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Pairs of Heron and right triangles with a common area and a common perimeter

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Abstract. A Heron triangle is one in which the side lengths and area are integers. An integral right triangle is an example of a Heron triangle. In this paper, we show that there are infinitely many pairs of integral right triangles and Heron triangles with a common area and common perimeter, continuing a line of similar results with other pairs of geometric shapes. This result is established using the theory of elliptic curves.

1. Introduction

All of us learn basic facts about triangles, rectangles, square, polygons, and so forth in our primary schools. At that time, we do not know how much mystery is still hidden in these geometric shapes. For example, the congruent number problem is easy enough for a schoolchild to understand, but is an active area of research today. The congruent number problem is: Given a positive integer n, does there exist a right triangle (with rational side lengths) whose area is n? Number theory has some deep connections with triangles and other polygons. From the point of view of arithmetic, the perimeter and area are fundamental characteristics of a rational triangle. Therefore, it is natural to try to classify rational triangles by their perimeters and/or areas. More generally, similar questions can be asked about other two-dimensional geometric objects. The question we concern ourselves with in this article, is to find pairs of shapes which have the same area and same perimeter. To the best of our knowledge, this kind of

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question was first posed when Bill Sands asked his colleague R. K. Guy if there were rectangles and triangles with integer side lengths with the property that they had the same perimeter and area. In 1995, GUY [10] showed that the answer was affirmative, but that there is no non-degenerate right triangle and rectangle pair with the same property. In that same paper, Guy also showed that there are infinitely many such isosceles triangle and rectangle pairs.

Since that time, there have been several other works dealing with finding infinitely many pairs of geometric objects which have the same perimeters and the same areas. We highlight some of the literature in this direction: two distinct Heron triangles by A. BREMNER [1], Heron triangle and rectangle pairs by R. K. GUY and BREMNER [2], integral right triangle and parallelogram pairs by Y. ZHANG [20], integral isosceles triangle-parallelogram and Heron triangle-rhombus pairs by DAS, JUYAL and MOODY [6], integral right triangle and rhombus pairs by S. CHERN [4], θ-triangle and ω-rational parallelogram pairs by LALIN and MA [15], and finally rational right and isosceles triangles by HIRAKAWA and MATSUMURA [11].

In this paper, we continue this line of study. We examine integer right triangles and Heron triangles which share a common area and common perimeter. Using the theory of elliptic curves, we are able to prove that there are infinitely many examples. As integral right triangles are Heron triangles, this will also yield a new proof of Bremner's result for two distinct Heron triangles. We conclude with some directions for future work.

2. Right triangle and a Heron triangle with the same area and perimeter

In this section, we prove the result that there are infinitely many pairs of Heron triangles and integral right triangle pairs which have both a common area and common perimeter. Recall that a Heron triangle is a triangle whose side lengths and area are all integers. Every Heron triangle can be parameterized by its side lengths which can be taken to be of the form

$$\left\{(m+n)(mn-k^2), m(m^2+k^2), n(m^2+k^2)\right\},$$

where $m, n, k \in \mathbb{Z}$, with $m, n \geq 1$ and $mn > k^2$ (see [3]).

Similarly, as is well-known, the side lengths of an integral right triangle may be parameterized by

$$\{(u^2 - v^2), 2uv, (u^2 + v^2)\}$$

for positive integers u, v, with $u > v$.

The main result of this paper is the following.

Theorem 2.1. *There are infinitely many integral right triangles and Heron triangles with the same area and perimeter.*

In order to prove Theorem 2.1, we will take the conditions that the two shapes have equal areas and equal perimeters, and produce a certain elliptic curve. This curve will have the property that some of its rational points will directly correspond to the property that the two shapes have equal areas and equal perimeters. First we prove some properties of this elliptic curve in the following proposition.

