exhibit a covariance matrix Γ that cannot be diagonalized by an OS matrix. We construct Γ such that its spectrum is different from that expected from its temperature and squeezing parameters. This prevents diagonalization by an OS matrix because if there exists an OS matrix A such that $D = A\Gamma A^T$ is diagonal, then the diagonal of D contains the spectrum and can be arranged to be of the form ($\nu_1 e^{2r_1}, \nu_1 e^{-2r_1}, \nu_2 e^{2r_2}, \nu_2 e^{-2r_2}$), where ν_1, ν_2 are the temperature parameters and r_1, r_2 are the squeezing parameters of Γ . To construct Γ , let c, s, τ be positive real numbers satisfying $c^2 - s^2 = 1$ and define

$$\Delta = \begin{pmatrix} 1+2\tau & 0 & 0 & 0\\ 0 & 1+2\tau & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$R = \begin{pmatrix} c & 0 & s & 0\\ 0 & c & 0 & -s\\ s & 0 & c & 0\\ 0 & -s & 0 & c \end{pmatrix}$$
$$\Gamma = R^{T} \Delta R.$$
(4.1)

Then R is symplectic and Δ is a thermal, diagonal covariance matrix. Therefore Γ is the covariance matrix of a Gaussian state. In this case the two temperature parameters are $\nu_1 = 1 + 2\tau$, $\nu_2 = 1$ and one can verify that the squeezing parameters are given by $r_1 = r_2 = \frac{1}{2} \ln(\frac{c+s}{c-s})$ by computing Q in the decomposition R = KQL, where K and L are OS and Q is diagonal. Because R is symmetric, $K = L^T$, so it suffices to check the spectrum of R. If Γ were OS diagonalizable, because the squeezing and temperature parameters do not change under OS transformations, its spectrum would be $((1 + 2\tau)\frac{c+s}{c-s}, (1 + 2\tau)\frac{c-s}{c+s}, \frac{c+s}{c-s}, \frac{c-s}{c+s})$. For $\tau > 0$ and s > 0, this consists of at least three distinct values. However, direct calculation of the spectrum of Γ shows that there are only two distinct eigenvalues $g_{\pm} = (1+\tau)(c^2+s^2) \pm \sqrt{4(1+\tau)^2c^2s^2+\tau^2}$, each with multiplicity 2. We conclude that Γ is not OS diagonalizable. Because not all OS orbits have diagonal representatives, an analysis of squeezing and thermal spectra of members of \mathcal{G} can not be reduced to an analysis of diagonal covariance matrices with the given spectrum.

5. DISCUSSION

We investigated the problem of what properties of an arbitrary multi-mode Gaussian state are determined by the total-photon-number distribution. We found that the photonnumber distribution determines the spectrum of the covariance matrix and the absolute displacement within each eigenspace. For pure states this implies that the distribution determines the squeezing parameters and the absolute displacement within each subspace of the phase space where the Gaussian state has the same amount of squeezing. The same holds for one mode in a mixed state, in which case the temperature parameter can also be determined. In general, we identified representatives for each equivalence class of Gaussian states with the same normal parameters and characterized the set of normal parameters of Gaussian states.

We established the mathematical relationship between photon-number probabilities and the normal parameters consisting of the spectrum and the displacement of a Gaussian state. Since the number of normal parameters is at most four times the number of modes, we conjecture that for a fixed number of modes not in vacuum it suffices to know a finite number of photon-number probabilities to calculate the normal parameters.

We assumed that the photon-number probabilities are exactly known. Experimental photon counters are lossy and noisy, and therefore they cannot perfectly resolve the number of photons in the state to be measured. However, this does not prevent determining the photon-number probabilities from the observed statistics. Loss and typical sources of noise can be modeled as a combination of a perfect photon counter followed by a classical noise process. Given the probabilities of outcomes after the classical noise process, it is often possible to invert the noise process to determine the original photon-number probabilities [44]. In particular, one can infer the needed probabilities from the output probabilities of lossy photon counters if the loss is known and is identical for all modes. In this case, the probability of detecting k photons o_k is given by $o_k = \sum_l M_{kl} p_l$, where p_l is the probability that there were l photons and M_{kl} is an upper triangular matrix with positive diagonal satisfying $\sum_{k\leq l} M_{kl} = 1$. Therefore M_{kl} is finitely invertible, and $p_l = \sum_l M_{lk}^{-1} o_k$. For one mode, M_{kl} can be derived from the representations of loss channels in Ref. [44]. If our conjecture that a finite number of probabilities suffices to infer the normal parameters holds, then photon counters whose output saturates for high photon counts are also sufficient.

Pure states and single-mode states are often the most experimentally relevant. In particular, the states of interest are often pure in applications, and it is desirable to have a simple method for verifying the squeezing and displacement parameters. In the situation where the experimental state is almost pure as revealed by the normal parameters that can be inferred from the photon-number probabilities, the squeezing parameters can also be verified. Further research is needed to determine the sensitivity of squeezing parameters to deviations from purity. Similarly, the stability and relevance of inferred parameters when the experimental state is slightly non-Gaussian needs to be investigated.

Our results assume that photon-number probabilities are exactly known or inferrable from measurements. In reality, photon-number probabilities are not known exactly, because of statistical uncertainties with finite data, and because of uncertainty in measurement device parameters such as losses. A goal of future research is to determine effective methods for inferring the normal parameters from finite data, where the inferred photon-number probabilities have statistical uncertainties.

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