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Nonterminating transformations and summations associated with some q-Mellin–Barnes integrals

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A B S T R A C T

In many cases one may encounter an integral which is of q-Mellin–Barnes type. These integrals are easily evaluated using theorems which have a long history dating back to Slater, Askey, Gasper, Rahman and others. We derive some interesting q-Mellin–Barnes integrals and using them we derive transformation and summation formulas for nonterminating basic hypergeometric functions. The cases which we treat include ratios of theta functions, the Askey–Wilson moments, nonterminating well-poised ${}_3\phi_2$, nonterminating very-well-poised ${}_5W_4$, ${}_8W_7$, products of two nonterminating ${}_2\phi_1$'s, square of a nonterminating well-poised ${}_2\phi_1$, a nonterminating ${}_{10}W_9$, two nonterminating ${}_{12}W_{11}$'s and several nonterminating summations which arise from the Askey–Roy and Gasper integrals.

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1. Preliminaries

There have existed in the past some very important q-Mellin–Barnes integrals. Some important examples include those given by Askey–Wilson [4, (2.1)], Nassrallah–Rahman [9, (6.3.9)] as well as those given in Askey–Roy [3, (2.8)] and in Gasper [8, (1.8)]. In this paper we take advantage of the powerful methods following the pioneering work of Bailey [5, Chapter 8], and his student Slater which were fully recapitulated by Gasper & Rahman in [9, Chapter 4]. We are able to use well-known formulas for certain highly symmetric basic hypergeometric functions to obtain new q-Mellin–Barnes integrals and from them derive a new class of transformation and summation formulas.

We adopt the following set notations: $\mathbb{N}_0 := \{0\}\cup\mathbb{N} = \{0, 1, 2, \ldots\}$, and we use the sets $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ which represent the integers, real numbers and complex numbers respectively, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, and $\mathbb{C}^\dagger := \{z \in \mathbb{C}^* : |z| < 1\}$. We also adopt the following notation and conventions. For any sequence $(a_1, \ldots, a_A)$ of length $A \in \mathbb{N}$, define the corresponding finite multiset $\mathbf{a} := \{a_1, \ldots, a_A\}$ (see [17, §1.2]) and for $1 \le k \le A$ define

$$\mathbf{a}_{[k]} := \mathbf{a} \setminus \{a_k\} = \{a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_A\}. \tag{1.1}$$

Furthermore, define the following multiset scalar product and sum notations, namely

$$b\,\mathbf{a} = \mathbf{a}\,b = b\,(\mathbf{a}) = (\mathbf{a})\,b := \{b\,a_1, b\,a_2, \ldots, b\,a_A\}, \tag{1.2}$$

$$\mathbf{a} + b = b + \mathbf{a} = (\mathbf{a}) + b = b + (\mathbf{a}) := \{a_1 + b, a_2 + b, \ldots, a_A + b\}, \tag{1.3}$$

where $b, a_1, \ldots, a_A \in \mathbb{C}$.

Remark 1.1. Observe in the following discussion we will often be referring to a collection of constants a, b, c, d, e, f. In such cases, which will be clear from context, then the constant e should not be confused with Euler's number e, the base of the natural logarithm, i.e., $\log \mathrm{e} = 1$. Observe the different (roman) typography for Euler's number.

We assume that the empty sum vanishes and the empty product is unity. We will also adopt the following symmetric sum notation.

Definition 1.2. For some function $f(a_1, \ldots, a_n; \mathbf{b})$, where $\mathbf{b}$ is some multiset of parameters. Then

$$\sum_{a_1; a_2, \ldots, a_n} f(a_1, a_2, \ldots, a_n; \mathbf{b}) := f(a_1, a_2, \ldots, a_n; \mathbf{b}) + \mathrm{idem}(a_1; a_2, \ldots, a_n), \tag{1.4}$$

where "$\mathrm{idem}(a_1; a_2, \ldots, a_n)$" after an expression stands for the sum of the $n - 1$ expressions obtained from the preceding expression by interchanging a_1 with each a_k, $k = 2, 3, \ldots, n$.

Definition 1.3. We adopt the following conventions for succinctly writing elements of sets. To indicate sequential positive and negative elements, we write

$$\pm a := \{a, -a\}.$$

We also adopt an analogous notation

$$\mathrm{e}^{\pm i\theta} := \{\mathrm{e}^{i\theta}, \mathrm{e}^{-i\theta}\}.$$

In the same vein, consider the numbers $f_s \in \mathbb{C}$ with $s \in \mathcal{S} \subset \mathbb{N}$, with $\mathcal{S}$ finite. Then, the notation $\{f_s\}$ represents the set of all complex numbers f_s such that $s \in \mathcal{S}$. Furthermore, consider some $p \in \mathcal{S}$, then the notation $\{f_s\}_{s\neq p}$ represents the sequence of all complex numbers f_s such that $s \in \mathcal{S}\backslash\{p\}$.

Consider $q \in \mathbb{C}^\dagger$, $n \in \mathbb{N}_0$. Define the sets

$$\Omega_q^n := \{q^{-k} : k \in \mathbb{N}_0,\ 0 \leq k \leq n-1\}, \tag{1.5}$$

$$\Omega_q := \Omega_q^\infty = \{q^{-k} : k \in \mathbb{N}_0\}, \tag{1.6}$$

$$\Upsilon_q := \{q^k : k \in \mathbb{Z}\}. \tag{1.7}$$

In order to obtain our derived identities, we rely on properties of the q-shifted factorial $(a;q)_n$. We refer to $(a;q)_n$ as a q-shifted factorial (it is also referred to as a q-Pochhammer symbol). For any $n \in \mathbb{N}_0$, $a, b, q \in \mathbb{C}$, the q-shifted factorial is defined as

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}). \tag{1.8}$$

One may also define

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \tag{1.9}$$

where $|q| < 1$. Furthermore, one has the following identities

$$(a^2;q)_\infty = (\pm a, \pm q^{\frac{1}{2}}a;q)_\infty \tag{1.10}$$

$$(a^2;q^2)_\infty = (\pm a;q)_\infty. \tag{1.11}$$

One also has

$$(q^{-n}a;q)_\infty = (q^{-n}a;q)_n(a;q)_\infty. \tag{1.12}$$

One also has the definition of the q-gamma function, namely [13, (1.9.1)]

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(1-q)^{x-1}(q^x;q)_\infty}, \tag{1.13}$$

and also the gamma function $\Gamma : \mathbb{C} \setminus -\mathbb{N}_0 \to \mathbb{C}$ defined in [15, (5.2.1)]. Note that [13, p. 13]

$$\lim_{q\to 1^-} \Gamma_q(x) = \Gamma(x). \tag{1.14}$$

We will also use the following notational product conventions, $a_k \in \mathbb{C}$, $k \in \mathbb{N}$, $n \in \mathbb{N}_0 \cup \{\infty\}$,

$$(a_1,\ldots,a_k;q)_n := (a_1;q)_n \cdots (a_k;q)_n, \tag{1.15}$$

$$\Gamma_q(a_1,\ldots,a_k) := \Gamma_q(a_1)\cdots\Gamma_q(a_k), \tag{1.16}$$

$$\Gamma(a_1,\ldots,a_k) := \Gamma(a_1)\cdots\Gamma(a_k). \tag{1.17}$$

The basic hypergeometric series, which we will often use, is defined for $z \in \mathbb{C}$, $q \in \mathbb{C}^\dagger$, $s \in \mathbb{N}_0$, $r \in \mathbb{N}_0 \cup \{-1\}$, $b_j \notin \Omega_q$, $j = 1,\ldots,s$, as [13, (1.10.1)]

$$ {}_{r+1}\phi_s\left(\begin{matrix} a_1,\ldots,a_{r+1} \\ b_1,\ldots,b_s \end{matrix};q,z\right) := \sum_{k=0}^\infty \frac{(a_1,\ldots,a_{r+1};q)_k}{(q,b_1,\ldots,b_s;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{s-r} z^k. \tag{1.18}$$

For $s > r$, ${}_{r+1}\phi_s$ is an entire function of z, for $s = r$ then ${}_{r+1}\phi_s$ is convergent for $|z| < 1$, and for $s < r$ the series is divergent unless it is terminating. Note that when we refer to a basic hypergeometric function with *arbitrary argument* z, we simply mean that the argument does not necessarily depend on the other parameters, namely the a_j's, b_j's nor q. However, for the arbitrary argument z, it very-well may be that the domain of the argument is restricted, such as for $|z| < 1$.

We will use the following notation ${}_{r+1}\phi_s^m$, $m \in \mathbb{Z}$ (originally due to van de Bult & Rains [18, p. 4]), for basic hypergeometric series when some parameter entries are equal to zero. Consider $p \in \mathbb{N}_0$. Then define

$$ {}_{r+1}\phi_s^{-p}\left(\begin{matrix} a_1,\ldots,a_{r+1} \\ b_1,\ldots,b_s \end{matrix};q,z\right) := {}_{r+p+1}\phi_s\left(\begin{matrix} a_1,a_2,\ldots,a_{r+1},\overbrace{0,\ldots,0}^{p} \\ b_1,b_2,\ldots,b_s \end{matrix};q,z\right), \tag{1.19}$$

$$ {}_{r+1}\phi_s^{p}\left(\begin{matrix} a_1,\ldots,a_{r+1} \\ b_1,\ldots,b_s \end{matrix};q,z\right) := {}_{r+1}\phi_{s+p}\left(\begin{matrix} a_1,a_2,\ldots,a_{r+1} \\ b_1,b_2,\ldots,b_s,\underbrace{0,\ldots,0}_{p} \end{matrix};q,z\right), \tag{1.20}$$

where $b_1,\ldots,b_s \notin \Omega_q \cup \{0\}$, and ${}_{r+1}\phi_s^0 = {}_{r+1}\phi_s$. The nonterminating basic hypergeometric series ${}_{r+1}\phi_s^m(\mathbf{a};\mathbf{b};q,z)$, $\mathbf{a} := \{a_1,\ldots,a_{r+1}\}$, $\mathbf{b} := \{b_1,\ldots,b_s\}$, is well-defined for $s-r+m \geq 0$. In particular ${}_{r+1}\phi_s^m$ is an entire function of z for $s-r+m > 0$, convergent for $|z| < 1$ for $s-r+m = 0$ and divergent if $s-r+m < 0$ unless it is terminating.

Note that we will move interchangeably between the van de Bult & Rains notation and the alternative notation with vanishing numerator and denominator parameters which are used on the right-hand sides of (1.19) and (1.20).

The geometric series is given by [1]

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \tag{1.21}$$

provided $|z| < 1$. The q-binomial theorem is given by [13, (1.11.1)]

$${}_1\phi_0\left(\begin{matrix} a \\ - \end{matrix}; q, z\right) = \frac{(az;q)_\infty}{(z;q)_\infty}, \tag{1.22}$$

provided $|q|, |z| < 1$ for convergence of the left-hand side nonterminating basic hypergeometric series.

1.1. The theta function and the partial theta function

The *theta function* $\vartheta(z;q)$ (sometimes referred to as a modified theta function [9, (11.2.1)]) is defined by Jacobi's triple product identity and is given by [9, (1.6.1)] (see also [14, (2.3)])

$$\vartheta(z;q) := (z, q/z; q)_\infty = \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} z^n, \tag{1.23}$$

where $z \neq 0$. Note that $\vartheta(q^n;q) = 0$ if $n \in \mathbb{Z}$. We will adopt the product convention for theta functions for $a_k \in \mathbb{C}$ for $k \in \mathbb{N}$, namely

$$\vartheta(a_1, \ldots, a_k; q) := \vartheta(a_1;q) \cdots \vartheta(a_k;q).$$

A particular ratio of theta function satisfies the following useful identity

$$\frac{(a, q/a; q)_\infty}{(qa, 1/a; q)_\infty} = \frac{\vartheta(a;q)}{\vartheta(qa;q)} = -a, \tag{1.24}$$

where $a \neq 0$.

The *partial theta function* $\Theta(z;q)$, described as such because it only involves the partial sum contribution for $n \geq 0$ in (1.23) as opposed to summing over all integers as in the theta function, is defined as follows with alternative representations.

Theorem 1.4. *Let $q \in \mathbb{C}^\dagger$, $p \in \mathbb{N}$, $z \in \mathbb{C}$, $|z| < 1$. Then*

$$\Theta(z;q) := \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} z^n = \frac{1}{(q;q)_\infty}\, {}_1\phi_0^1\left(\begin{matrix} q \\ - \end{matrix}; q, z\right) \tag{1.25}$$

$$= \frac{1}{(q;q)_\infty}\ {}_1\phi_0^p\left(\begin{matrix} q^{1/p} \\ - \end{matrix}; q^{1/p}, (-1)^{p-1}z\right) \tag{1.26}$$

$$= (z;q)_\infty\ {}_0\phi_1^{-2}\left(\begin{matrix} - \\ z \end{matrix}; q, q\right) \tag{1.27}$$

$$= \frac{(z;q)_\infty}{(q;q)_\infty}\ {}_0\phi_1\left(\begin{matrix} - \\ z \end{matrix}; q, qz\right) \tag{1.28}$$

$$= \frac{(z;q)_\infty}{(\pm q;q)_\infty}\ {}_4\phi_3\left(\begin{matrix} \pm i\sqrt{z}, \pm i\sqrt{qz} \\ -q, \pm z \end{matrix}; q, q\right). \tag{1.29}$$

Proof. The representation (1.25) follows the definition of nonterminating basic hypergeometric series (1.18). The representation (1.26) follows from direct substitution using (1.18) and (1.20). The representations (1.27), (1.28) follow from [13, (1.13.8-9)]. The representation (1.29) follows from Andrews & Warnaar's formula for a product of partial theta functions [2, Theorem 1.1]

$$\Theta(a;q)\Theta(b;q) = \frac{(a,b;q)_\infty}{(q;q)_\infty}\ {}_4\phi_3\left(\begin{matrix} \pm\sqrt{ab}, \pm\sqrt{\frac{ab}{q}} \\ a, b, \frac{ab}{q} \end{matrix}; q, q\right), \tag{1.30}$$

with the substitutions $(a,b) \mapsto (z,-q)$ and the identity [15, (20.4.3)]

$$\Theta(-q;q) = (-q,-q;q)_\infty. \tag{1.31}$$

This completes the proof. □

1.2. Some theorems involving q-Mellin–Barnes integrals

Now we present a result which allows one to evaluate integrals of products and ratios of infinite q-shifted factorials in terms of sums of non-terminating basic hypergeometric functions. The following result is a special case ($t = 1$) of the more general result which appears in [6, Theorem 2.1]. Note that we adopt a representation for the contour integral as in [9, (4.9.3)]. However, there are several other alternative integral representations which can be used (see [9, §4.9]).