Proposition 2.2. *Let $\tilde{E}$ be the elliptic curve defined over $\mathbb{Q}(k)$ by*

$$\tilde{E}(k) : y^2 = x^3 + \tilde{A}(k)x^2 + \tilde{B}(k)x, \tag{1}$$

where

$$\tilde{A}(k) = -24k^3 - 2k^2 - 12k - 2,$$

and

$$\tilde{B}(k) = (k+1)^2(16k^4 - 8k^3 + 17k^2 + 10k + 1).$$

The rank of $\tilde{E}(\mathbb{Q}(k))$ is 2, and

$$\tilde{P}_1 = \left((k+1)^2, 4k(k-1)(k+1)^2\right), \qquad \tilde{P}_2 = \left((k-1)^2, 4k(k-1)(k^2+2)\right),$$

are generators of $\tilde{E}(\mathbb{Q}(k))$.

Proof of Proposition 2.2. The discrimininant of $\tilde{E}$ is

$$512k^4(k+1)^4(k^2+1)(16k^4 - 8k^3 + 17k^2 + 10k + 1)^2.$$

It is easy to verify that $\tilde{P}_1$ and $\tilde{P}_2$ are rational points on $\tilde{E}(\mathbb{Q}(k))$. Now, we show that P_1 and P_2 are generators. The key result needed is the Gusić and Tadić specialization theorem [9, Theorem 1.3]. The theorem deals with elliptic curves given by an equation $y^2 = x^3 + A(k)x^2 + B(k)x$, where $A, B \in \mathbb{Z}[k]$, with exactly one non-trivial 2-torsion point over $\mathbb{Q}(k)$. If $k_0 \in \mathbb{Q}$ satisfies the condition that for every nonconstant square-free divisor h of $B(k)$ or $A(k)^2 - 4B(k)$ in $\mathbb{Z}[k]$ that the rational number $h(k_0)$ is not a square in $\mathbb{Q}$, then the specialized curve $E(k_0)$ is elliptic and the specialization homomorphism at k_0 is injective. If, additionally, there exist $P_1, P_2 \in E(\mathbb{Q}(k))$ such that $P_1(k_0), P_2(k_0)$ are the free generators of $E(k_0)(\mathbb{Q})$, then $E(\mathbb{Q}(k))$ and $E(k_0)(\mathbb{Q})$ have the same rank r, and P_1, P_2 are the free generators of $E(\mathbb{Q}(k))$.

Using SAGE [17], we can compute for the specialized curve at $k_0 = 5$ the rank 2. The generators are $(36, 2880)$, and $(16, 2160)$. These are the same points as evaluating $\tilde{P}_1$ and $\tilde{P}_2$ at $k_0 = 5$. Checking the conditions of Gusić and Tadić specialization theorem, a calculation shows $k_0 = 5$ satisfies the squarefree requirements. We therefore have that the specialization is injective, and hence the rank of $\tilde{E}(k)$ is at most 2. But the specialization at $k_0 = 5$ shows $\tilde{P}_1$ and $\tilde{P}_2$ are linearly independent, and hence the rank is 2 over $\mathbb{Q}(k)$, and necessarily $\tilde{P}_1$ and $\tilde{P}_2$ are the generators. □

We now give the proof of the main result: infinitely many Heron and integral right triangles with both the same area and same perimeter.

Proof of Theorem 2.1. We know that the sides of a Heron triangle may be parameterized by

$$s_1 = n(m^2 + k^2), \qquad s_2 = m(n^2 + k^2), \qquad s_3 = (m + n)(mn - k^2),$$

for some positive integers m, n, k, with $mn > k^2$. The area is then $mnk(m + n)(mn - k^2)$ and the perimeter is $2mn(m + n)$.

An integral right triangle has its sides parameterized by

$$t_1 = u^2 - v^2, \qquad t_2 = 2uv, \qquad t_3 = u^2 + v^2,$$

for positive integers u, v (with $u > v$), and where $t_1^2 + t_2^2 = t_3^2$. The area is $uv(u^2 - v^2)$ and the perimeter is $2u(u + v)$.

