

New Bounds on the Multiplicative Complexity of Boolean functions

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Abstract

Multiplicative Complexity (MC) is defined as the minimum number of AND gates required to implement a function with a circuit over the basis {AND, XOR, NOT}. This complexity measure is relevant for many advanced cryptographic protocols such as fully homomorphic encryption, multi-party computation, and zero-knowledge proofs, where processing AND gates is more expensive than processing XOR gates. Although there is no known asymptotically efficient technique to compute the MC of a random Boolean function, bounds on the MC of Boolean functions are successfully used to show existence of Boolean functions with a particular MC. In 2000, Boyar et al. [1] showed that, for all $n \geq 0$, at most $2^{k^2+2k+2kn+n+1}$ n -variable Boolean functions with can be computed with k AND gates, This bound is used to prove the existence of a 7-variable Boolean functions with MC greater than 7. In this paper, we improve the Boyar et al. bound by a factor of $2^{3k}3^k$.

1 Introduction

Multiplicative Complexity (MC) is defined as the minimum number of AND gates required to implement a function with a circuit over the basis {AND, XOR, NOT}. This complexity measure is relevant for many advanced cryptographic protocols (e.g., [2]), fully homomorphic encryption (e.g., [3]), and zero-knowledge proofs (e.g., [4]), where processing nonlinear gates such as AND, NAND, is more expensive than processing linear gates such as XOR. These protocols benefit from new symmetric-key primitives that can be implemented with small number of AND gates for these applications (e.g., Rasta [5], LowMC [6]).

There is no known asymptotically efficient technique to compute the MC of a random Boolean function. Boyar et al. [1] showed that the MC of an n -variable random Boolean function is at least $2^{n/2} - \mathcal{O}(n)$ with high probability. For arbitrary n , it is known that under standard cryptographic assumptions, it is not possible to compute the MC in polynomial time in the length of the truth table [7]. The *degree bound* states that the MC of a Boolean function having degree d is at least $d - 1$ [8]. This bound is commonly used to prove that a given Boolean function implementation is optimal.

For up to 6 variables, the MC of each Boolean function has been established in [9, 10]. There are also known bounds for special classes of Boolean functions. The MC of affine Boolean functions is zero. Mirwald and Schnorr [11] showed that the MC of a quadratic function f is k , iff f is affine equivalent to the canonical form $\bigoplus_{i=1}^k x_{2i-1}x_{2i}$. This implies the MC of quadratic functions is at most $\lfloor \frac{n}{2} \rfloor$. Turan and Peralta [12] improved the bounds on MC of cubic Boolean functions. Brandão et al. [13] studied the MC of symmetric Boolean functions and constructed circuits for all such functions with up to 25 variables. In 2017, Find et al. [14] characterized the Boolean functions with MC 2 by using the fact

that MC is invariant with respect to affine transformations. In 2020, Çalik et al. extended the result to Boolean functions with MC up to 4 [15]. In 2022, Häner and Soeken [16] showed the MC of interval checking.

A particular value of interest is the number of n -variable Boolean functions with MC k , denoted $\lambda(n, k)$. In [1], it is shown that $\lambda(n, k) \leq 2^{k^2+2k+2kn+n+1}$. In 2002, Fischer and Peralta [17] showed that $\lambda(n, 1)$ is equal to $2\binom{2^n}{3}$. In 2017, Find et al. [14] showed that

$$\lambda(n, 2) = 2^n(2^n - 1)(2^n - 2)(2^n - 4) \left(\frac{2}{21} + \frac{2^n - 8}{12} + \frac{2^n - 8}{360} \right). \quad (1)$$

Çalik and Turan [15] studied the Boolean functions with MC 3 and 4, and provided a closed formula for $\lambda(n, 3)$ and $\lambda(n, 4)$, by summing the sizes of all the affine equivalence classes with MC 3 (total of 24 classes) and 4 (total of 1277 classes).

For large values of n and k , the bound $\lambda(n, k) \leq 2^{k^2+2k+2kn+n+1}$ is essentially tight, but it is unclear to what extent this is true for small constant values of k . In this paper, we improve the Boyar et al. bound and provide new bounds on the maximum multiplicative complexity for n -variable Boolean functions.