Theorem 1.5. *Let $q \in \mathbb{C}^\dagger$, $m \in \mathbb{Z}$, $\sigma \in (0,\infty)$, $\mathbf{a} := \{a_1,\ldots,a_A\}$, $\mathbf{b} := \{b_1,\ldots,b_B\}$, $\mathbf{c} := \{c_1,\ldots,c_C\}$, $\mathbf{d} := \{d_1,\ldots,d_D\}$ be sets of non-zero complex numbers with cardinality $A,B,C,D \in \mathbb{N}_0$ (not all zero) respectively with $|c_k| < \sigma$, $|d_l| < 1/\sigma$, for any $a_i,b_j,c_k,d_l \in \mathbb{C}^*$ elements of $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$, and $z := \mathrm{e}^{i\psi}$. Define the q-Mellin–Barnes integral*

$$G_m := G_m(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d};\sigma,q) := \frac{(q;q)_\infty}{2\pi}\left(\frac{1}{\sigma}\right)^m \int_{-\pi}^{\pi} \frac{(\mathbf{b}\frac{\sigma}{z}, \mathbf{a}\frac{z}{\sigma};q)_\infty}{(\mathbf{d}\frac{\sigma}{z}, \mathbf{c}\frac{z}{\sigma};q)_\infty}\mathrm{e}^{im\psi}\mathrm{d}\psi, \tag{1.32}$$

such that the integral exists. In the integrand, we have adopted the multiset scalar product notation (1.2)*. Then*

$$G_m(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d};\sigma,q) = G_{-m}(\mathbf{b},\mathbf{a},\mathbf{d},\mathbf{c};\sigma,q), \tag{1.33}$$

if $|c_k|, |d_l| < \min\{1/\sigma, \sigma\}$*. Furthermore, let* $d_l c_k \notin \Omega_q$*. If* $D \geq B$*,* $d_l/d_{l'} \notin \Omega_q$*,* $l \neq l'$*, then*

$$G_m = \sum_{k=1}^{D} \frac{(d_k\mathbf{a}, \mathbf{b}/d_k; q)_\infty d_k^m}{(d_k\mathbf{c}, \mathbf{d}_{[k]}/d_k; q)_\infty} {}_{B+C}\phi_{A+D-1}^{C-A} \left(\begin{matrix} d_k\mathbf{c}, qd_k/\mathbf{b} \\ d_k\mathbf{a}, qd_k/\mathbf{d}_{[k]} \end{matrix} ; q, q^m (qd_k)^{D-B} \frac{b_1\cdots b_B}{d_1\cdots d_D} \right), \tag{1.34}$$

and/or if $C \geq A$*,* $c_k/c_{k'} \notin \Omega_q$*,* $k \neq k'$*, then*

$$G_m = \sum_{k=1}^{C} \frac{(c_k\mathbf{b}, \mathbf{a}/c_k; q)_\infty c_k^{-m}}{(c_k\mathbf{d}, \mathbf{c}_{[k]}/c_k; q)_\infty} {}_{A+D}\phi_{B+C-1}^{D-B} \left(\begin{matrix} c_k\mathbf{d}, qc_k/\mathbf{a} \\ c_k\mathbf{b}, qc_k/\mathbf{c}_{[k]} \end{matrix} ; q, q^{-m} (qc_k)^{C-A} \frac{a_1\cdots a_A}{c_1\cdots c_C} \right), \tag{1.35}$$

where the nonterminating basic hypergeometric series in (1.34) *(resp.* (1.35)*) is entire if* $D > B$ *(resp.* $C > A$*), convergent for* $|q^m b_1\cdots b_B| < |d_1\cdots d_D|$ *if* $D = B$ *(resp.* $|q^{-m}a_1\cdots a_A| < |c_1\cdots c_C|$ *if* $C = A$*), and divergent otherwise.*

Proof. See proof of [6, Theorem 2.1]. □

One can convert the integral in the above theorem to a form which is more similar to that which appears in Mellin–Barnes integrals by replacing the infinite q-shifted factorials with q-gamma functions using (1.13).

Corollary 1.6. *Let* $q \in \mathbb{C}^\dagger$*,* $m \in \mathbb{Z}$*,* $\mathbf{a} := \{a_1,\ldots,a_A\}$*,* $\mathbf{b} := \{b_1,\ldots,b_B\}$*,* $\mathbf{c} := \{c_1,\ldots,c_C\}$*,* $\mathbf{d} := \{d_1,\ldots,d_D\}$ *be sets of non-zero complex numbers with cardinality* $A,B,C,D \in \mathbb{N}_0$ *(not all zero) respectively,* $\Sigma a_j := \sum_{j=1}^{A} a_j$*,* $\Sigma b_j := \sum_{j=1}^{B} b_j$*,* $\Sigma c_j := \sum_{j=1}^{C} c_j$*,* $\Sigma d_j := \sum_{j=1}^{D} d_j$*, and* $|q|^\sigma \in (0,\infty)$*,* $|q^{c_k}|, |q^{d_l}| < \min\{|q|^\sigma, |q|^{-\sigma}\}$*,* $d_l + c_k \notin -\mathbb{N}_0$*, for any* $a_i, b_j, c_k, d_l \in \mathbb{C}^*$ *elements of* $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$*. Define*

$$\begin{aligned} I_m &:= I_m(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d};\sigma;q) \\ &:= \int_{\frac{\pi}{\log q}}^{-\frac{\pi}{\log q}} \frac{\Gamma_q(\mathbf{d}+\sigma-ix, \mathbf{c}-\sigma+ix)}{\Gamma_q(\mathbf{b}+\sigma-ix, \mathbf{a}-\sigma+ix)} q^{imx} (1-q)^{(ix-\sigma)(C-D+B-A)} \,\mathrm{d}x. \end{aligned} \tag{1.36}$$

In the integrand we have adopted the multiset scalar sum notation (1.3)*. Then if* $D \geq B$*,* $d_l - d_{l'} \notin -\mathbb{N}_0$*,* $l \neq l'$*, one has*

$$I_m = \frac{2\pi(1-q)q^{m\sigma}}{-\log q}\sum_{k=1}^{D}\frac{\Gamma_q(d_k+\mathbf{c},\mathbf{d}_{[k]}-d_k)}{\Gamma_q(d_k+\mathbf{a},\mathbf{b}-d_k)}q^{md_k}(1-q)^{d_k(C-D+B-A)}$$
$$\times {}_{B+C}\phi_{A+D-1}^{C-A}\left(\begin{matrix}q^{d_k+\mathbf{c}}, q^{1+d_k-\mathbf{b}}\\ q^{d_k+\mathbf{a}}, q^{1+d_k-\mathbf{d}_{[k]}}\end{matrix};q,q^{m+(D-B)(1+d_k)+\Sigma b_j-\Sigma d_j}\right), \quad (1.37)$$

and if $C \geq A$, $c_k - c_{k'} \notin -\mathbb{N}_0$, $k \neq k'$, *one has*

$$I_m = \frac{2\pi(1-q)q^{m\sigma}}{-\log q}\sum_{k=1}^{C}\frac{\Gamma_q(c_k+\mathbf{d},\mathbf{c}_{[k]}-c_k)}{\Gamma_q(c_k+\mathbf{b},\mathbf{a}-c_k)}q^{-mc_k}(1-q)^{-c_k(C-D+B-A)}$$
$$\times {}_{A+D}\phi_{B+C-1}^{D-B}\left(\begin{matrix}q^{c_k+\mathbf{d}}, q^{1+c_k-\mathbf{a}}\\ q^{c_k+\mathbf{b}}, q^{1+c_k-\mathbf{c}_{[k]}}\end{matrix};q,q^{-m+(C-A)(1+c_k)+\Sigma a_j-\Sigma c_j}\right). \quad (1.38)$$

Proof. Using (1.34), (1.35), we respectively start along the lines of Askey & Roy [3, p. 368] and use the map $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\sigma,\mathrm{e}^{i\psi}) \mapsto (q^{\mathbf{a}},q^{\mathbf{b}},q^{\mathbf{c}},q^{\mathbf{d}},q^{\sigma},q^{ix})$. This completes the proof. □

Note that

$$\lim_{q\to 1^-}\frac{-\log q}{1-q} = 1. \quad (1.39)$$

So certainly in the case where all infinite q-shifted factorials are composed of parameters which do not have leading negative factors, we can convert the integral in Theorem 1.9 to one which resembles a Mellin–Barnes integral in the $q \to 1^-$ limit (1.14). It is this reason that we refer to these integrals as q-Mellin–Barnes integrals. It is also clear that there are situations where the $q \to 1^-$ limit either vanishes or is perhaps not well-defined. This is a technicality that may or may not be easily addressed.

Now consider the situation where $D = B$ and $C = A$. This produces the following result.

Corollary 1.7. *Let* $q \in \mathbb{C}^\dagger$, $m \in \mathbb{Z}$, $\mathbf{a} := \{a_1,\ldots,a_A\}$, $\mathbf{b} := \{b_1,\ldots,b_B\}$, $\mathbf{c} := \{c_1,\ldots,c_A\}$, $\mathbf{d} := \{d_1,\ldots,d_B\}$ *be sets of non-zero complex numbers with cardinality* $A,B,C,D \in \mathbb{N}_0$ *(not all zero) respectively,* $\Sigma a_j := \sum_{j=1}^{A} a_j$, $\Sigma b_j := \sum_{j=1}^{B} b_j$, $\Sigma c_j := \sum_{j=1}^{A} c_j$, $\Sigma d_j := \sum_{j=1}^{B} d_j$, *and* $|q|^\sigma \in (0,\infty)$, $|q^{c_k}|,|q^{d_l}| < \min\{|q|^\sigma,|q|^{-\sigma}\}$, $d_l + c_k \notin -\mathbb{N}_0$, *for any* $a_i,b_j,c_k,d_l \in \mathbb{C}^*$ *elements of* $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$. *Then*

$$\int_{\frac{\pi}{\log q}}^{-\frac{\pi}{\log q}}\frac{\Gamma_q(\mathbf{d}+\sigma-ix,\mathbf{c}-\sigma+ix)}{\Gamma_q(\mathbf{b}+\sigma-ix,\mathbf{a}-\sigma+ix)}q^{imx}\,\mathrm{d}x = \frac{2\pi(1-q)q^{m\sigma}}{-\log q}\mathsf{A}. \quad (1.40)$$

In the integrand we have adopted the multiset scalar sum notation (1.3). *If* $d_l - d_{l'} \notin -\mathbb{N}_0$, $l \neq l'$ *one has*

$$\mathsf{A} = \sum_{k=1}^{B} \frac{\Gamma_q(d_k + \mathbf{c}, \mathbf{d}_{[k]} - d_k) q^{md_k}}{\Gamma_q(d_k + \mathbf{a}, \mathbf{b} - d_k)} {}_{A+B}\phi_{A+B-1}\left(\begin{matrix} q^{d_k + \mathbf{c}}, q^{1+d_k-\mathbf{b}} \\ q^{d_k+\mathbf{a}}, q^{1+d_k-\mathbf{d}_{[k]}} \end{matrix} ; q, q^{m+\Sigma b_j - \Sigma d_j} \right), \tag{1.41}$$

and if $c_k - c_{k'} \notin -\mathbb{N}_0$, $k \neq k'$ *one has*

$$\mathsf{A} = \sum_{k=1}^{A} \frac{\Gamma_q(c_k + \mathbf{d}, \mathbf{c}_{[k]} - c_k) q^{-mc_k}}{\Gamma_q(c_k + \mathbf{b}, \mathbf{a} - c_k)} {}_{A+B}\phi_{A+B-1}\left(\begin{matrix} q^{c_k + \mathbf{d}}, q^{1+c_k-\mathbf{a}} \\ q^{c_k+\mathbf{b}}, q^{1+c_k-\mathbf{c}_{[k]}} \end{matrix} ; q, q^{-m+\Sigma a_j - \Sigma c_j} \right), \tag{1.42}$$

where $|q^{m+\Sigma b_j - \Sigma d_j}| < 1$ *and* $|q^{-m+\Sigma a_j - \Sigma c_j}| < 1$ *respectively.*

We now take the limit as $q \to 1^-$ and obtain the following result.

Corollary 1.8. *Let* $\mathbf{a} := \{a_1, \ldots, a_A\}$, $\mathbf{b} := \{b_1, \ldots, b_B\}$, $\mathbf{c} := \{c_1, \ldots, c_A\}$, $\mathbf{d} := \{d_1, \ldots, d_B\}$, $A, B \in \mathbb{N}_0$ *(not both zero) respectively, for any* $a_i, b_j, c_k, d_l \in \mathbb{C}^*$ *elements of* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, $\Sigma a_j := \sum_{j=1}^{A} a_j$, $\Sigma b_j := \sum_{j=1}^{B} b_j$, $\Sigma c_j := \sum_{j=1}^{A} c_j$, $\Sigma d_j := \sum_{j=1}^{B} d_j$, $\sigma \in (0, \infty)$. *Define*

$$\mathsf{B} := \mathsf{B}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(\mathbf{d} + \sigma - ix)\Gamma(\mathbf{c} - \sigma + ix)}{\Gamma(\mathbf{b} + \sigma - ix)\Gamma(\mathbf{a} - \sigma + ix)} \, \mathrm{d}x. \tag{1.43}$$

Then

$$\mathsf{B} = \sum_{k=1}^{B} \frac{\Gamma(d_k + \mathbf{c}, \mathbf{d}_{[k]} - d_k)}{\Gamma(d_k + \mathbf{a}, \mathbf{b} - d_k)} {}_{A+B}F_{A+B-1}\left(\begin{matrix} d_k + \mathbf{c}, 1 + d_k - \mathbf{b} \\ d_k + \mathbf{a}, 1 + d_k - \mathbf{d}_{[k]} \end{matrix} ; 1 \right) \tag{1.44}$$

$$= \sum_{k=1}^{A} \frac{\Gamma(c_k + \mathbf{d}, \mathbf{c}_{[k]} - c_k)}{\Gamma(c_k + \mathbf{b}, \mathbf{a} - c_k)} {}_{A+B}F_{A+B-1}\left(\begin{matrix} c_k + \mathbf{d}, 1 + c_k - \mathbf{a} \\ c_k + \mathbf{b}, 1 + c_k - \mathbf{c}_{[k]} \end{matrix} ; 1 \right), \tag{1.45}$$

where $\Re(\Sigma a_j + \Sigma b_j - \Sigma c_j - \Sigma d_j - 1) > 0$, *so that the generalized hypergeometric series are convergent.*

Proof. Starting with Corollary 1.7 and taking the limit $q \to 1^-$ using (1.39) completes the proof. □

If one can write a basic hypergeometric function with a specific argument as a symmetric sum of two nonterminating basic hypergeometric functions with argument q, then there is the following useful consequence of Theorem 1.5.

Theorem 1.9. *Let* $q \in \mathbb{C}^\dagger$, $\mathbf{a} := \{a_1, \ldots, a_A\}$, $\mathbf{c} := \{c_1, \ldots, c_C\}$, *be multisets of non-zero complex numbers with cardinality* $A, C \in \mathbb{N}_0$ *(not both zero) respectively,* $\mathbf{d} := \{d_1, d_2\}$,

$c_k d_l \notin \Omega_q$, $\sigma \in (0,\infty)$, $d_1, d_2 \in \mathbb{C}^*$, such that $|c_k| < \sigma$, $|d_1|, |d_2| < 1/\sigma$, for any $c_k \in \mathbf{c}$. Define

$$H(\mathbf{a},\mathbf{c},\mathbf{d};q) := \sum^{d_1;d_2} \frac{(d_1\mathbf{a};q)_\infty}{\left(\frac{d_2}{d_1}, d_1\mathbf{c};q\right)_\infty} {}_C\phi_{A+1}^{C-A-2}\left(\begin{matrix} d_1\mathbf{c} \\ d_1\mathbf{a}, qd_1/d_2 \end{matrix};q,q\right) \tag{1.46}$$

$$= \frac{(d_1\mathbf{a};q)_\infty}{\left(\frac{d_2}{d_1}, d_1\mathbf{c};q\right)_\infty} {}_C\phi_{A+1}^{C-A-2}\left(\begin{matrix} d_1\mathbf{c} \\ d_1\mathbf{a}, qd_1/d_2 \end{matrix};q,q\right)$$

$$+ \frac{(d_2\mathbf{a};q)_\infty}{\left(\frac{d_1}{d_2}, d_2\mathbf{c};q\right)_\infty} {}_C\phi_{A+1}^{C-A-2}\left(\begin{matrix} d_2\mathbf{c} \\ d_2\mathbf{a}, qd_2/d_1 \end{matrix};q,q\right), \tag{1.47}$$

where $d_l/d_{l'} \notin \Omega_q$, $l \neq l'$, and if $C \geq A+2$,

$$J(\mathbf{a},\mathbf{c},\mathbf{d};f,q) := \sum_{k=1}^{C} \frac{\vartheta(fc_kd_1, \frac{f}{c_kd_2};q)(\mathbf{a}/c_k;q)_\infty}{(c_k\mathbf{d}, \mathbf{c}_{[k]}/c_k;q)_\infty}$$

$$\times {}_{A+2}\phi_{C-1}\left(\begin{matrix} c_k\mathbf{d}, qc_k/\mathbf{a} \\ qc_k/\mathbf{c}_{[k]} \end{matrix};q, \frac{q(qc_k)^{C-A-2}a_1\cdots a_A}{d_1d_2c_1\cdots c_C}\right), \tag{1.48}$$

where $c_k/c_{k'} \notin \Omega_q$, $k \neq k'$, and ${}_{A+2}\phi_{C-1}$ is convergent for $C = A+2$ if $|qa_1\cdots a_A| < |d_1d_2c_1\cdots c_C|$, and is an entire function if $C > A+2$. Then adopting the multiset scalar product notation (1.2) (from now on when we adopt these notations their meaning should be clear) one has,

$$\int_{-\pi}^{\pi} \frac{((fd_1, \frac{q}{f}d_2)\frac{\sigma}{z}, (\frac{f}{d_2}, \frac{q}{fd_1}, \mathbf{a})\frac{z}{\sigma};q)_\infty}{\left((d_1,d_2)\frac{\sigma}{z}, \mathbf{c}\frac{z}{\sigma};q\right)_\infty} \mathrm{d}\psi = \frac{2\pi\vartheta(f, f\frac{d_1}{d_2};q)}{(q;q)_\infty} H(\mathbf{a},\mathbf{c},\mathbf{d};q) \tag{1.49}$$

$$= \frac{2\pi}{(q;q)_\infty} J(\mathbf{a},\mathbf{c},\mathbf{d};f,q), \quad (C \geq A+2), \tag{1.50}$$

where $z = \mathrm{e}^{i\psi}$, and none of the arguments of the modified theta functions are equal to some q^m, $m \in \mathbb{Z}$.