We set the perimeters and areas equal:

$$m^2n + mn^2 - u^2 - uv = 0, \tag{2}$$

$$k^3m^2n + k^3mn^2 - km^3n^2 - km^2n^3 + u^3v - uv^3 = 0. \tag{3}$$

We solve for v in the first equation (2), and substitute it into the second equation (3). Assuming that $mn(m + n) \neq 0$, then this yields

$$m^4n^2 + 2m^3n^3 + m^2n^4 - k^3u^2 + kmnu^2 - 3m^2nu^2 - 3mn^2u^2 + 2u^4 = 0. \tag{4}$$

This equation in m, n, u, and k is affine, and so does not represent a curve. If we fix variables, then the resulting equations can be curves of different genus. For example, fixing $m = 1$, $n = 1$ results in a curve of genus 4. Of interest to us, we observe that fixing $n = 1$, $k = 2$ yields a genus 1 curve in m and u.

We note that equation (4) is symmetric with regards to the variables m and n. We will choose to simplify by setting $n = 1$. Then we see that the equation has rational points $(m, u) = (0, 0)$ and $(-1, 0)$, hence it is thus an elliptic curve in m and u. A Weierstrass equation for this curve is given by

$$E(k) : y^2 = x^3 + A(k)x + B(k),$$

where

$$A(k) = -11k^6 - k^5/2 - 529/48k^4 - 3k^3/4 - 19k^2/24 - k/4 - 1/48,$$

$$B(k) = (-1/864)(12k^3+k^2+6k+1)(1008k^6-24k^5+1007k^4-36k^3-38k^2-12k-1).$$

We include the maps of the transformation in the Appendix, as they are unwieldy, though straightforward to compute (for example, it can be computed via the *Weierstrassform* command in Maple [14]). $E(k)$ is an elliptic curve provided that

$$k(k + 1)(k^2 + 1)(16k^4 - 8k^3 + 17k^2 + 10k + 1) \neq 0.$$

This curve $E(k)$ has a rational two-torsion point

$$(-2k^3 - k^2/6 - k - 1/6, 0).$$

We also note two rational points P_1 and P_2, which lie on the elliptic curve $E(k)$:

$$P_1 = (-2k^3 + k^2/12 - k/2 + 1/12, k(k - 1)(k + 1)^2/2),$$

and

$$P_2 = (-2k^3 + k^2/12 - 3k/2 + 1/12, k(k - 1)(k^2 + 2)/2).$$

We transform $E(k)$, so that the 2-torsion point is translated to become $(0, 0)$ and the coefficients are in $\mathbb{Z}[k]$. This simple linear transformation is an isomorphism which yields

$$\tilde{E} : y^2 = x^3 + \tilde{A}x^2 + \tilde{B}x,$$

with

$$\tilde{A} = -24k^3 - 2k^2 - 12k - 2,$$

$$\tilde{B} = (k + 1)^2(16k^4 - 8k^3 + 17k^2 + 10k + 1).$$

The two rational points become

$$\tilde{P}_1 = ((k+1)^2, 4k(k-1)(k+1)^2), \qquad \tilde{P}_2 = ((k-1)^2, 4k(k-1)(k^2+2)).$$

By Proposition 2.2, these two points are the generators of $\tilde{E}$ and hence have infinite order. If we trace back through the transformations, each rational point on $\tilde{E}$ will potentially lead to a Heron triangle and right triangle pair which have the same area and same perimeter. We say potentially, because we need to ensure that the side lengths are positive. As a result of our analysis, we will prove there are infinitely many such rational pairs with the same area and perimeter.