2 Preliminaries

2.1 Boolean functions

Let \mathbb{F}_2 be the finite field with two elements. An n -variable Boolean function f is a mapping from \mathbb{F}_2^n to \mathbb{F}_2 . Let B_n be the set of n -variable Boolean functions and B_n^c be the set of n -variable cubic Boolean functions.

The *algebraic normal form* (ANF) of f is the multivariate polynomial $f(x_1, \dots, x_n) = \sum_{u \in \mathbb{F}_2^n} a_u x^u$, where $a_u \in \mathbb{F}_2$ and $x^u = x_1^{u_1} x_2^{u_2} \dots x_n^{u_n}$ is a *monomial* containing the variables x_i where $u_i = 1$. The degree of the monomial x^u is the number of variables appearing in x^u . The *degree* of a Boolean function, denoted $\deg(f)$, is the highest degree among the monomials appearing in its ANF.

Two functions $f, g \in B_n$ are *affine equivalent* if f can be written as

$$f(\mathbf{x}) = g(A\mathbf{x} + \mathbf{a}) + \mathbf{b}^\top \mathbf{x} + c \quad (2)$$

where A is a non-singular $n \times n$ matrix over \mathbb{F}_2 , \mathbf{a}, \mathbf{b} are column vectors in \mathbb{F}_2^n , and $c \in \mathbb{F}_2$. We use $[f]$ to denote the affine equivalence class of the function f . Degree and MC are invariant under affine transformations.

2.2 Boolean Circuits

A *Boolean circuit* C with n inputs and m outputs is a directed acyclic graph, where the inputs and the gates are the nodes, and the edges correspond to the Boolean-valued *wires*. The *fanin* and *fanout* of a node is the number of wires going in and out of the node, respectively. The nodes with fanin zero are called the *input nodes* and are labeled with an input variable from $\{x_1, \dots, x_n\}$. The circuits considered in this study only contain gates from the complete basis (AND, XOR, NOT) and have exactly one node with fanout zero (i.e., $m = 1$), which is called the *output node*. For our purposes, we assume AND gates have fan-in two, but XOR gates have arbitrary fan-in > 0 .

MC	Bound	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$	$n=12$	$n=13$	$n=14$	$n=15$	$n=16$
1	Exact	16.34	19.38	22.38	25.40	28.41	31.41	34.41	37.41	40.41	43.41	46.41
1	Bound	22	25	28	31	34	37	40	43	47	50	53
1	Improved	22	25	28	31	34	37	40	43	47	50	53
2	Exact	26.13	31.30	36.38	41.42	46.44	51.45	56.45	61.45	66.46	71.46	76.46
2	Bound	39	44	49	54	59	64	69	74	79	84	89
3	Exact	38.03	45.64	52.92	60.05	67.12	74.15	81.17	88.18	95.18	102.18	109.18
3	Bound	58	65	72	79	86	93	100	107	114	121	128
4	Exact	52.81	63.15	71.94	80.29	88.46	96.56	104.63	112.70	120.82	129.02	137.35
4	Bound	79	88	97	106	115	124	133	142	151	160	169

Table 1: Number of Boolean functions with MC 1, 2, 3, and 4 compared to the Boyar et al. bound [1] on a log scale with base 2

3 Number of Boolean functions with MC k

3.1 Previous results

Boyar et al. [1] showed that $\lambda(n, k) \leq 2^{k^2+2k+2kn+n+1}$, for all $n \geq 0$. To compute this bound, the authors considered an abstraction of Boolean circuits having binary AND gates and XOR gates with unbounded inputs are considered. Each AND gate is assumed to input a subset of input variables, outputs of AND gates (that are topologically located before the gate) and the constant function $\mathbf{1}$, i.e., for the i th AND gate a_i the (right and the left) input is subset of $\{x_1, \dots, x_n, a_1, a_2, \dots, a_{i-1}, \mathbf{1}\}$. Hence, for a_i , there are $2^{n+1+(i-1)} = 2^{n+i}$ possible choices of its left and right inputs. The bound increases by $2(n+k)+3$ for each addition of the new AND gate to the circuit. (See Table 1 for comparison of the bound to the exact values.)