Proof. See proof of [6, Theorem 2.4]. □

Theorem 1.10. *Let $q \in \mathbb{C}^\dagger$, $\mathbf{a} := \{a_1,\ldots,a_A\}$, $\mathbf{c} := \{c_1,\ldots,c_C\}$, be multisets of non-zero complex numbers with cardinality $A, C \in \mathbb{N}_0$ (not both zero) respectively, $\mathbf{d} := \{d_1, d_2\}$, $c_k + d_l \notin -\mathbb{N}_0$, $|q|^\sigma \in (0,\infty)$, $q^{d_1}, q^{d_2} \in \mathbb{C}^*$, such that $|q^{c_k}| < |q|^\sigma$, $|q^{d_1}|, |q^{d_2}| < |q|^{-\sigma}$, for any $c_k \in \mathbf{c}$, and fractional powers take their principal values. Then*

$$\int_{\frac{\pi}{\log q}}^{-\frac{\pi}{\log q}} \frac{(1-q)^{(C-A-2)(ix-\sigma)}\Gamma_q(\mathbf{d}+\sigma-ix,\mathbf{c}-\sigma+ix)}{\Gamma_q((d_1+f,d_2+1-f)+\sigma-ix,(f-d_2,1-d_1-f,\mathbf{a})-\sigma+ix)}\,\mathrm{d}x$$
$$= \frac{2\pi(1-q)}{-\log(q)\Gamma_q(f,1-f,d_1-d_2+f,d_2-d_1+1-f)}$$
$$\times \mathop{\Big[\!\!\Big]}^{d_1;d_2} (1-q)^{d_1(C-A-2)}\frac{\Gamma_q(\mathbf{c}+d_1,d_2-d_1)}{\Gamma_q(\mathbf{a}+d_1)}\ {}_C\phi_{A+1}^{C-A-2}\left(\begin{matrix} q^{\mathbf{c}+d_1} \\ q^{\mathbf{a}+d_1},q^{1+d_1-d_2}\end{matrix};q,q\right) \tag{1.51}$$
$$= \frac{2\pi(1-q)}{-\log q}\sum_{k=1}^{C} \frac{(1-q)^{-c_k(C-A-2)}\Gamma_q(c_k+\mathbf{d},\mathbf{c}_{[k]}-c_k)}{\Gamma_q(\mathbf{a}-c_k,1-c_k-d_1-f,1+c_k+d_2-f,c_k+d_1+f,-c_k-d_2+f)}$$
$$\times\ {}_{A+2}\phi_{C-1}\left(\begin{matrix} q^{c_k+\mathbf{d}},q^{1-\mathbf{a}+c_k} \\ q^{1+c_k-\mathbf{c}_{[k]}}\end{matrix};q,q^{1+(C-A-2)(1+c_k)+\Sigma a_j-\Sigma c_j-d_1-d_2}\right). \tag{1.52}$$

Proof. Using (1.49), (1.50), we respectively start along the lines of Askey & Roy [3, p. 368] and use the map $(\mathbf{a},\mathbf{c},\mathbf{d},\sigma,\mathrm{e}^{i\psi})\mapsto(q^{\mathbf{a}},q^{\mathbf{c}},q^{\mathbf{d}},q^{\sigma},q^{ix})$ and the definition of the q-gamma function (1.13). This completes the proof. □

By assuming that $C=A+2$, then the problematic $(1-q)^{C-A-2}$ terms become unity. This produces the following result.

Theorem 1.11. *Let $q\in\mathbb{C}^\dagger$, $\mathbf{a}:=\{a_1,\ldots,a_A\}$, $\mathbf{c}:=\{c_1,\ldots,c_{A+2}\}$, be sets of non-zero complex numbers with cardinality $A\in\mathbb{N}_0$, $\mathbf{d}:=\{d_1,d_2\}$, $c_k+d_l\notin-\mathbb{N}_0$, $|q|^\sigma\in(0,\infty)$, $d_1,d_2\in\mathbb{C}^*$, such that $|q^{c_k}|<|q|^\sigma$, $|q^{d_1}|,|q^{d_2}|<|q|^{-\sigma}$, for any $c_k\in\mathbf{c}$, and fractional powers take their principal values. Then*

$$\int_{\frac{\pi}{\log q}}^{-\frac{\pi}{\log q}} \frac{\Gamma_q(\mathbf{d}+\sigma-ix,\mathbf{c}-\sigma+ix)}{\Gamma_q((d_1+f,d_2+1-f)+\sigma-ix,(f-d_2,1-d_1-f,\mathbf{a})-\sigma+ix)}\,\mathrm{d}x$$
$$= \frac{2\pi(1-q)}{-\log(q)\Gamma_q(f,1-f,d_1-d_2+f,d_2-d_1+1-f)}$$
$$\times \mathop{\Big[\!\!\Big]}^{d_1;d_2} \frac{\Gamma_q(\mathbf{c}+d_1,d_2-d_1)}{\Gamma_q(\mathbf{a}+d_1)}\ {}_{A+2}\phi_{A+1}\left(\begin{matrix} q^{\mathbf{c}+d_1} \\ q^{\mathbf{a}+d_1},q^{1+d_1-d_2}\end{matrix};q,q\right) \tag{1.53}$$
$$= \frac{2\pi(1-q)}{-\log q}\sum_{k=1}^{C} \frac{\Gamma_q(c_k+\mathbf{d},\mathbf{c}_{[k]}-c_k)}{\Gamma_q(\mathbf{a}-c_k,1-c_k-d_1-f,1+c_k+d_2-f,c_k+d_1+f,-c_k-d_2+f)}$$
$$\times\ {}_{A+2}\phi_{A+1}\left(\begin{matrix} q^{c_k+\mathbf{d}},q^{1-\mathbf{a}+c_k} \\ q^{1+c_k-\mathbf{c}_{[k]}}\end{matrix};q,q^{1+\Sigma a_j-\Sigma c_j-d_1-d_2}\right). \tag{1.54}$$

We now take the limit as $q\to1^-$ to obtain the following result.

Theorem 1.12. *Let* $\mathbf{a} := \{a_1, \dots, a_A\}$, $\mathbf{c} := \{c_1, \dots, c_{A+2}\}$, *be multisets of non-zero complex numbers with cardinality* $A \in \mathbb{N}_0$, $\mathbf{d} := \{d_1, d_2\}$, $c_k + d_l \notin -\mathbb{N}_0$, $\sigma \in (0, \infty)$, $d_1, d_2 \in \mathbb{C}^*$, *such that* $|c_k| < \sigma$, $|d_1|, |d_2| < 1/\sigma$, *for any* $c_k \in \mathbf{c}$, $\Re(\Sigma a_j - \Sigma c_j + 1 - d_1 - d_2) > 0$. *Then*

$$\int_{-\infty}^{\infty} \frac{\Gamma(\mathbf{d}+\sigma-ix, \mathbf{c}-\sigma+ix)\,\mathrm{d}x}{\Gamma((d_1+f, d_2+1-f)+\sigma-ix, (f-d_2, 1-d_1-f, \mathbf{a})-\sigma+ix)}$$

$$= \frac{2\pi}{\Gamma(f, 1-f, d_1-d_2+f, d_2-d_2+1-f)} \left[\!\!\left[\frac{\Gamma(\mathbf{c}+d_1, d_2-d_1)}{\Gamma(\mathbf{a}+d_1)} \right]\!\!\right]^{d_1;d_2}$$

$$\times {}_{A+2}F_{A+1}\left(\begin{matrix} \mathbf{c}+d_1 \\ \mathbf{a}+d_1, 1+d_1-d_2 \end{matrix}; 1 \right) \tag{1.55}$$

$$= 2\pi \sum_{k=1}^{C} \frac{\Gamma(c_k+\mathbf{d}, \mathbf{c}_{[k]}-c_k)}{\Gamma(\mathbf{a}-c_k, 1-c_k-d_1-f, 1+c_k+d_2-f, c_k+d_1+f, -c_k-d_2+f)}$$

$$\times {}_{A+2}F_{A+1}\left(\begin{matrix} c_k+\mathbf{d}, 1-\mathbf{a}+c_k \\ 1+c_k-\mathbf{c}_{[k]} \end{matrix}; 1 \right). \tag{1.56}$$

Proof. Start with Theorem 1.11 and letting $q \to 1^-$ produces (1.55), (1.56). The condition for convergence of generalized hypergeometric functions with argument unity is given by [15, (16.2.2)]. This completes the proof. □

Remark 1.13. As just indicated, it is often feasible to convert integrals of products of infinite q-shifted factorials to integrals of products of q-gamma functions. This makes a direct q-analogue with Mellin–Barnes integrals for the integrals in question. In some cases we have undertaken this recasting for the integrals which appear below. For instance in Corollary 4.2 we recast the integral of a well-poised ${}_3\phi_2$ in terms of an integral of products of terms given by Γ_q and Γ_{q^2}. In Theorem 6.1, we are able to write the integral of a very-well poised ${}_8W_7$ as an integral of products of terms given by Γ_q, and in this case a clear $q \to 1^-$ limit exists and is computed. Other cases such as Theorems 5.1, 8.1, 8.3 and 9.4 can also be written as products of terms involving Γ_q and Γ_{q^2}, and for Theorem 9.1 it can be recast similarly as above but also including terms of the form Γ_{q^3}. However, we will leave these recastings to the reader.

2. A q-Mellin–Barnes integral for a ratio of theta functions

If one would like to integrate a ratio of an arbitrary product of theta functions as a q-Mellin–Barnes integral then Theorem 1.5 provides a powerful tool to evaluate this integral which provides insight into the connection between theta functions and partial theta functions. This will be seen in the following theorem.

Theorem 2.1. *Let $q \in \mathbb{C}^\dagger$, $\sigma \in (0,\infty)$, $\mathbf{b} \in {\mathbb{C}^*}^B$, $\mathbf{d} \in {\mathbb{C}^*}^D$ with $D \geq B$ such that $|q|/\sigma < |d_k| < 1/\sigma$ for $k = 1, ..., D$, and $d_l/d_k \notin \Upsilon_q$ for any $d_l, d_k \in \mathbf{d}$ with $l \neq k$. Then*

$$\int_{-\pi}^{\pi} \frac{\vartheta(\mathbf{b}\frac{\sigma}{z};q)}{\vartheta(\mathbf{d}\frac{\sigma}{z};q)} \mathrm{e}^{im\psi}\,\mathrm{d}\psi = \frac{2\pi\sigma^m}{(q;q)_\infty} G_m\left(\frac{q}{\mathbf{b}}, \mathbf{b}, \frac{q}{\mathbf{d}}, \mathbf{d}; \sigma, q\right), \tag{2.1}$$

where $z = \mathrm{e}^{i\psi}$. If $D \geq B$ then

$$G_m = \frac{1}{(q;q)_\infty} \sum_{k=1}^{D} \frac{\vartheta(\mathbf{b}/d_k;q) d_k^m}{\vartheta(\mathbf{d}_{[k]}/d_k;q)}\ {}_1\phi_0^{D-B}\left(\begin{matrix} q \\ - \end{matrix}; q, q^m (qd_k)^{D-B} \frac{b_1\cdots b_B}{d_1\cdots d_D}\right), \tag{2.2}$$

which is an entire function and for $D = B$, there is a specialized sum using the geometric series (1.21), provided $|b_1 \cdots b_B| < |d_1 \cdots d_D|$

$$G_m = \frac{1}{(q;q)_\infty} \sum_{k=1}^{D} \frac{\vartheta(\mathbf{b}/d_k;q) d_k^m}{\vartheta(\mathbf{d}_{[k]}/d_k;q)} \frac{1}{1 - q^m \frac{b_1\cdots b_B}{d_1\cdots d_B}}. \tag{2.3}$$

Moreover, for $D > B$, one also has

$$G_m = \frac{(q^{D-B};q^{D-B})_\infty}{(q;q)_\infty} \sum_{k=1}^{D} \frac{\vartheta(\mathbf{b}/d_k;q) d_k^m}{\vartheta(\mathbf{d}_{[k]}/d_k;q)} \Theta\left(-\frac{q^m(-qd_k)^{D-B} b_1\cdots b_B}{d_1\cdots d_D}; q^{D-B}\right). \tag{2.4}$$

Proof. Start with the integral on the left-hand side of (2.1) and replace the theta function with its definition (1.23) in terms of infinite q-shifted factorials. Then we can easily identify the multisets $\mathbf{a} = q/\mathbf{b}$, $\mathbf{c} = q/\mathbf{d}$. Direct substitution of these multisets using Theorem 1.5 provides (2.1). The utilization of (1.34) with these multisets provides (2.2). Due to the symmetric nature of the arguments (1.35) yields the same expression in terms of nonterminating basic hypergeometric functions. The function which appears in the representation of the q-Mellin–Barnes integral of a ratio of theta functions

$$g_p(z;q) := {}_1\phi_0^{D-B}\left(\begin{matrix} q \\ - \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \left((-1)^n q^{\binom{n}{2}}\right)^{D-B} z^n$$

is connected to the partial theta function (see §1.1). The necessary relation is given by

$$g_p(z;q) = (q^{D-B};q^{D-B})_\infty \Theta((-1)^{D-B-1} z; q^{D-B}). \tag{2.5}$$

Inserting this relation in (2.2) yields (2.4). If $D = B$ then the nonterminating basic hypergeometric series can be evaluated using the geometric series (1.21). This provides the form of (2.3) which completes the proof. □

3. Symmetric representation of the Askey–Wilson moments

Define the Askey–Wilson weight function

$$w(x;\mathbf{a}|q) := \frac{(\mathrm{e}^{\pm 2i\theta};q)_\infty}{(\mathbf{a}\mathrm{e}^{\pm i\theta};q)_\infty} = \frac{(\pm\mathrm{e}^{\pm i\theta}, \pm q^{\frac12}\mathrm{e}^{\pm i\theta};q)_\infty}{(\mathbf{a}\mathrm{e}^{\pm i\theta};q)_\infty}, \tag{3.1}$$

where $\mathbf{a} := \{a,b,c,d\}$, $x = \cos\theta \in [-1,1]$, $a,b,c,d \notin \Omega_q$ and we have used the identity (1.10). Then, the moments of the Askey–Wilson polynomials are given by

$$\mu_n = \frac{(q,ab,\ldots,cd;q)_\infty}{4\pi(abcd;q)_\infty}\int_{-\pi}^{\pi} w(x;\mathbf{a}|q)\cos^n\theta\,\mathrm{d}\theta, \tag{3.2}$$

where $\{ab,\ldots,cd\} := \{ab,ac,ad,bc,bd,cd\}$, and μ_n has been normalized so that $\mu_0 = 1$ (see [12, p. 170]).

Due to the $z = \mathrm{e}^{i\theta}$ dependence of the second equality of the Askey–Wilson weight function (3.1) and a judicious use of the binomial theorem, the moments of the Askey–Wilson polynomials (3.2) are given by a q-Mellin–Barnes integral. Using the method of integral representations for nonterminating basic hypergeometric functions (see Theorem 1.5) we are able to obtain a form symmetric in the parameters a, b, c, d for the moments of the Askey–Wilson polynomials.