Concretely, the sides of the Heron triangle (resulting from P_1) are

$$s_1 = k^4-2k^3+4k^2-2k+1, \qquad s_2 = (k^2+1)(k^2-k+1), \qquad s_3 = -(k-1)(k^2-k+2),$$

and the sides of the right triangle are

$$t_1 = k(k-1)(k^2-k+2), \qquad t_2 = 2k^2-2k+2, \qquad t_3 = (k^2+1)(k^2-2k+2).$$

The common perimeter is

$$2k^4 - 4k^3 + 8k^2 - 6k + 4,$$

and the common area is

$$k(k-1)(k^2-k+2)(k^2-k+1).$$

For P_2, the resulting sides of the Heron triangle are (after scaling)

$$s_1 = k^6+k^4+2k^3+k^2+4, \qquad s_2 = (k^3+2)(k^2+1)(k-1), \qquad s_3 = (k^3+k+1)(k^2+2),$$

and the sides of the right triangle are

$$t_1 = k(k^2+2)(k^3+2), \qquad t_2 = -2(k^3+k+1)(k-1), \qquad t_3 = k^6+2k^4+2k^3+2k^2+2.$$

The common perimeter is

$$2(k^3+k+1)(k^3+2),$$

and the common area is

$$-k(k^2+2)(k^3+2)(k^3+k+1)(k-1).$$

The Heron triangles are not right triangles, as can be checked by the converse to the Pythagorean theorem (except for $k = \pm 1$). It is trivial to observe that the side lengths are integers, provided $k \in \mathbb{Z}$.

In order to make geometric sense, it should be checked that the sides of the above triangles are all of positive length. We first examine the case for triangles resulting from the point P_1. It is easy to check that s_1 and s_2 are postive for any value of k, and that $s_3 > 0$ for $k < 1$. Similarly, we see $t_1 > 0$ for $k < 0$ and $k > 1$, and that t_2 and t_3 are always positive. Thus, for $k < 0$, the side lengths of the Heron and right triangles will have positive length. For example, when $k = -2$, the Heron triangle sides are $[53, 35, 24]$, and the right triangle sides are $[48, 14, 50]$. The common perimeter is 112, with the common area 336. See Figure 1.

A similar analysis carried out on the triangles corresponding to P_2 yields that at least two sides will always have opposite sign, hence it is not possible in this case to have actual Heron and right triangles, despite the formulas for their areas and perimeters being equal. For example, for $k = 2$, the Heron triangle sides are $[104, 50, 66]$, and the right triangle side lengths are computed to be $[120, -22, 122]$. The common perimeter would be 220, and the common area ± 1320.

Any rational point Q on $\tilde{E}$ can be expressed as an integral linear combination of P_1 and P_2 added to a torsion point (such as $(0, 0)$ or the identity element). Each such point Q will potentially lead to a Heron and right triangle pair with the same perimeter and same area. An easy check can then be done to determine for what values of k the side lengths are positive.

Now we use a result of Poincaré–Hurwitz [19, p. 78] about the density of rational points. Their theorem states that if an elliptic curve $E(\mathbb{Q})$ has positive rank and at most one torsion point of order two, then the set $E(\mathbb{Q})$ is dense in $E(\mathbb{R})$. We can check that $\tilde{E}(k)$ has another point of order two if and only if $\tilde{B}^2 - 4\tilde{A} = 0$, or equivalently,

$$256k^{12} + 768k^{11} + 1120k^{10} + 1968k^9 + 3233k^8 + 3432k^7 \\ + 3196k^6 + 3192k^5 + 2342k^4 + 1080k^3 + 228k^2 + 72k + 9 = 0.$$

There are no rational solutions, so $\tilde{E}$ only has one point of order two. Thus, by the Poincaré–Hurwitz result, once we have one such point (for a fixed value of k), we will have infinitely many. □

Another question that could be asked: When is the right triangle primitive? Using the triangles corresponding to P_1, it is easy to see that t_1, t_2 and t_3 are all divisible by 2 when k is an integer. For the Heron triangle, s_1, s_2 and s_3 will be