The $\lambda(n, k)$ values for which the exact values are known are listed below:

- [17] $\lambda(n, 1) = 2^{\binom{2^n}{3}}$
- [7] $\lambda(n, 2) = 2^n(2^n - 1)(2^n - 2)(2^n - 4) \left(\frac{2}{21} + \frac{2^n - 8}{12} + \frac{2^n - 8}{360} \right)$.
- [15] $\lambda(n, 3) = \sum_{d=4}^6 \left(2^{n-d} \prod_{i=0}^{d-1} \frac{2^n - 2^i}{2^d - 2^i} \beta(d, 3) \right)$ where

$$\begin{aligned} \beta(4, 3) &= 32\,768, \\ \beta(5, 3) &= 775\,728\,128, \\ \beta(6, 3) &= 183\,894\,007\,808. \end{aligned}$$

- [15] $\lambda(n, 4) = \sum_{d=5}^8 \left(2^{n-d} \prod_{i=0}^{d-1} \frac{2^n - 2^i}{2^d - 2^i} \beta(d, 4) \right)$ where

$$\begin{aligned} \beta(5, 4) &= 3\,515\,396\,096, \\ \beta(6, 4) &= 7\,944\,313\,921\,970\,176, \\ \beta(7, 4) &= 8\,217\,135\,092\,528\,316\,416, \\ \beta(8, 4) &= 5\,502\,415\,308\,673\,798\,144. \end{aligned}$$

3.2 Improving the Boyar et al. bound

Next, we present two observations on Boolean circuits to improve the bound.

1. *Elimination of equivalent inputs* Let f_1 and f_2 be n -bit Boolean functions representing the left and right inputs to an AND gate, respectively, and let f_3 be the Boolean

function XORed to the output to the AND gate. The output of the AND gate is $f_1 * f_2 + f_3$. It is easy to see that the following inputs also produce the same output as f_1, f_2 , and f_3 .

$$\begin{aligned}
(f_1, f_2, f_3) &\rightarrow f_1 * f_2 + f_3 \\
(f_2, f_1, f_3) &\rightarrow f_1 * f_2 + f_3 \\
(f_1 + f_2, f_2, f_3 + f_2) &\rightarrow f_1 * f_2 + f_2 + f_2 + f_3 = f_1 * f_2 + f_3 \\
(f_2, f_1 + f_2, f_3 + f_2) &\rightarrow f_2 * f_1 + f_2 + f_2 + f_3 = f_1 * f_2 + f_3 \\
(f_1, f_2 + f_1, f_3 + f_1) &\rightarrow f_2 * f_1 + f_1 + f_3 + f_1 = f_1 * f_2 + f_3 \\
(f_2 + f_1, f_1, f_3 + f_1) &\rightarrow f_2 * f_1 + f_1 + f_3 + f_1 = f_1 * f_2 + f_3
\end{aligned}$$

In the Boyar et al. bound each of these cases are counted separately.

2. *Elimination of the constant 1 function.* Boolean functions can be partitioned into those f for which $f(0) = 0$ and those f for which $f(0) = 1$. One set can be mapped bijectively into the other by the transformation $g(\mathbf{x}) = f(\mathbf{x}) + 1$. A function $f(\mathbf{x})$ for which $f(0) = 0$ can be computed by a circuit which is both optimal with respect to multiplicative complexity and has no negations. Thus, considering circuits that do not have the constant **1** as input would produce the same set of Boolean functions. Boyar et al. bound computes the two inputs (f_1, f_2, f_3) , and $(f_1, f_2 + 1, f_3 + f_1)$ separately, although they both results in the same output.

Using the observations given above, the bound on the number of n -variable Boolean functions that can be generated using k AND gates, over the basis $\{\text{AND}, \text{XOR}, \text{NOT}\}$, can be improved to

$$\begin{aligned}
\lambda(n, k) &\leq 2^{n+k+1} \prod_{i=1}^k \frac{1}{6} (2^{n+i-1})^2 \\
&\leq 2^{(n+k+1)} 2^{(2nk-2k)} 6^{-k} 2^{\sum_{i=1}^k (2i)} \\
&\leq 2^{k^2-k+2nk+n+1} 3^{-k}.
\end{aligned}$$

4 Discussion

In this note, we improved the Boyar et al. [1] the bound on the number of n -variable Boolean functions that can be generated using k AND gates by a factor of $2^{3k} 3^k$, which can provide bounds on the maximum MC across all n -variable Boolean functions. The bounds will be provided in the extended version of the paper.

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