Theorem 3.1. *Let $n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $a,b,c,d \in \mathbb{C}^*$, $a,b,c,d \notin \Omega_q$. Then, the moments of the Askey–Wilson polynomials can be given by*

$$\mu_n = \frac{(ab,\ldots,cd;q)_\infty}{2^{n+1}(abcd;q)_\infty}\sum_{k=0}^{n}\binom{n}{k} \times \overset{a;b,c,d}{\text{Ⅱ}} \frac{(\frac{1}{a^2};q)_\infty a^{|n-2k|}}{(ab,ac,ad,\frac{b}{a},\frac{c}{a},\frac{d}{a};q)_\infty}\,{}_6W_5\left(a^2;ab,ac,ad;q,\frac{q^{1+|n-2k|}}{abcd}\right), \tag{3.3}$$

where $a/b, a/c, a/d, b/c, b/d, c/d \notin \Upsilon_q$.

Proof. Start with the left-hand side of (3.3) and take account of (3.2). Applying the binomial theorem [15, (1.2.2)] to the $\cos^n\theta$ produces

$$\cos^n\theta = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\mathrm{e}^{i\theta(n-2k)}. \tag{3.4}$$

Applying Theorem 1.5 with $m = n-2k \in \mathbb{N}_0$ with cardinalities $(A,B,C,D) = (4,4,4,4)$, given by

$$\mathbf{a}=\mathbf{b}:=\{\pm 1,\pm\sqrt{q}\}\,,\ \mathbf{c}=\mathbf{d}:=\{a,b,c,d\}\,, \tag{3.5}$$

produces the right-hand side of (3.3) by substituting the variables in (1.32). This completes the proof. □

Remark 3.2. In Kim & Stanton (2014) [12, Theorem 2.10], a representation for the Askey–Wilson moments which are symmetric in the parameters a,b,c,d is given. Let $t\in\mathbb{C}$. Then

$$\begin{aligned}\mu_n=&\sum_{k=0}^{n}(-q)^k\frac{(ta,tb,tc,td;q)_k}{(t^2,abcd;q)_k}\,{}_8W_7\left(\frac{t^2}{q};q^{-k},\frac{t}{a},\frac{t}{b},\frac{t}{c},\frac{t}{d};q,q^kabcd\right)\\&\times\sum_{s=0}^{n+1}\left(\binom{n}{s}-\binom{n}{s-1}\right)\sum_{p=0}^{n-2s-k}\begin{bmatrix}k+p\\k\end{bmatrix}_q\begin{bmatrix}n-2s-p\\k\end{bmatrix}_q q^{k(2s+p-n)+\binom{k}{2}}t^{2p+2s-n},\end{aligned} \tag{3.6}$$

where the q-binomial coefficient [13, (1.9.4)] is defined by

$$\begin{bmatrix}n\\k\end{bmatrix}_q:=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}. \tag{3.7}$$

It is interesting to note that both (3.6) and (3.3) are symmetric in the parameters a, b, c, d. Define $\mathbf{4}:=\{1,2,3,4\}$. The representation (3.6) is a finite sum over a lattice given by

$$\{0,\ldots,n\}\times\{0,\ldots,n+1\}\times\{0,\ldots,n-2s-k\}, \tag{3.8}$$

for each terminating very-well-poised ${}_8W_7$. On the other hand (3.3) is a finite sum over a rectangular lattice given by $\{0,\ldots,n\}\times\mathbf{4}$ for each nonterminating very-well-poised ${}_6W_5$.

Remark 3.3. It was pointed out by one of the referees that the original proof of the Askey–Wilson integral (see the Askey–Wilson Memoirs of the AMS article [4, (2.1)])

$$\int_0^{\pi}w(x;\mathbf{a}|q)\,\mathrm{d}\theta=\int_0^{\pi}\frac{(\mathrm{e}^{\pm 2i\theta};q)_\infty}{(\mathbf{a}\mathrm{e}^{\pm i\theta};q)_\infty}\,\mathrm{d}\theta=\frac{2\pi(abcd;q)_\infty}{(q,ab,\ldots,cd;q)_\infty}, \tag{3.9}$$

was accomplished by summing 4 very-well poised ${}_6W_5$'s and using an elliptic function identity. Furthermore, this should correspond to Theorem 3.1 for the constant $n=0$. The correspondence of our formula for the Askey–Wilson moments (Theorem 3.1) with their evaluation of the Askey–Wilson integral (for $n=0$ as a symmetric sum of four ${}_6W_5$'s) is made clear by the identity [6, Corollary 2.12] (where the elliptic function identity that Askey & Wilson refer to can be found), namely

$$\overset{a;b,c,d}{\int\!\!\!\!\!\int}\frac{(\frac{1}{a^2};q)_\infty\,{}_6W_5\big(a^2;ab,ac,ad;q,\frac{q}{abcd}\big)}{\big(ab,ac,ad,\frac{b}{a},\frac{c}{a},\frac{d}{a};q\big)_\infty}=\frac{2(abcd;q)_\infty}{(ab,\ldots,cd;q)_\infty},\tag{3.10}$$

where $|q|<|a_{1234}|<1$. As was indicated on [4, p. 11], the restriction $|q|<|abcd|$ can be removed by analytic continuation. In fact, one can see that the analytic continuation of the integral follows directly from (3.2) since the only restriction on the convergence of the integral is that $|q|,|a|,|b|,|c|,|d|<1$.

4. Three, five and six-term transformations for a nonterminating well-poised ${}_3\phi_2$

Let $q,z\in\mathbb{C}^\dagger$, $\tau\in(0,\infty)$, $a,b,c,h\in\mathbb{C}^*$, $\frac{qa}{b},\frac{qa}{c}\notin\Omega_q$, $h,h\frac{qa}{bcz}\notin\Upsilon_q$. In [6, Corollary 2.15], we presented an integral for a nonterminating well-poised ${}_3\phi_2$, namely

$$\begin{aligned}{}_3\phi_2\left(\begin{matrix}a,b,c\\ \frac{qa}{b},\frac{qa}{c}\end{matrix};q,z\right)&=\frac{(q,a,\frac{qa}{bc};q)_\infty}{2\pi\vartheta(h,h\frac{qa}{bcz};q)(\frac{qa}{b},\frac{qa}{c};q)_\infty}\\&\times\int_{-\pi}^{\pi}\frac{((\frac{1}{h}\sqrt{\frac{bcz}{a}},h\sqrt{\frac{a}{bcz}})\frac{\tau}{w},(qh\sqrt{\frac{a}{bcz}},\frac{q}{h}\sqrt{\frac{bcz}{a}},q\sqrt{\frac{abz}{c}},q\sqrt{\frac{acz}{b}},\frac{(bcz)^{\frac{3}{2}}}{q\sqrt{a}})\frac{w}{\tau};q)_\infty}{((\sqrt{\frac{a}{bcz}},\sqrt{\frac{bcz}{q^2a}})\frac{\tau}{w},(\pm\sqrt{bcz},\pm\sqrt{qbcz},q\sqrt{\frac{az}{bc}})\frac{w}{\tau};q)_\infty}\mathrm{d}\eta,\end{aligned}\tag{4.1}$$

where $w=\mathrm{e}^{i\eta}$ and the maximum modulus of the denominator factors in the integrand is less than unity. This integral (4.1) followed from the following transformation of a nonterminating well-poised ${}_3\phi_2$ with arbitrary argument $z\in\mathbb{C}^\dagger$, in terms of a sum of two nonterminating ${}_5\phi_4(q,q)$ basic hypergeometric series, presented in cf. [9, (III.35)]

$$\begin{aligned}{}_3\phi_2\left(\begin{matrix}a,b,c\\ \frac{qa}{b},\frac{qa}{c}\end{matrix};q,z\right)&=\frac{(\frac{bcz}{q};q)_\infty}{(\frac{bcz}{qa};q)_\infty}\,{}_5\phi_4\left(\begin{matrix}\pm\sqrt{a},\pm\sqrt{qa},\frac{qa}{bc}\\ \frac{qa}{b},\frac{qa}{c},\frac{bcz}{q},\frac{q^2a}{bcz}\end{matrix};q,q\right)\\&+\frac{(a,bz,cz,\frac{qa}{bc};q)_\infty}{(\frac{qa}{b},\frac{qa}{c},z,\frac{qa}{bcz};q)_\infty}\,{}_5\phi_4\left(\begin{matrix}\pm\frac{bcz}{q\sqrt{a}},\pm\frac{bcz}{\sqrt{qa}},z\\ bz,cz,\frac{bcz}{a},\frac{b^2c^2z^2}{q^2a}\end{matrix};q,q\right).\end{aligned}\tag{4.2}$$

Remark 4.1. In order to simplify the constraints for the nonterminating infinite q-shifted factorials, modified theta functions and nonterminating basic hypergeometric series expressions which we will present below, we will avoid adding the constraints which must occur in order to prevent vanishing denominator factors which are not defined. For example, in (4.1) one must require the constraints

$$\frac{qa}{b},\frac{qa}{c}\notin\Omega_q,\quad h,h\frac{qa}{bcz}\notin\Upsilon_q,$$

and in (4.2) one must require the constraints

$$z,bz,cz,\frac{qa}{b},\frac{qa}{c},\frac{bcz}{q},\frac{b^2c^2z^2}{q^2a}\notin\Omega_q,\quad\frac{bcz}{a}\notin\Upsilon_q.$$

Since it is obvious and sometimes tedious to know for which values this happens, we will avoid inserting such constraints in the results below.

Corollary 4.2. *Let $q \in \mathbb{C}^\dagger$, $a, b, c, z \in \mathbb{C}^*$, $|q^z| < 1$. Then*

$$
\begin{aligned}
{}_3\phi_2\left(\begin{matrix} q^a, q^b \\ q^c \end{matrix}; q, q^z\right) &= \frac{-\log(q)(1+q)^{b+c+z-2\tau-\frac{3}{2}}}{2\pi(-q, \pm q; q)_\infty (1-q)^{b+c+z-2\tau+\frac{3}{2}}} \\
&\times \frac{\Gamma_q(f, 1-f, 1+f+a-b-c-z, 1+b+c+z-f-1-a, 1+a-b, 1+a-c)}{\Gamma_q(a, 1+a-b-c)} \\
&\times \int_{\frac{\pi}{\log q}}^{-\frac{\pi}{\log q}} \left(\frac{1+q}{1-q}\right)^{2ix} \frac{\Gamma_{q^2}((\frac{b+c+z}{2}, \frac{b+c+z+1}{2})+ix-\tau)\Gamma_q(\frac{a-b-c-z}{2}+\tau-ix)}{\Gamma_q((\frac{b+c+z-a}{2}-f, f+\frac{a-b-c-z}{2})+\tau-ix, 1+f+\frac{a-b-c-z}{2}+ix-\tau)} \\
&\times \frac{\Gamma_q(\frac{b+c+z-a}{2}-1+\tau-ix, 1+\frac{a+z-b-c}{2}+ix-\tau)}{\Gamma_q((1-f+\frac{b+c+z-a}{2}, 1+\frac{a+c+z-b}{2}, 1+\frac{a+b+z-c}{2}, \frac{3(b+c+z)}{2}-\frac{a}{2}-1)+ix-\tau)}\, \mathrm{d}x. \qquad (4.3)
\end{aligned}
$$

Proof. Start with (4.1) let $(a, b, c, z) \mapsto (q^a, q^b, q^c, q^z)$, and use the definition of the q-gamma function (1.13), and also (1.11) to convert the $(\pm a; q)_\infty$ terms. This completes the proof. □

One can now use Theorem 1.9 and (4.1) to derive a six-term representation of a nonterminating very-well-poised ${}_3\phi_2$ with arbitrary argument.

Theorem 4.3. *Let $q, z \in \mathbb{C}^\dagger$, $a, b, c, h \in \mathbb{C}^*$, and we assume there are no vanishing denominator factors (see Remark 4.1), e.g., $qa/b, qa/c \notin \Omega_q$, and $h, ha/(bcz) \notin \Upsilon_q$. Then*

$$
\begin{aligned}
&{}_3\phi_2\left(\begin{matrix} a, b, c \\ \frac{qa}{b}, \frac{qa}{c} \end{matrix}; q, z\right) \\
&= \frac{\vartheta(hz^{-1}, h\frac{qa}{bc}; q)(a, b, c, \frac{b^2c^2z^2}{q^2a}; q)_\infty}{\vartheta(h, h\frac{qa}{bcz}; q)(\frac{qa}{b}, \frac{qa}{c}, \frac{b^2c^2}{q^2a}, z; q)_\infty} {}_5\phi_4\left(\begin{matrix} \frac{q}{b}, \frac{q}{c}, \frac{qa}{bc}, z, \frac{q^3a}{b^2c^2z} \\ \pm\frac{\sqrt{q^3a}}{bc}, \pm\frac{q^2\sqrt{a}}{bc} \end{matrix}; q, q\right) \\
&\quad + \frac{(a, \frac{qa}{bc}; q)_\infty}{2\vartheta(h, h\frac{qa}{bcz}; q)(\frac{qa}{b}, \frac{qa}{c}; q)_\infty} \\
&\quad \times \coprod_{\pm\sqrt{a}} \left\{ \frac{\vartheta(h\sqrt{a}, h\frac{q\sqrt{a}}{bcz}; q)(\frac{q\sqrt{a}}{b}, \frac{q\sqrt{a}}{c}, \frac{bcz}{q\sqrt{a}}; q)_\infty}{(\sqrt{a}, \frac{q\sqrt{a}}{bc}, \frac{bcz}{q\sqrt{a}}; q)_\infty} {}_5\phi_4\left(\begin{matrix} \sqrt{a}, \frac{b}{\sqrt{a}}, \frac{c}{\sqrt{a}}, \frac{bcz}{q\sqrt{a}}, \frac{q^2\sqrt{a}}{bcz} \\ -q, \pm\sqrt{q}, \frac{bc}{\sqrt{a}} \end{matrix}; q, q\right) \right. \\
&\quad \left. - \frac{\vartheta(h\sqrt{qa}, h\frac{\sqrt{qa}}{bcz}; q)(\frac{\sqrt{qa}}{b}, \frac{\sqrt{qa}}{c}, \frac{bcz}{\sqrt{q^3a}}; q)_\infty}{(\sqrt{qa}, \frac{\sqrt{qa}}{bc}, \frac{bcz}{\sqrt{qa}}; q)_\infty} {}_5\phi_4\left(\begin{matrix} \sqrt{qa}, \frac{\sqrt{q}b}{\sqrt{a}}, \frac{\sqrt{q}c}{\sqrt{a}}, \frac{bcz}{\sqrt{qa}}, \frac{\sqrt{q^5a}}{bcz} \\ -q, \pm q^{\frac{3}{2}}, \sqrt{q}\frac{bc}{\sqrt{a}} \end{matrix}; q, q\right) \right\}.
\end{aligned}
\qquad (4.4)
$$

Proof. In the integrand of (4.1), the multisets of parameters $\{\mathbf{a}, \mathbf{c}, \mathbf{d}\}$ with cardinalities $(A, C, D) = (3, 5, 2)$ are given by

$$\mathbf{a} := \left\{ q\sqrt{\frac{abz}{c}}, q\sqrt{\frac{acz}{b}}, \sqrt{\frac{b^3c^3z^3}{qa}} \right\}, \ \mathbf{c} := \left\{ \pm\sqrt{bcz}, \pm\sqrt{qbcz}, q\sqrt{\frac{az}{bc}} \right\}, \tag{4.5}$$

$$\mathbf{d} := \left\{ \sqrt{\frac{a}{bcz}}, \sqrt{\frac{bcz}{q^2a}} \right\}. \tag{4.6}$$

Now we use (1.50) with the multisets of parameters given in (4.5), (4.6) using (1.49). This completes the proof. □

By making a judicious choice for $h = q^n z$, then since $\vartheta(q^n; q) = 0$ for all $n \in \mathbb{Z}$, the six-term transformation reduces to a five-term transformation.