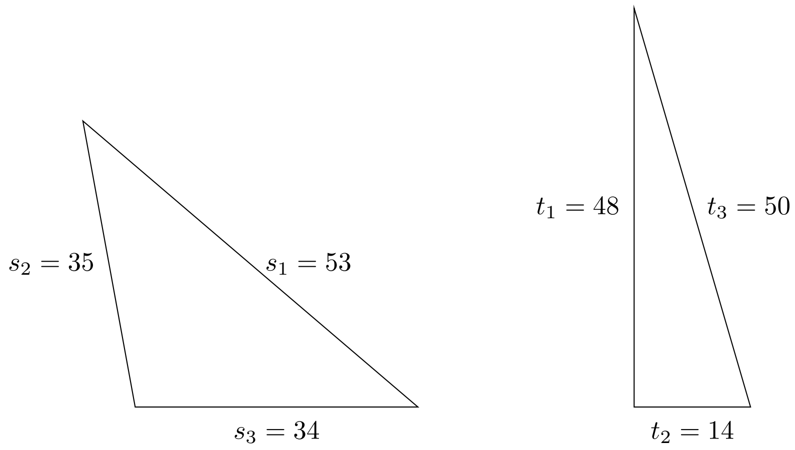

Figure 1. An example of a Heron triangle and integral right triangle, both of which have a perimeter of 112 and an area of 336.

even only when k is odd. So we could consider only odd k, and then divide the side lengths of both triangles by 2. Note under this assumption that we always have $t_3/2 - t_1/2 = 1$. Thus, the resulting integral right triangle will have two sides which differ by 1, and hence must be primitive.

We end this section with an easy corollary. In [1], Bremner found two parameter families of pairs of Heron triangles with equal perimeter and area. Using Theorem 2.1, we have another proof that there are infinitely many pairs of Heron triangle pairs with equal perimeters and equal area, since an integral right triangle is a Heron triangle. We also observed that the Heron triangles obtained above are not right triangles, showing they are distinct.

Corollary 2.3. *There are infinitely many pairs of distinct Heron triangles with the same area and same perimeter.*

Remark 2.4. *We note that there are no (non-trivial) integral right triangle pairs with the same area and same perimeter.*

To see this, we set the side lengths for the two right triangles as $\{u^2-v^2, 2uv, u^2+v^2\}$ and $\{p^2-q^2, 2pq, p^2+q^2\}$. Setting the perimeters equal, we can then solve for v:

$$v = (p^2 + pq - u^2)/u.$$

Substituting this expression for v into the equation for equal areas yields

$$-2p(p-u)(p+u)(p+q)(p^2+2pq+q^2-2u^2) = 0.$$

The trivial solutions are for $p = \pm u$, $p = 0$, $p = -q$, and lead to degenerate triangles. The only non-trivial solutions would be when $p^2 + 2pq + q^2 - 2u^2 = 0$. However, this can be re-written as $(p+q)^2 = 2u^2$, or equivalently, $((p+q)/u)^2 = 2$. Since a rational square root of 2 does not exist, we see that there can be no such desired pair of right triangles.

3. Conclusion and directions for future work

We have proved that there are infinitely many pairs of Heron triangle and integral right triangle pairs with common area and common perimeter. We have also shown that there are infinitely many pairs of Heron triangles with the same area and same perimeter, while there does not exist any such right triangle pairs. A related question one can try to solve is the same problem for pairs of isosceles and (non)-isosceles Heron triangles.

It would also be interesting to study the curve family $\tilde{E}(k)$ in (1). We showed the rank is 2 over $\tilde{E}(\mathbb{Q}(k))$, but for specific values of k, the rank (over $\mathbb{Q}$) can be higher. For example, when $k = -1929, -1582, -1563, -933, -745$, the rank is 6. Can the rank be higher?

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Appendix A. The maps to and from the Weierstrass equation

Recall that in the proof of Theorem 2.1 we had the following equation:

$$C(k): m^4 + 2m^3 + m^2 - k^3u^2 + kmu^2 - 3m^2u^2 - 3mu^2 + 2u^4 = 0.$$

We can map this curve to and from a Weierstrass equation

$$E(k): y^2 = x^3 + A(k)x + B(k),$$

via the following maps (computed via Maple [14]).