Theorem 4.4. *Let $q, z \in \mathbb{C}^\dagger$, $a, b, c \in \mathbb{C}^*$, and we assume there are no vanishing denominator factors (see Remark 4.1). Then*

$$\begin{aligned}
{}_3\phi_2\left(\begin{matrix} a, b, c \\ \frac{qa}{b}, \frac{qa}{b} \end{matrix}; q, z \right) &= \frac{(a; q)_\infty}{2\vartheta(z; q)(\frac{qa}{b}, \frac{qa}{c}, \frac{bc}{a}; q)_\infty} \\
&\times \coprod^{\pm\sqrt{a}} \Bigg\{ \frac{\vartheta(\sqrt{a}z; q)(\frac{q\sqrt{a}}{b}, \frac{q\sqrt{a}}{c}, \frac{bc}{\sqrt{a}}; q)_\infty}{(\sqrt{a}; q)_\infty} {}_5\phi_4\left(\begin{matrix} \sqrt{a}, \frac{b}{\sqrt{a}}, \frac{c}{\sqrt{a}}, \frac{bcz}{q\sqrt{a}}, \frac{q^2\sqrt{a}}{bcz} \\ -q, \pm\sqrt{q}, \frac{bc}{\sqrt{a}} \end{matrix}; q, q \right) \\
&- \frac{\vartheta(\sqrt{qa}z; q)(\frac{\sqrt{qa}}{b}, \frac{\sqrt{qa}}{c}, \frac{\sqrt{q}\,bc}{\sqrt{a}}, \frac{bcz}{\sqrt{q^3a}}; q)_\infty}{(\sqrt{qa}, \frac{bcz}{\sqrt{qa}}; q)_\infty} {}_5\phi_4\left(\begin{matrix} \sqrt{qa}, \frac{\sqrt{q}b}{\sqrt{a}}, \frac{\sqrt{q}c}{\sqrt{a}}, \frac{bcz}{\sqrt{qa}}, \frac{\sqrt{q^5a}}{bcz} \\ -q, \pm q^{\frac{3}{2}}, \sqrt{q}\frac{bc}{\sqrt{a}} \end{matrix}; q, q \right) \Bigg\}.
\end{aligned} \tag{4.7}$$

Proof. Choose $h = q^n z$ in Theorem 4.3. Then since $\vartheta(q^n; q) = 0$ for all $n \in \mathbb{Z}$, the six-sformation reduces to a five-term transformation. Then replacing the infinite q-shifted factorials with arguments involving q^n and q^{-n} using (1.9), (1.12), the factors involving n all cancel, which completes the proof. □

Similarly, if one chooses a $h = q^n \frac{bc}{qa}$ then the six-term transformation reduces to a five-formation.

Theorem 4.5. *Let $q, z \in \mathbb{C}^\dagger$, $a, b, c \in \mathbb{C}^*$, and we assume there are no vanishing denominator factors (see Remark 4.1). Then*

$${}_3\phi_2\left(\begin{matrix} a, b, c \\ \frac{qa}{b}, \frac{qa}{b} \end{matrix}; q, z \right) = \frac{(a, \frac{qa}{bc}; q)_\infty}{2\vartheta(z^{-1}, \frac{bc}{qa}; q)(\frac{qa}{b}, \frac{qa}{c}; q)_\infty}$$

$$\times \coprod^{\pm\sqrt{a}} \Bigg\{ \frac{\vartheta(\frac{bc}{q\sqrt{a}}, \frac{1}{\sqrt{az}}; q)(\frac{q\sqrt{a}}{b}, \frac{q\sqrt{a}}{c}; q)_\infty}{(\sqrt{a}, \frac{q\sqrt{a}}{bc}; q)_\infty} {}_5\phi_4\left(\begin{matrix} \sqrt{a}, \frac{b}{\sqrt{a}}, \frac{c}{\sqrt{a}}, \frac{bcz}{q\sqrt{a}}, \frac{q^2\sqrt{a}}{bcz} \\ -q, \pm\sqrt{q}, \frac{bc}{\sqrt{a}} \end{matrix}; q, q\right)$$

$$- \frac{\vartheta(\frac{bc}{\sqrt{qa}}, \frac{1}{\sqrt{qaz}}; q)(\frac{\sqrt{qa}}{b}, \frac{\sqrt{qa}}{c}, \frac{bcz}{\sqrt{q^3 a}}; q)_\infty}{(\sqrt{qa}, \frac{\sqrt{qa}}{bc}, \frac{bcz}{\sqrt{qa}}; q)_\infty} {}_5\phi_4\left(\begin{matrix} \sqrt{qa}, \frac{\sqrt{q}\, b}{\sqrt{a}}, \frac{\sqrt{q}\, c}{\sqrt{a}}, \frac{bcz}{\sqrt{qa}}, \frac{\sqrt{q^5 a}}{bcz} \\ -q, \pm q^{\frac{3}{2}}, \frac{\sqrt{q}bc}{\sqrt{a}} \end{matrix}; q, q\right) \Bigg\}. \tag{4.8}$$

Proof. Choose $h = q^n \frac{bc}{qa}$ in Theorem 4.3. Then since $\vartheta(q^n; q) = 0$ for all $n \in \mathbb{Z}$, the six-term transformation reduces to a five-term transformation. Then replacing the infinite q-shifted factorials with arguments involving q^n and q^{-n} using (1.9), (1.12), the factors involving n all cancel, which completes the proof. □

5. Three, five and six-term transformations for nonterminating very-well-poised ${}_5W_4$

In this section we present a q-Mellin–Barnes integral for a nonterminating very-well-poised ${}_5W_4$.

Theorem 5.1. *Let $q, z \in \mathbb{C}^\dagger$, $\tau \in (0, \infty)$, $a, b, c, h \in \mathbb{C}^*$, $h\sqrt{\frac{qa}{bcz}} \notin \Upsilon_q$, and we assume there are no vanishing denominator factors (see Remark 4.1) and the maximum modulus of the denominator factors in the integrand is less than unity. Then*

$${}_5W_4(a; b, c; q, z) = \frac{(q, qa, \frac{qa}{bc}, \frac{b^2c^2z^2}{qa}; q)_\infty}{2\pi\vartheta(h, h\frac{qa}{bcz}; q)(\frac{qa}{b}, \frac{qa}{c}, \frac{b^2c^2z^2}{a}; q)_\infty}$$

$$\times \int_{-\pi}^{\pi} \frac{((\frac{q}{h}\sqrt{\frac{bcz}{qa}}, h\sqrt{\frac{qa}{bcz}})\frac{\tau}{w}, (h\sqrt{\frac{qa}{bcz}}, \frac{q}{h}\sqrt{\frac{bcz}{qa}}, \sqrt{\frac{qacz}{b}}, \sqrt{\frac{qabz}{c}}, \frac{\sqrt{q}(bcz)^{\frac{3}{2}}}{\sqrt{a}})\frac{w}{\tau}; q)_\infty}{((\sqrt{\frac{qa}{bcz}}, \sqrt{\frac{bcz}{qa}})\frac{\tau}{w}, (\pm\sqrt{bcz}, \pm\sqrt{qbcz}, \sqrt{\frac{qaz}{bc}})\frac{w}{\tau}; q)_\infty} \mathrm{d}\eta, \tag{5.1}$$

where $w = \mathrm{e}^{i\eta}$.

Proof. The integral for a nonterminating very-well-poised ${}_5W_4$ with arbitrary argument z (5.1) follows from the following transformation of a very-well-poised ${}_5W_4$ in terms of a sum of two nonterminating balanced ${}_5\phi_4(q, q)$ basic hypergeometric series [9, (3.4.4)]

$${}_5W_4(a; b, c; q, z) = \frac{(\frac{b^2c^2z^2}{qa}, qbcz; q)_\infty}{(\frac{b^2c^2z^2}{a}, \frac{bcz}{qa}; q)_\infty} {}_5\phi_4\left(\begin{matrix} \pm\sqrt{qa}, \pm q\sqrt{a}, \frac{qa}{bc} \\ \frac{qa}{b}, \frac{qa}{c}, qbcz, \frac{q^2 a}{bcz} \end{matrix}; q, q\right)$$

$$+ \frac{(qa, bz, cz, \frac{qa}{bc}; q)_\infty}{(\frac{qa}{b}, \frac{qa}{c}, z, \frac{qa}{bcz}; q)_\infty} {}_5\phi_4\left(\begin{matrix} \pm\frac{bcz}{\sqrt{qa}}, \pm\frac{bcz}{\sqrt{a}}, z \\ bz, cz, \frac{b^2c^2z^2}{a}, \frac{bcz}{a} \end{matrix}; q, q\right). \tag{5.2}$$

Now apply Theorem 1.9 with cardinalities $(A, C, D) = (3, 5, 2)$, given by

$$\mathbf{a} := \left\{\sqrt{\frac{qabz}{c}}, \sqrt{\frac{qacz}{b}}, \sqrt{\frac{qb^3c^3z^3}{a}}\right\}, \ \mathbf{c} := \left\{\pm\sqrt{bcz}, \pm\sqrt{qbcz}, \sqrt{\frac{qaz}{bc}}\right\}, \tag{5.3}$$

$$\mathbf{d} := \left\{\sqrt{\frac{qa}{bcz}}, \sqrt{\frac{bcz}{qa}}\right\}, \tag{5.4}$$

which generates the integral in (5.1) using (1.49). Clearly $h, h\frac{qa}{bcz} \notin \Omega_q$ since then one would have vanishing denominator factors which are not defined. Similarly one must avoid vanishing denominator factors for other infinite q-shifted factorials. Furthermore, the denominator factors in the integrand must have maximum modulus less than unity so that the integral converges. This completes the proof. $\square$

Now we apply (1.50) using the parameters **a**, **c**, **d** defined in (5.3), (5.4). Since $C = 5$, we generate a six-term transformation for the general nonterminating very-well-poised ${}_5W_4$. This is given in the following theorem.

Theorem 5.2. *Let $q, z \in \mathbb{C}^\dagger$, $a, b, c, h \in \mathbb{C}^*$, $w = e^{i\eta}$, and we assume there are no vanishing denominator factors (see Remark 4.1). Then, one has the following six-term transformation for a nonterminating very-well-poised ${}_5W_4$ with argument z:*

$$\begin{aligned}
{}_5W_4(a;b,c;q,z) &= \frac{(qa, \frac{b^2c^2z^2}{qa}, \frac{qa}{bc};q)_\infty}{\vartheta(h, h\frac{qa}{bcz};q)(\frac{b^2c^2z^2}{a}, \frac{qa}{b}, \frac{qa}{c};q)_\infty} \\
&\times\Bigg(\frac{\vartheta(h\sqrt{qa}, h\frac{\sqrt{qa}}{bcz};q)(\frac{\sqrt{qa}}{b}, \frac{\sqrt{qa}}{c}, bcz\sqrt{\frac{q}{a}};q)_\infty}{(-1, \pm\sqrt{q}, \sqrt{qa}, \frac{\sqrt{qa}}{bc}, \frac{bcz}{\sqrt{qa}};q)_\infty}\, {}_5\phi_4\left(\begin{matrix}\sqrt{qa}, \frac{qb}{\sqrt{qa}}, \frac{qc}{\sqrt{qa}}, \frac{bcz}{\sqrt{qa}}, \frac{\sqrt{qa}}{bcz} \\ -q, \pm\sqrt{q}, \frac{\sqrt{q}bc}{\sqrt{a}}\end{matrix};q,q\right) \\
&+\frac{\vartheta(-h\sqrt{qa}, -h\frac{\sqrt{qa}}{bcz};q)(-\frac{\sqrt{qa}}{b}, -\frac{\sqrt{qa}}{c}, -bcz\sqrt{\frac{q}{a}};q)_\infty}{(-1, \pm\sqrt{q}, -\sqrt{qa}, -\frac{\sqrt{qa}}{bc}, -\frac{bcz}{\sqrt{qa}};q)_\infty} \\
&\times{}_5\phi_4\left(\begin{matrix}-\sqrt{qa}, \frac{-\sqrt{q}b}{\sqrt{a}}, \frac{-\sqrt{q}c}{\sqrt{a}}, \frac{-bcz}{\sqrt{qa}}, \frac{-\sqrt{qa}}{bcz} \\ -q, \pm\sqrt{q}, -\frac{\sqrt{q}bc}{\sqrt{a}}\end{matrix};q,q\right) \\
&+\frac{\vartheta(hq\sqrt{a}, h\frac{\sqrt{a}}{bcz};q)(\frac{\sqrt{a}}{b}, \frac{\sqrt{a}}{c}, \frac{bcz}{\sqrt{a}};q)_\infty}{(-1, \pm\frac{1}{\sqrt{q}}, q\sqrt{a}, \frac{\sqrt{a}}{bc}, \frac{bcz}{\sqrt{a}};q)_\infty}\, {}_5\phi_4\left(\begin{matrix}q\sqrt{a}, \frac{qb}{\sqrt{a}}, \frac{qc}{\sqrt{a}}, \frac{bcz}{\sqrt{a}}, \frac{q\sqrt{a}}{bcz} \\ -q, \pm q^{\frac{3}{2}}, \frac{qbc}{\sqrt{a}}\end{matrix};q,q\right) \\
&+\frac{\vartheta(-hq\sqrt{a}, -h\frac{\sqrt{a}}{bcz};q)(\frac{-\sqrt{a}}{b}, -\frac{\sqrt{a}}{c}, -\frac{bcz}{\sqrt{a}};q)_\infty}{(-1, \pm\frac{1}{\sqrt{q}}, -q\sqrt{a}, -\frac{\sqrt{a}}{bc}, -\frac{bcz}{\sqrt{a}};q)_\infty} \\
&\times{}_5\phi_4\left(\begin{matrix}-q\sqrt{a}, -\frac{qb}{\sqrt{a}}, -\frac{qc}{\sqrt{a}}, -\frac{bcz}{\sqrt{a}}, -\frac{q\sqrt{a}}{bcz} \\ -q, \pm q^{\frac{3}{2}}, -\frac{qbc}{\sqrt{a}}\end{matrix};q,q\right) \\
&+\frac{\vartheta(\frac{h}{z}, h\frac{qa}{bc};q)(b, c, \frac{b^2c^2z}{a};q)_\infty}{(z, \frac{qa}{bc}, \frac{b^2c^2}{qa};q)_\infty}\, {}_5\phi_4\left(\begin{matrix}\frac{q}{b}, \frac{q}{c}, \frac{qa}{bc}, z, \frac{qa}{b^2c^2z} \\ \pm\frac{q\sqrt{a}}{bc}, \pm\frac{q^{\frac{3}{2}}\sqrt{a}}{bc}\end{matrix};q,q\right)\Bigg).
\end{aligned} \tag{5.5}$$

One can force the last term to vanish by setting for $n\in\mathbb{Z}$, $h=q^nz^{-1}$ or $h=q^{n-1}\frac{bc}{a}$, providing a naturally symmetric five-term transformation for a nonterminating very-well-poised ${}_5W_4$ with argument z.

Theorem 5.3. *Let $n\in\mathbb{Z}$, $q,z\in\mathbb{C}^\dagger$, $a,b,c\in\mathbb{C}^*$, $w=\mathrm{e}^{i\eta}$, and we assume there are no vanishing denominator factors (see Remark 4.1). Then, one has the following five-term transformations for a nonterminating very-well-poised ${}_5W_4$ with argument z:*

$$
\begin{aligned}
{}_5W_4(a;b,c;q,z) &= \frac{(qa,\frac{b^2c^2z^2}{qa},\frac{qa}{bc};q)_\infty}{2\vartheta(q^nz,q^{n+1}\frac{a}{bc};q)(-q,\frac{b^2c^2z^2}{a},\frac{qa}{b},\frac{qa}{c};q)_\infty}\\
&\times\Bigg(\frac{\vartheta(q^{n+\frac12}z\sqrt{a},\frac{q^{n+\frac12}\sqrt{a}}{bc};q)(\frac{\sqrt{qa}}{b},\frac{\sqrt{qa}}{c},bcz\sqrt{\frac{q}{a}};q)_\infty}{(\pm\sqrt{q},\sqrt{qa},\frac{\sqrt{qa}}{bc},\frac{bcz}{\sqrt{qa}};q)_\infty}\,{}_5\phi_4\left(\begin{matrix}\sqrt{qa},\frac{qb}{\sqrt{qa}},\frac{qc}{\sqrt{qa}},\frac{bcz}{\sqrt{qa}},\frac{\sqrt{qa}}{bcz}\\ -q,\pm\sqrt{q},\frac{\sqrt{q}bc}{\sqrt{a}}\end{matrix};q,q\right)\\
&+\frac{\vartheta(-q^{n+\frac12}z\sqrt{a},\frac{-q^{n+\frac12}\sqrt{a}}{bc};q)(\frac{-\sqrt{qa}}{b},\frac{-\sqrt{qa}}{c},-bcz\sqrt{\frac{q}{a}};q)_\infty}{(\pm\sqrt{q},-\sqrt{qa},-\frac{\sqrt{qa}}{bc},-\frac{bcz}{\sqrt{qa}};q)_\infty}\\
&\times{}_5\phi_4\left(\begin{matrix}-\sqrt{qa},\frac{-\sqrt{q}b}{\sqrt{a}},\frac{-\sqrt{q}c}{\sqrt{a}},\frac{-bcz}{\sqrt{qa}},\frac{-\sqrt{qa}}{bcz}\\ -q,\pm\sqrt{q},-\frac{\sqrt{q}bc}{\sqrt{a}}\end{matrix};q,q\right)\\
&+\frac{\vartheta(-q^{n+1}z\sqrt{a},-q^n\frac{\sqrt{a}}{bc};q)(\frac{-\sqrt{a}}{b},-\frac{\sqrt{a}}{c},-\frac{bcz}{\sqrt{a}};q)_\infty}{(\pm\frac{1}{\sqrt{q}},-q\sqrt{a},-\frac{\sqrt{a}}{bc},-\frac{bcz}{\sqrt{a}};q)_\infty}\\
&\times{}_5\phi_4\left(\begin{matrix}-q\sqrt{a},-\frac{qb}{\sqrt{a}},-\frac{qc}{\sqrt{a}},-\frac{bcz}{\sqrt{a}},-\frac{q\sqrt{a}}{bcz}\\ -q,\pm q^{\frac32},-\frac{qbc}{\sqrt{a}}\end{matrix};q,q\right)\\
&+\frac{\vartheta(q^{n+1}z\sqrt{a},q^n\frac{\sqrt{a}}{bc};q)(\frac{\sqrt{a}}{b},\frac{\sqrt{a}}{c},\frac{bcz}{\sqrt{a}};q)_\infty}{(\pm\frac{1}{\sqrt{q}},q\sqrt{a},\frac{\sqrt{a}}{bc},\frac{bcz}{\sqrt{a}};q)_\infty}\,{}_5\phi_4\left(\begin{matrix}q\sqrt{a},\frac{qb}{\sqrt{a}},\frac{qc}{\sqrt{a}},\frac{bcz}{\sqrt{a}},\frac{q\sqrt{a}}{bcz}\\ -q,\pm q^{\frac32},\frac{qbc}{\sqrt{a}}\end{matrix};q,q\right)\Bigg) \qquad (5.6)\\
&= \frac{(qa,\frac{b^2c^2z^2}{qa},\frac{qa}{bc};q)_\infty}{2\vartheta(q^{n-1}\frac{bc}{a},\frac{q^n}{z};q)(-q,\frac{b^2c^2z^2}{a},\frac{qa}{b},\frac{qa}{c};q)_\infty}\\
&\times\Bigg(\frac{\vartheta(q^{n-\frac12}\frac{bc}{\sqrt{a}},\frac{q^{n-\frac12}}{\sqrt{a}z};q)(\frac{\sqrt{qa}}{b},\frac{\sqrt{qa}}{c},bcz\sqrt{\frac{q}{a}};q)_\infty}{(\pm\sqrt{q},\sqrt{qa},\frac{\sqrt{qa}}{bc},\frac{bcz}{\sqrt{qa}};q)_\infty}\,{}_5\phi_4\left(\begin{matrix}\sqrt{qa},\frac{qb}{\sqrt{qa}},\frac{qc}{\sqrt{qa}},\frac{bcz}{\sqrt{qa}},\frac{\sqrt{qa}}{bcz}\\ -q,\pm\sqrt{q},\frac{\sqrt{q}bc}{\sqrt{a}}\end{matrix};q,q\right)\\
&+\frac{\vartheta(-q^{n-\frac12}\frac{bc}{\sqrt{a}},\frac{-q^{n-\frac12}}{\sqrt{a}z};q)(\frac{-\sqrt{qa}}{b},\frac{-\sqrt{qa}}{c},-bcz\sqrt{\frac{q}{a}};q)_\infty}{(\pm\sqrt{q},-\sqrt{qa},\frac{-\sqrt{qa}}{bc},\frac{-bcz}{\sqrt{qa}};q)_\infty}\\
&\times{}_5\phi_4\left(\begin{matrix}-\sqrt{qa},\frac{-\sqrt{q}b}{\sqrt{a}},\frac{-\sqrt{q}c}{\sqrt{a}},\frac{-bcz}{\sqrt{qa}},\frac{-\sqrt{qa}}{bcz}\\ -q,\pm\sqrt{q},-\frac{\sqrt{q}bc}{\sqrt{a}}\end{matrix};q,q\right)
\end{aligned}
$$

$$
+\frac{\vartheta(-q^n\frac{bc}{\sqrt{a}},-\frac{q^{n-1}}{\sqrt{a}z};q)(\frac{-\sqrt{a}}{b},-\frac{\sqrt{a}}{c},-\frac{bcz}{\sqrt{a}};q)_\infty}{(\pm\frac{1}{\sqrt{q}},-q\sqrt{a},-\frac{\sqrt{a}}{bc},-\frac{bcz}{\sqrt{a}};q)_\infty}
$$

$$
\times{}_5\phi_4\left(\begin{matrix}-q\sqrt{a},-\frac{qb}{\sqrt{a}},-\frac{qc}{\sqrt{a}},-\frac{bcz}{\sqrt{a}},-\frac{q\sqrt{a}}{bcz}\\ -q,\pm q^{\frac32},-\frac{qbc}{\sqrt{a}}\end{matrix};q,q\right)
$$

$$
+\frac{\vartheta(q^n\frac{bc}{\sqrt{a}},\frac{q^{n-1}}{\sqrt{a}z};q)(\frac{\sqrt{a}}{b},\frac{\sqrt{a}}{c},\frac{bcz}{\sqrt{a}};q)_\infty}{(\pm\frac{1}{\sqrt{q}},q\sqrt{a},\frac{\sqrt{a}}{bc},\frac{bcz}{\sqrt{a}};q)_\infty}\,{}_5\phi_4\left(\begin{matrix}q\sqrt{a},\frac{qb}{\sqrt{a}},\frac{qc}{\sqrt{a}},\frac{bcz}{\sqrt{a}},\frac{q\sqrt{a}}{bcz}\\ -q,\pm q^{\frac32},\frac{qbc}{\sqrt{a}}\end{matrix};q,q\right)\Bigg)\Bigg). \tag{5.7}
$$

Proof. Start by inserting $h\in\{q^nz,q^{n-1}\frac{bc}{a}\}$ respectively in Theorem 5.2. This forces the last term to vanish producing a five-term transformation for a nonterminating very-well-poised ${}_5W_4$ with arbitrary argument z. Note that we have used the identity $(-1;q)_\infty=2(-q;q)_\infty$. This completes the proof. □

Remark 5.4. Note that one can also choose $h\in\pm q^n\left\{\frac{1}{\sqrt{qa}},\frac{bcz}{\sqrt{a}},\frac{1}{q\sqrt{a}},\frac{bcz}{\sqrt{a}}\right\}$, in Theorem 5.2 with $n\in\mathbb{Z}$ and this will also produce four-term transformation formulas for nonterminating very-well-poised ${}_5W_4$. However, we leave the representation of these transformation formulas to the reader.

6. Three, four and five-term transformations for the nonterminating very-well-poised ${}_8W_7$

By starting with Bailey's transformation of a sum of two nonterminating balanced ${}_4\phi_3$ basic hypergeometric functions expressed as a very-well-poised ${}_8W_7$ we derive an integral representation for the nonterminating very-well-poised ${}_8W_7$.

Theorem 6.1. *Let $q\in\mathbb{C}^\dagger$, $a,b,c,d,e,f,h\in\mathbb{C}^*$, $\sigma\in(0,\infty)$, such that $|q^2a^2|<|bcdef|$, $h\sqrt{\frac{def}{qa}}\notin\Upsilon_q$. Then, one has the following integral representation for a nonterminating very-well-poised ${}_8W_7$:*

$$
{}_8W_7\left(a;b,c,d,e,f;q,\frac{q^2a^2}{bcdef}\right)=\frac{(q,qa,\frac{qa}{bc},\frac{qa}{de},\frac{qa}{df},\frac{qa}{ef},d,e,f;q)_\infty}{2\pi\vartheta(h,h\frac{def}{qa};q)(\frac{qa}{b},\frac{qa}{c},\frac{qa}{d},\frac{qa}{e},\frac{qa}{f};q)_\infty}
$$

$$
\times\int_{-\pi}^{\pi}\frac{((h\sqrt{\frac{def}{qa}},\frac{q}{h}\sqrt{\frac{qa}{def}})\frac{\sigma}{z},(h\sqrt{\frac{def}{qa}},\frac{q}{h}\sqrt{\frac{qa}{def}},\frac{(qa)^{\frac32}}{b\sqrt{def}},\frac{(qa)^{\frac32}}{c\sqrt{def}})\frac{z}{\sigma};q)_\infty}{((\sqrt{\frac{def}{qa}},\sqrt{\frac{qa}{def}})\frac{\sigma}{z},(\sqrt{\frac{qad}{ef}},\sqrt{\frac{qae}{df}},\sqrt{\frac{qaf}{de}},\frac{(qa)^{\frac32}}{bc\sqrt{def}})\frac{z}{\sigma};q)_\infty}\,\mathrm{d}\psi, \tag{6.1}
$$

where $z=\mathrm{e}^{i\psi}$ and the maximum modulus of the denominator factors in the integrand is less than unity and we assume there are no vanishing denominator factors (see Remark 4.1).

Proof. We start with Bailey's transformation of a nonterminating very-well-poised ${}_8W_7$ [15, (17.9.16)]

$$
\begin{aligned}
{}_8W_7\left(a;b,c,d,e,f;q,\frac{q^2a^2}{bcdef}\right) &= \frac{(\frac{qa}{de},\frac{qa}{df},\frac{qa}{ef},qa;q)_\infty}{(\frac{qa}{def},\frac{qa}{d},\frac{qa}{e},\frac{qa}{f};q)_\infty}\,{}_4\phi_3\left(\begin{matrix}\frac{qa}{bc},d,e,f\\ \frac{qa}{b},\frac{qa}{c},\frac{def}{a}\end{matrix};q,q\right)\\
&+\frac{(\frac{q^2a^2}{bdef},\frac{q^2a^2}{cdef},\frac{qa}{bc},qa,d,e,f;q)_\infty}{(\frac{q^2a^2}{bcdef},\frac{def}{qa},\frac{qa}{b},\frac{qa}{c},\frac{qa}{d},\frac{qa}{e},\frac{qa}{f};q)_\infty}\,{}_4\phi_3\left(\begin{matrix}\frac{q^2a^2}{bcdef},\frac{qa}{de},\frac{qa}{df},\frac{qa}{ef}\\ \frac{q^2a^2}{bdef},\frac{q^2a^2}{cdef},\frac{q^2a}{def}\end{matrix};q,q\right),
\end{aligned}
\tag{6.2}
$$

and applying Theorem 1.9 with

$$
\begin{aligned}
\mathbf{a} &:= \left\{\frac{(qa)^{\frac32}}{b\sqrt{def}},\frac{(qa)^{\frac32}}{c\sqrt{def}}\right\},\ \mathbf{c} := \left\{\sqrt{\frac{qad}{ef}},\sqrt{\frac{qae}{df}},\sqrt{\frac{qaf}{de}},\frac{(qa)^{\frac32}}{bc\sqrt{def}}\right\},\\
\mathbf{d} &:= \left\{\sqrt{\frac{def}{qa}},\sqrt{\frac{qa}{def}}\right\},
\end{aligned}
\tag{6.3}
$$

completes the proof. □

In the following, we will adopt a generalization of Bailey's original W notation for a nonterminating very-well-poised ${}_7F_6$ of argument unity (see for instance, [10, p. 2]). We define ${}_{r+1}\mathsf{W}_r(a;\mathbf{b})$, where $|\mathbf{b}| = r-1$ as follows:

$$
{}_{r+1}\mathsf{W}_r(a;\mathbf{b}) := {}_{r+1}F_r\left(\begin{matrix}a,\frac{a}{2}+1,b_1,\ldots,b_{r-1}\\ \frac{a}{2},1+a-b_1,\ldots,1+a-b_{r-1}\end{matrix};1\right),
\tag{6.4}
$$

which is absolutely convergent if [15, (16.2.2)]

$$
\Re(2a-(b_1+\cdots+b_{r-1})+2)>0.
\tag{6.5}
$$

Here the connection to Bailey's original W notation for a ${}_7F_6$ is given as follows

$$
\begin{aligned}
W(a;b,c,d,e,f) &:= {}_7\mathsf{W}_6(a;b,c,d,e,f)\\
&= {}_7F_6\left(\begin{matrix}a,\frac{a}{2}+1,b,c,d,e,f\\ \frac{a}{2},1+a-b,1+a-c,1+a-d,1+a-e,1+a-f\end{matrix};1\right).
\end{aligned}
\tag{6.6}
$$

Theorem 6.2. *Let $a,b,c,d,e,f\in\mathbb{C}$, $\sigma\in(0,\infty)$, $h\in\mathbb{C}\setminus\mathbb{Z}$, $d+e+f-a-1, 2+a-d-e-f, a-b+1, a-c+1, a-d+1, a-e+1, a-f+1 \notin -\mathbb{N}_0$. Then*

$$
\begin{aligned}
W(a;b,c,d,e,f) &= \frac{1}{2\pi}\Gamma(h,1-h,h+d+e+f-a-1,2+a-h-d-e-f)\\
&\times\frac{\Gamma(a-b+1,a-c+1,a-d+1,a-e+1,a-f+1)}{\Gamma(a+1,a-b-c+1,a-d-e+1,a-d-f+1,a-e-f+1,d,e,f)}
\end{aligned}
$$

$$\times \int_{-\infty}^{\infty} \frac{\Gamma(\frac{d+e+f-a-1}{2}+\sigma-ix, \frac{a+1-d-e-f}{2}+\sigma-ix, \frac{3}{2}(a+1)-b-c-\frac{d+e+f}{2}-\sigma+ix)}{\Gamma(h+\frac{d+e+f-a-1}{2}\mp\sigma\pm ix, 1-h+\frac{a+1-d-e-f}{2}\mp\sigma\pm ix)}$$
$$\times \frac{\Gamma(\frac{a+1+f-d-e}{2}-\sigma+ix, \frac{a+1+d-e-f}{2}-\sigma+ix, \frac{a+1+e-d-f}{2}-\sigma+ix)}{\Gamma(\frac{3}{2}(a+1)-b-\frac{d+e+f}{2}+ix-\sigma, \frac{3}{2}(a+1)-c-\frac{d+e+f}{2}+ix-\sigma)}\,\mathrm{d}x. \tag{6.7}$$

Proof. First use the map $(a,b,c,d,e,f,h,\mathrm{e}^{i\psi}) \mapsto (q^a,q^b,q^c,q^d,q^e,q^f,q^h,q^{ix})$ in (6.1). This converts the ${}_8W_7$ to

$${}_8W_7\left(q^a;q^b,q^c,q^d,q^e,q^f;q,q^{2a+2-b-c-d-e-f}\right), \tag{6.8}$$

which in the limit as $q\to 1^-$ becomes $W(a;b,c,d,e,f)$. On the right-hand side of (6.1), the infinite q-shifted factorials can be converted to q-gamma functions using [9, (I.35)]. Upon taking the limit as $q\to 1^-$ using This completes the proof. □

Now we present a theorem which gives a representation for a nonterminating very-well-poised ${}_8W_7$ given as a sum of four balanced ${}_4\phi_3(q)$'s.