We have $\phi : C(k) \to E(k)$, where $\phi(m,u) = (x,y)$ with

$$\begin{aligned}
x = {} & \frac{1}{12m(m+1)}(12k^4u - 12k^3m^2 - 24k^3mu - 12k^3m - 12k^3u + k^2m^2 \\
& - 12k^2mu + 24km^3 - 36km^2u + 24ku^3 - 24m^4 + 24m^3u + 48m^2u^2 \\
& - 48mu^3 + k^2m + 30km^2 - 48kmu - 48m^3 + 36m^2u + 48mu^2 \\
& - 24u^3 + 6km - 23m^2 + 12mu + m),
\end{aligned}$$

and

$$\begin{aligned}
y = {} & \frac{1}{2m(m+1)}(4k^6u + k^5u - 4k^4m^2 + 4k^4mu + 8k^3m^3 - 12k^3m^2u \\
& + 8k^3u^3 - 4k^4m + 6k^4u + 12k^3m^2 - 13k^3mu + 4k^2m^3 - 9k^2m^2u \\
& + 2k^2u^3 + 4km^4 - 24km^2u^2 + 16kmu^3 + 8m^5 - 8m^4u - 16m^3u^2 \\
& + 16m^2u^3 + 4k^3m + k^3u + 4k^2m^2 - 7k^2mu + 12km^3 - 6km^2u \\
& - 24kmu^2 + 12ku^3 + 20m^4 - 16m^3u - 24m^2u^2 + 16mu^3 + 8km^2 \\
& - 7kmu + 16m^3 - 9m^2u - 8mu^2 + 2u^3 + 4m^2 - u).
\end{aligned}$$

The maps back are given by $\psi : E(k) \to C(k)$, where $\psi(x,y) = (m,u) =$ with

$$(m,u) = \left(\frac{f_1(x,y,k)}{f_2(x,y,k)}, \frac{f_3(x,y,k)}{f_4(x,y,k)}\right).$$

Here

$$\begin{aligned}
f_1(x,y,k) = {} & -19008k^{10} - 5040k^9 - 636k^8 - 26784k^7x - 8601k^7 - 6192k^6x \\
& - 2592k^6y + 3425k^6 + 1350k^5x - 720k^5y - 13824k^4x^2 - 2745k^5 \\
& - 6858k^4x - 2562k^4y - 3456k^3xy - 639k^4 + 2700k^3x - 1080k^3y \\
& + 288k^2xy - 2592kx^3 + 621k^3 - 1332k^2x + 1140k^2y - 1728kxy \\
& + 864x^3 - 864x^2y - 165k^2 + 270kx - 360ky + 288xy + 21k - 18x \\
& + 30y - 1,
\end{aligned}$$

$$\begin{aligned}
f_2(x,y,k) = {} & 2(12k^3 - k^2 + 6k + 6x - 1)(1008k^6 - 2280k^5 + 1199k^4 + 288k^3x \\
& - 1116k^3 - 1176k^2x + 154k^2 + 144kx - 144x^2 + 12k - 24x - 1),
\end{aligned}$$

$$\begin{aligned}
f_3(x,y,k) = {} & -5184k^{10} + 1008k^9 - 7596k^8 - 9504k^7x + 2021k^7 - 432k^6x \\
& - 432k^6y + 18k^6 - 9522k^5x + 1368k^5y - 5184k^4x^2 + 1563k^5 \\
& - 648k^4x - 291k^4y - 432k^3x^2 - 864k^3xy + 1440k^4 - 684k^3x
\end{aligned}$$

$$
\begin{aligned}
&+ 324k^3y - 2592k^2x^2 + 936k^2xy - 864kx^3 + 555k^3 - 216k^2x \\
&- 834k^2y - 432kx^2 - 432kxy - 432x^2y + 90k^2 - 18kx - 180ky \\
&+ 72xy + 5k - 3y,
\end{aligned}
$$

$$
\begin{aligned}
f_4(x, y, k) = {}&(12k^3 + k^2 + 6k + 6x + 1)(1008k^6 + 2280k^5 + 1199k^4 + 288k^3x \\
&+ 1116k^3 + 1176k^2x + 154k^2 + 144kx - 144x^2 - 12k + 24x - 1).
\end{aligned}
$$

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