Theorem 6.3. *Let $q\in\mathbb{C}^\dagger$, $a,b,c,d,e,f,h\in\mathbb{C}^*$ such that $|q^2a^2|<|bcdef|$, and we assume there are no vanishing denominator factors (see Remark 4.1). Then*

$${}_8W_7\left(a;b,c,d,e,f;q,\frac{q^2a^2}{bcdef}\right) = \frac{(qa;q)_\infty}{\vartheta(h,h\frac{def}{qa};q)(\frac{qa}{b},\frac{qa}{c},\frac{qa}{d},\frac{qa}{e},\frac{qa}{f};q)_\infty}$$
$$\times\left(\frac{\vartheta(h\frac{qa}{bc},h\frac{bcdef}{q^2a^2};q)(\frac{qa}{de},\frac{qa}{df},\frac{qa}{ef},b,c,d,e,f;q)_\infty}{(\frac{q^2a^2}{bcdef},\frac{bcd}{qa},\frac{bce}{qa},\frac{bcf}{qa};q)_\infty}\,{}_4\phi_3\left(\begin{matrix}\frac{q^2a^2}{bcdef},\frac{qa}{bc},\frac{q}{b},\frac{q}{c}\\ \frac{q^2a}{bcd},\frac{q^2a}{bce},\frac{q^2a}{bcf}\end{matrix};q,q\right)\right.$$
$$+\frac{\vartheta(hf,h\frac{de}{qa};q)(\frac{qa}{bc},\frac{qa}{bf},\frac{qa}{cf},\frac{qa}{df},\frac{qa}{ef},d,e;q)_\infty}{(\frac{qa}{bcf},\frac{d}{f},\frac{e}{f};q)_\infty}\,{}_4\phi_3\left(\begin{matrix}\frac{qa}{de},\frac{bf}{a},\frac{cf}{a},f\\ \frac{bcf}{a},\frac{qf}{d},\frac{qf}{e}\end{matrix};q,q\right)$$
$$+\frac{\vartheta(hd,h\frac{ef}{qa};q)(\frac{qa}{bc},\frac{qa}{bd},\frac{qa}{cd},\frac{qa}{de},\frac{qa}{df},e,f;q)_\infty}{(\frac{qa}{bcd},\frac{e}{d},\frac{f}{d};q)_\infty}\,{}_4\phi_3\left(\begin{matrix}\frac{qa}{ef},\frac{bd}{a},\frac{cd}{a},d\\ \frac{bcd}{a},\frac{qd}{e},\frac{qd}{f}\end{matrix};q,q\right)$$
$$\left.+\frac{\vartheta(he,h\frac{df}{qa};q)(\frac{qa}{bc},\frac{qa}{be},\frac{qa}{ce},\frac{qa}{de},\frac{qa}{ef},d,f;q)_\infty}{(\frac{qa}{bce},\frac{d}{e},\frac{f}{e};q)_\infty}\,{}_4\phi_3\left(\begin{matrix}\frac{qa}{df},\frac{be}{a},\frac{ce}{a},e\\ \frac{bce}{a},\frac{qe}{d},\frac{qe}{f}\end{matrix};q,q\right)\right). \tag{6.9}$$

Proof. Applying **a**, **c** and **d** in (6.3) to (1.50) produces the result. □

Remark 6.4. If one takes either d,e,f equal to q^{-n} for some $n\in\mathbb{N}_0$ then (6.9) becomes Watson's q-analogue of Whipple's theorem [15, (17.9.15)].

Remark 6.5. The authors failed to obtain Bailey's transformation of a nonterminating very-well-poised ${}_8W_7$ as a sum of two balanced ${}_4\phi_3$'s [15, (17.9.16)] as a limit case of (6.9).

Furthermore, they were neither unable to express (6.9) as a sum of two nonterminating very-well-poised ${}_8W_7$'s (using [9, (III.37)]).

As the parameter h is free, one may choose $h = q^n bc/(qa)$ or $h = q^{n+2}a^2/(bcdef)$ in (6.9). Then the first term in (6.9) vanishes and they are left with a symmetric sum of three ${}_4\phi_3$'s as a representation of a nonterminating very-well-poised ${}_8W_7$, namely the following theorem.

Remark 6.6. One might also consider the limit of (6.9) as $h \to \infty$. However, this limit produces a multiplicative elliptic function in proportion, which is doubly periodic on the entire complex plane. Therefore this limit does not exist.

Theorem 6.7. *Let $n \in \mathbb{Z}$, $q \in \mathbb{C}^\dagger$, $a, b, c, d, e, f \in \mathbb{C}^*$ such that $|q^2a^2| < |bcdef|$, and we assume there are no vanishing denominator factors (see Remark 4.1). Then*

$$
{}_8W_7\left(a; b, c, d, e, f; q, \frac{q^2a^2}{bcdef}\right) = \frac{(qa, \frac{qa}{bc}; q)_\infty}{\vartheta(\frac{q^{n-1}bc}{a}, \frac{q^{n-2}bcdef}{a^2}; q)(\frac{qa}{b}, \frac{qa}{c}, \frac{qa}{d}, \frac{qa}{e}, \frac{qa}{f}; q)_\infty}
$$
$$
\times \sum^{d;e,f} \frac{\vartheta(\frac{q^{n-1}bcd}{a}, \frac{q^{n-2}bcef}{a^2}; q)(\frac{qa}{bd}, \frac{qa}{cd}, \frac{qa}{de}, \frac{qa}{df}, e, f; q)_\infty}{(\frac{qa}{bcd}, \frac{e}{d}, \frac{f}{d}; q)_\infty} {}_4\phi_3\left(\begin{matrix} \frac{qa}{ef}, \frac{bd}{a}, \frac{cd}{a}, d \\ \frac{bcd}{a}, \frac{qd}{e}, \frac{qd}{f} \end{matrix}; q, q\right). \quad (6.10)
$$

Proof. Choose $h = q^{n-1}bc/a$ in (6.9). This causes the first term to vanish and one is left with a symmetric sum of three nonterminating balanced ${}_4\phi_3$'s as a representation of the nonterminating very-well-poised ${}_8W_7$, namely the following theorem. □

Alternatively one could have chosen $h = q^2a^2/(bcdef)$ in (6.9). This produces the following result.

Theorem 6.8. *Let $n \in \mathbb{Z}$, $q \in \mathbb{C}^\dagger$, $a, b, c, d, e, f \in \mathbb{C}^*$ such that $|q^2a^2| < |bcdef|$, and we assume there are no vanishing denominator factors (see Remark 4.1). Then*

$$
{}_8W_7\left(a; b, c, d, e, f; q, \frac{q^2a^2}{bcdef}\right) = \frac{(qa, \frac{qa}{bc}; q)_\infty}{\vartheta(\frac{q^{n+2}a^2}{bcdef}, \frac{q^{n+1}a}{bc}; q)(\frac{qa}{b}, \frac{qa}{c}, \frac{qa}{d}, \frac{qa}{e}, \frac{qa}{f}; q)_\infty}
$$
$$
\times \sum^{d;e,f} \frac{\vartheta(\frac{q^{n+2}a^2}{bcef}, \frac{q^{n+1}a}{bcd}; q)(\frac{qa}{bd}, \frac{qa}{cd}, \frac{qa}{de}, \frac{qa}{df}, e, f; q)_\infty}{(\frac{qa}{bcd}, \frac{e}{d}, \frac{f}{d}; q)_\infty} {}_4\phi_3\left(\begin{matrix} \frac{qa}{ef}, \frac{bd}{a}, \frac{cd}{a}, d \\ \frac{bcd}{a}, \frac{qd}{e}, \frac{qd}{f} \end{matrix}; q, q\right). \quad (6.11)
$$

Proof. Replace $h = q^{n+2}a^2/(bcdef)$ in (6.9) and simplifying completes the proof. □

Remark 6.9. Note that one can also choose for $n \in \mathbb{Z}$, $h \in q^n \left\{\frac{1}{d}, \frac{1}{e}, \frac{1}{f}, \frac{qa}{de}, \frac{qa}{df}, \frac{qa}{ef}\right\}$, and then the five-term representation of the nonterminating very-well-poised ${}_8W_7$ in (6.9) reduces to a four-term transformation. However, these representations are not symmetric and we leave their depictions to the reader.

7. Summation and transformation formulas for nonterminating balanced very-well-poised ${}_8W_7$

By starting with Bailey's three-term transformation formula for a nonterminating very-well-poised ${}_8W_7$, we are able to prove a generalized q-beta integral which will be useful to generate further transformation and summation formulas in the special case where the nonterminating very-well-poised ${}_8W_7$ are also balanced.

Theorem 7.1. *Let $q \in \mathbb{C}^\dagger$, $\sigma \in (0,\infty)$, $a,b,c,d,e,h \in \mathbb{C}^*$, $h, h\sqrt{\frac{a}{b}} \notin \Upsilon_q$. Then, one has the following q-Mellin–Barnes integral:*

$$\int_{-\pi}^{\pi} \frac{((h\sqrt{\frac{a}{b}}, \frac{q}{h}\sqrt{\frac{b}{a}})\frac{\sigma}{z}, (h\sqrt{\frac{a}{b}}, \frac{q}{h}\sqrt{\frac{b}{a}}, \pm\sqrt{b}, \frac{q}{c}\sqrt{ab}, \frac{q}{d}\sqrt{ab}, \frac{q}{e}\sqrt{ab}, (\frac{b}{a})^{\frac{3}{2}} cde)\frac{z}{\sigma}; q)_\infty}{((\sqrt{\frac{a}{b}}, \sqrt{\frac{b}{a}})\frac{\sigma}{z}, (\pm q\sqrt{b}, \sqrt{ab}, c\sqrt{\frac{b}{a}}, d\sqrt{\frac{b}{a}}, e\sqrt{\frac{b}{a}}, \frac{b^{\frac{3}{2}}}{\sqrt{a}}, \frac{qa^{\frac{3}{2}}}{cde\sqrt{b}})\frac{z}{\sigma}; q)_\infty} \mathrm{d}\psi$$
$$= 2\pi \frac{\vartheta(h, h\frac{a}{b}; q)(\frac{qa}{c}, \frac{qa}{cd}, \frac{qa}{ce}, \frac{qa}{de}, \frac{bcd}{a}, \frac{bce}{a}, \frac{bde}{a}; q)_\infty}{(q, \frac{qa}{c}, \frac{bc}{a}, \frac{bd}{a}, \frac{be}{a}, b, c, d, e, \frac{qa}{cde}, \frac{qa^2}{bcde}; q)_\infty}, \tag{7.1}$$

where $z = \mathrm{e}^{i\psi}$ and the maximum modulus of the denominator factors in the integrand is less than unity and we assume there are no vanishing denominator factors (see Remark 4.1).

Proof. Start with Bailey's three-term transformation of a nonterminating very-well-poised ${}_8W_7$ [9, (III.37)]. Then as in the discussion surrounding [9, (2.11.7)], replace f using the substitution $qa^2 = bcdef$. This converts the nonterminating very-well-poised ${}_8W_7$ with argument bd/a to a nonterminating very-well-poised ${}_6W_5$ with the same argument. This nonterminating very-well-poised ${}_6W_5$ can then be summed using the nonterminating sum for a nonterminating very-well-poised ${}_6W_5$ [9, (II.20)]

$${}_6W_5\left(a; b, c, d; q, \frac{qa}{bcd}\right) = \frac{\left(qa, \frac{qa}{bc}, \frac{qa}{bd}, \frac{qa}{cd}; q\right)_\infty}{\left(\frac{qa}{b}, \frac{qa}{c}, \frac{qa}{d}, \frac{qa}{bcd}; q\right)_\infty}. \tag{7.2}$$

The remaining two nonterminating very-well-poised ${}_8W_7$'s both now have argument q and the application of Theorem 1.9 with cardinalities $(A, C, D) = (6, 8, 2)$, given by

$$\mathbf{a} := \left\{\pm\sqrt{b}, \frac{q}{c}\sqrt{ab}, \frac{q}{d}\sqrt{ab}, \frac{q}{e}\sqrt{ab}, \left(\frac{b}{a}\right)^{\frac{3}{2}} cde\right\}, \tag{7.3}$$

$$\mathbf{c} := \left\{\sqrt{ab}, \pm q\sqrt{b}, \frac{b^{\frac{3}{2}}}{\sqrt{a}}, c\sqrt{\frac{b}{a}}, d\sqrt{\frac{b}{a}}, e\sqrt{\frac{b}{a}}, \frac{qa^{\frac{3}{2}}}{cde\sqrt{b}}\right\}, \tag{7.4}$$

$$\mathbf{d} := \left\{\sqrt{\frac{a}{b}}, \sqrt{\frac{b}{a}}\right\}, \tag{7.5}$$

generates the integral in (7.1). This completes the proof. □

Now that we've generated the definite integral in Theorem 7.1, we can apply (1.50) to this definite integral to obtain in principle, a new summation theorem which is an eight-term sum of nonterminating very-well-poised ${}_8W_7$'s with argument q. However, when one applies (1.50), two of the terms vanish because of leading factors of q^{-1} in the list of numerator infinite q-shifted factorials. So we are left with a summation formula for six-terms each of which are nonterminating very-well-poised ${}_8W_7$'s with argument q. This is given as follows.

Theorem 7.2. *Let $q \in \mathbb{C}^\dagger$, $a, b, c, d, e, h \in \mathbb{C}^*$, and we assume there are no vanishing denominator factors (see Remark 4.1). Then, one has the following six-term summation formulas for nonterminating balanced very-well-poised ${}_8W_7$ with argument q:*

$$\begin{aligned}
&\frac{\vartheta(ha, \frac{h}{b}; q)(\pm\frac{1}{\sqrt{a}}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{bcde}{a^2}; q)_\infty}{(\pm\frac{q}{\sqrt{a}}, a, b, \frac{b}{a}, \frac{c}{a}, \frac{d}{a}, \frac{e}{a}, \frac{qa}{bcde}; q)_\infty}\, {}_8W_7\left(a; b, c, d, e, \frac{qa^2}{bcde}; q, q\right)\\
&\quad + \frac{\vartheta(h\frac{qa^2}{bcde}, h\frac{cde}{qa}; q)(\pm\frac{bcde}{qa^{\frac{3}{2}}}, \frac{bcd}{a}, \frac{bce}{a}, \frac{bde}{a}, \frac{b^2c^2d^2e^2}{qa^3}; q)_\infty}{(\pm\frac{bcde}{a^{\frac{3}{2}}}, \frac{qa^2}{bcde}, \frac{qa}{cde}, \frac{bcde}{qa}, \frac{b^2cde}{qa^2}, \frac{bc^2de}{qa^2}, \frac{bcd^2e}{qa^2}, \frac{bcde^2}{qa^2}; q)_\infty}\\
&\qquad\qquad \times {}_8W_7\left(\frac{q^2a^3}{b^2c^2d^2e^2}; \frac{qa^2}{bcde}, \frac{qa}{bcd}, \frac{qa}{bce}, \frac{qa}{bde}, \frac{qa}{cde}; q, q\right)\\
&\quad + \mathop{\left[\!\left[\;\right]\!\right]}^{b;c,d,e} \frac{\vartheta(hb, h\frac{a}{b^2}; q)(\pm\frac{\sqrt{a}}{b}, \frac{qa}{bc}, \frac{qa}{bd}, \frac{qa}{be}, \frac{cde}{a}; q)_\infty}{(\pm\frac{q\sqrt{a}}{b}, b, \frac{b^2}{a}, \frac{a}{b}, \frac{c}{b}, \frac{d}{b}, \frac{e}{b}, \frac{qa^2}{b^2cde}; q)_\infty}\, {}_8W_7\left(\frac{b^2}{a}; b, \frac{bc}{a}, \frac{bd}{a}, \frac{be}{a}, \frac{qa}{cde}; q, q\right)\\
&\quad = \frac{\vartheta(h, h\frac{a}{b}; q)(\frac{qa}{cd}, \frac{qa}{ce}, \frac{qa}{de}, \frac{bcd}{a}, \frac{bce}{a}, \frac{bde}{a}; q)_\infty}{(b, c, d, e, \frac{bc}{a}, \frac{bd}{a}, \frac{be}{a}, \frac{qa}{cde}, \frac{qa^2}{bcde}; q)_\infty}.
\end{aligned} \tag{7.6}$$

Proof. Starting with Theorem 7.1, we apply (1.50). This produces an eight-term sum of nonterminating very-well-poised ${}_8W_7$'s with argument q. However two of the terms vanish because of the appearance of leading factors of q^{-1} in the numerator infinite q-shifted factorials. This leaves us with six-terms each of which are nonterminating very-well-poised ${}_8W_7$'s with argument q, which completes the proof. □

Corollary 7.3. *Let $n \in \mathbb{Z}$, $q \in \mathbb{C}^\dagger$, $a, b, c, d, e \in \mathbb{C}^*$, and we assume there are no vanishing denominator factors (see Remark 4.1). Then, one produces the following six-term transformation formulas for nonterminating balanced very-well-poised ${}_8W_7$ with argument q:*

$$
\begin{aligned}
0 &= \frac{\vartheta(q^n a, \frac{q^n}{b}; q)(\pm\frac{1}{\sqrt{a}}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{bcde}{a^2}; q)_\infty}{(\pm\frac{q}{\sqrt{a}}, a, b, \frac{b}{a}, \frac{c}{a}, \frac{d}{a}, \frac{e}{a}, \frac{qa}{bcde}; q)_\infty} {}_8W_7\left(a; b, c, d, e, \frac{qa^2}{bcde}; q, q\right) \\
&+ \frac{\vartheta(q^{n+1}\frac{a^2}{bcde}, q^{n-1}\frac{cde}{a}; q)(\pm\frac{bcde}{qa^{\frac{3}{2}}}, \frac{bcd}{a}, \frac{bce}{a}, \frac{bde}{a}, \frac{b^2c^2d^2e^2}{qa^3}; q)_\infty}{(\pm\frac{bcde}{a^{\frac{3}{2}}}, \frac{qa^2}{bcde}, \frac{qa}{cde}, \frac{bcde}{qa}, \frac{b^2cde}{qa^2}, \frac{bc^2de}{qa^2}, \frac{bcd^2e}{qa^2}, \frac{bcde^2}{qa^2}; q)_\infty} \\
&\qquad \times {}_8W_7\left(\frac{q^2a^3}{b^2c^2d^2e^2}; \frac{qa^2}{bcde}, \frac{qa}{bcd}, \frac{qa}{bce}, \frac{qa}{bde}, \frac{qa}{cde}; q, q\right) \\
&+ \coprod_{b;c,d,e} \frac{\vartheta(q^n b, q^n\frac{a}{b^2}; q)(\pm\frac{\sqrt{a}}{b}, \frac{qa}{bc}, \frac{qa}{bd}, \frac{qa}{be}, \frac{cde}{a}; q)_\infty}{(\pm\frac{q\sqrt{a}}{b}, b, \frac{b^2}{a}, \frac{a}{b}, \frac{c}{b}, \frac{d}{b}, \frac{e}{b}, \frac{qa^2}{b^2cde}; q)_\infty} {}_8W_7\left(\frac{b^2}{a}; b, \frac{bc}{a}, \frac{bd}{a}, \frac{be}{a}, \frac{qa}{cde}; q, q\right),
\end{aligned}
\tag{7.7}
$$

$$
\begin{aligned}
0 &= \frac{\vartheta(q^n b, \frac{q^n}{a}; q)(\pm\frac{1}{\sqrt{a}}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{bcde}{a^2}; q)_\infty}{(\pm\frac{q}{\sqrt{a}}, a, b, \frac{b}{a}, \frac{c}{a}, \frac{d}{a}, \frac{e}{a}, \frac{qa}{bcde}; q)_\infty} {}_8W_7\left(a; b, c, d, e, \frac{qa^2}{bcde}; q, q\right) \\
&+ \frac{\vartheta(q^{n+1}\frac{a}{cde}, q^{n-1}\frac{bcde}{a^2}; q)(\pm\frac{bcde}{qa^{\frac{3}{2}}}, \frac{bcd}{a}, \frac{bce}{a}, \frac{bde}{a}, \frac{b^2c^2d^2e^2}{qa^3}; q)_\infty}{(\pm\frac{bcde}{a^{\frac{3}{2}}}, \frac{qa^2}{bcde}, \frac{qa}{cde}, \frac{bcde}{qa}, \frac{b^2cde}{qa^2}, \frac{bc^2de}{qa^2}, \frac{bcd^2e}{qa^2}, \frac{bcde^2}{qa^2}; q)_\infty} \\
&\qquad \times {}_8W_7\left(\frac{q^2a^3}{b^2c^2d^2e^2}; \frac{qa^2}{bcde}, \frac{qa}{bcd}, \frac{qa}{bce}, \frac{qa}{bde}, \frac{qa}{cde}; q, q\right) \\
&+ \coprod_{b;c,d,e} \frac{\vartheta(\frac{q^n b^2}{a}, \frac{q^n}{b}; q)(\pm\frac{\sqrt{a}}{b}, \frac{qa}{bc}, \frac{qa}{bd}, \frac{qa}{be}, \frac{cde}{a}; q)_\infty}{(\pm\frac{q\sqrt{a}}{b}, b, \frac{b^2}{a}, \frac{a}{b}, \frac{c}{b}, \frac{d}{b}, \frac{e}{b}, \frac{qa^2}{b^2cde}; q)_\infty} {}_8W_7\left(\frac{b^2}{a}; b, \frac{bc}{a}, \frac{bd}{a}, \frac{be}{a}, \frac{qa}{cde}; q, q\right).
\end{aligned}
\tag{7.8}
$$

Proof. Taking $h = q^n$ and $h = q^n\frac{b}{a}$, respectively in Theorem 7.2 produces transformation formulas for nonterminating very-well-poised ${}_8W_7$ with argument q given by (7.7), (7.8). This completes the proof. □

Corollary 7.4. *Let $n \in \mathbb{Z}$, $q \in \mathbb{C}^\dagger$, $a, b, c, d, e \in \mathbb{C}^*$, and we assume there are no vanishing denominator factors (see Remark 4.1). Then, one has the following six-term summation formulas for nonterminating balanced very-well-poised ${}_8W_7$ with argument q:*

$$
\begin{aligned}
&\frac{\vartheta(q^{n+1}\frac{a}{bcde}, q^{n-1}\frac{cde}{a^2}; q)(\pm\frac{bcde}{qa^{\frac{3}{2}}}, \frac{bcd}{a}, \frac{bce}{a}, \frac{bde}{a}, \frac{b^2c^2d^2e^2}{qa^3}; q)_\infty}{(\pm\frac{bcde}{a^{\frac{3}{2}}}, \frac{qa^2}{bcde}, \frac{qa}{cde}, \frac{bcde}{qa}, \frac{b^2cde}{qa^2}, \frac{bc^2de}{qa^2}, \frac{bcd^2e}{qa^2}, \frac{bcde^2}{qa^2}; q)_\infty} \\
&\qquad \times {}_8W_7\left(\frac{q^2a^3}{b^2c^2d^2e^2}; \frac{qa^2}{bcde}, \frac{qa}{bcd}, \frac{qa}{bce}, \frac{qa}{bde}, \frac{qa}{cde}; q, q\right)
\end{aligned}
$$

$$+\sum^{b;c,d,e}\frac{\vartheta(q^n\frac{b}{a},\frac{q^n}{b^2};q)(\pm\frac{\sqrt{a}}{b},\frac{qa}{bc},\frac{qa}{bd},\frac{qa}{be},\frac{cde}{a};q)_\infty}{(\pm\frac{q\sqrt{a}}{b},b,\frac{b^2}{a},\frac{a}{b},\frac{c}{b},\frac{d}{b},\frac{e}{b},\frac{qa^2}{b^2cde};q)_\infty}\,{}_8W_7\left(\frac{b^2}{a};b,\frac{bc}{a},\frac{bd}{a},\frac{be}{a},\frac{qa}{cde};q,q\right)$$
$$=\frac{\vartheta(\frac{q^n}{a},\frac{q^n}{b};q)(\frac{qa}{cd},\frac{qa}{ce},\frac{qa}{de},\frac{bcd}{a},\frac{bce}{a},\frac{bde}{a};q)_\infty}{(b,c,d,e,\frac{bc}{a},\frac{bd}{a},\frac{be}{a},\frac{qa}{cde},\frac{qa^2}{bcde};q)_\infty}, \tag{7.9}$$

$$\frac{\vartheta(q^{n+1}\frac{a^2}{cde},q^{n-1}\frac{bcde}{a};q)(\pm\frac{bcde}{qa^{\frac{3}{2}}},\frac{bcd}{a},\frac{bce}{a},\frac{bde}{a},\frac{b^2c^2d^2e^2}{qa^3};q)_\infty}{(\pm\frac{bcde}{a^{\frac{3}{2}}},\frac{qa^2}{bcde},\frac{qa}{cde},\frac{bcde}{qa},\frac{b^2cde}{qa^2},\frac{bc^2de}{qa^2},\frac{bcd^2e}{qa^2},\frac{bcde^2}{qa^2};q)_\infty}$$
$$\times\,{}_8W_7\left(\frac{q^2a^3}{b^2c^2d^2e^2};\frac{qa^2}{bcde},\frac{qa}{bcd},\frac{qa}{bce},\frac{qa}{bde},\frac{qa}{cde};q,q\right)$$
$$+\sum^{b;c,d,e}\frac{\vartheta(q^nb^2,q^n\frac{a}{b};q)(\pm\frac{\sqrt{a}}{b},\frac{qa}{bc},\frac{qa}{bd},\frac{qa}{be},\frac{cde}{a};q)_\infty}{(\pm\frac{q\sqrt{a}}{b},b,\frac{b^2}{a},\frac{a}{b},\frac{c}{b},\frac{d}{b},\frac{e}{b},\frac{qa^2}{b^2cde};q)_\infty}\,{}_8W_7\left(\frac{b^2}{a};b,\frac{bc}{a},\frac{bd}{a},\frac{be}{a},\frac{qa}{cde};q,q\right)$$
$$=\frac{\vartheta(q^na,q^nb;q)(\frac{qa}{cd},\frac{qa}{ce},\frac{qa}{de},\frac{bcd}{a},\frac{bce}{a},\frac{bde}{a};q)_\infty}{(b,c,d,e,\frac{bc}{a},\frac{bd}{a},\frac{be}{a},\frac{qa}{cde},\frac{qa^2}{bcde};q)_\infty}. \tag{7.10}$$

Proof. Take $h\in\{q^na^{-1},q^nb\}$, with $n\in\mathbb{Z}$, in Theorem 7.2 and the first term following the symmetric sum vanishes and one obtains six-term summation formulas which produce (7.9), (7.10) respectively. This completes the proof. □

Corollary 7.5. *Let $n\in\mathbb{Z}$, $q\in\mathbb{C}^\dagger$, $a,b,c,d,e\in\mathbb{C}^*$, and we assume there are no vanishing denominator factors (see Remark 4.1). Then, one has the following six-term summation formulas for nonterminating balanced very-well-poised ${}_8W_7$ with argument q:*

$$\frac{\vartheta(q^{n-1}\frac{bcde}{a},q^{n-1}\frac{cde}{a^2};q)(\pm\frac{1}{\sqrt{a}},\frac{q}{c},\frac{q}{d},\frac{q}{e},\frac{bcde}{a^2};q)_\infty}{(\pm\frac{q}{\sqrt{a}},a,b,\frac{b}{a},\frac{c}{a},\frac{d}{a},\frac{e}{a},\frac{qa}{bcde};q)_\infty}\,{}_8W_7\left(a;b,c,d,e,\frac{qa^2}{bcde};q,q\right)$$
$$+\sum^{b;c,d,e}\frac{\vartheta(q^{n-1}\frac{b^2cde}{a^2},q^{n-1}\frac{cde}{ab};q)(\pm\frac{\sqrt{a}}{b},\frac{qa}{bc},\frac{qa}{bd},\frac{qa}{be},\frac{cde}{a};q)_\infty}{(\pm\frac{q\sqrt{a}}{b},b,\frac{b^2}{a},\frac{a}{b},\frac{c}{b},\frac{d}{b},\frac{e}{b},\frac{qa^2}{b^2cde};q)_\infty}$$
$$\times{}_8W_7\left(\frac{b^2}{a};b,\frac{bc}{a},\frac{bd}{a},\frac{be}{a},\frac{qa}{cde};q,q\right)$$
$$=\frac{\vartheta(q^{n-1}\frac{bcde}{a^2},q^{n-1}\frac{cde}{a};q)(\frac{qa}{cd},\frac{qa}{ce},\frac{qa}{de},\frac{bcd}{a},\frac{bce}{a},\frac{bde}{a};q)_\infty}{(b,c,d,e,\frac{bc}{a},\frac{bd}{a},\frac{be}{a},\frac{qa}{cde},\frac{qa^2}{bcde};q)_\infty}, \tag{7.11}$$

$$\frac{\vartheta(q^{n+1}\frac{a^2}{cde},q^{n+1}\frac{a}{bcde};q)(\pm\frac{1}{\sqrt{a}},\frac{q}{c},\frac{q}{d},\frac{q}{e},\frac{bcde}{a^2};q)_\infty}{(\pm\frac{q}{\sqrt{a}},a,b,\frac{b}{a},\frac{c}{a},\frac{d}{a},\frac{e}{a},\frac{qa}{bcde};q)_\infty}\,{}_8W_7\left(a;b,c,d,e,\frac{qa^2}{bcde};q,q\right)$$

$$
+ \mathop{\Big[\!\Big]}^{b;c,d,e} \frac{\vartheta(q^{n+1}\frac{ab}{cde}, q^{n+1}\frac{a^2}{b^2cde}; q)(\pm\frac{\sqrt{a}}{b}, \frac{qa}{bc}, \frac{qa}{bd}, \frac{qa}{be}, \frac{cde}{a}; q)_\infty}{(\pm\frac{q\sqrt{a}}{b}, b, \frac{b^2}{a}, \frac{a}{b}, \frac{c}{b}, \frac{d}{b}, \frac{e}{b}, \frac{qa^2}{b^2cde}; q)_\infty}
$$

$$
\times {}_8W_7\left(\frac{b^2}{a}; b, \frac{bc}{a}, \frac{bd}{a}, \frac{be}{a}, \frac{qa}{cde}; q, q\right)
$$

$$
= \frac{\vartheta(q^{n+1}\frac{a}{cde}, q^{n+1}\frac{a^2}{bcde}; q)(\frac{qa}{cd}, \frac{qa}{ce}, \frac{qa}{de}, \frac{bcd}{a}, \frac{bce}{a}, \frac{bde}{a}; q)_\infty}{(b, c, d, e, \frac{bc}{a}, \frac{bd}{a}, \frac{be}{a}, \frac{qa}{cde}, \frac{qa^2}{bcde}; q)_\infty}. \tag{7.12}
$$

Proof. Take $h \in \{q^{n-1}\frac{bcde}{a^2}, q^{n+1}\frac{a}{cde}\}$ with $n \in \mathbb{Z}$, in Theorem 7.2 and the second term following the symmetric sum vanishes and one obtains six-term summation formulas which produce (7.11), (7.12) respectively. This completes the proof. □

Remark 7.6. Note that one can also choose $h \in q^n\left\{\frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}, \frac{b^2}{a}, \frac{c^2}{a}, \frac{d^2}{a}, \frac{e^2}{a}\right\}$, in Theorem 7.2 with $n \in \mathbb{Z}$ and this will also produce six-term summation formulas for nonterminating balanced very-well-poised ${}_8W_7$. However, we leave the representation of these summation formulas to the reader.

8. Gasper & Rahman's product formula for a product of two nonterminating ${}_2\phi_1$'s and for the square of a nonterminating well-poised ${}_2\phi_1$

This section follows from two formulas which can be found in Gasper & Rahman, namely [9, (8.8.18)] (8.1).

$$
{}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) {}_2\phi_1\left(\begin{matrix} a, \frac{qa}{c} \\ \frac{qa}{b} \end{matrix}; q, z\right)
$$

$$
= \frac{(az, \frac{abz}{c}; q)_\infty}{(z, \frac{bz}{c}; q)_\infty} {}_6\phi_5\left(\begin{matrix} a, \frac{c}{b}, \pm\sqrt{\frac{ac}{b}}, \pm\sqrt{\frac{qac}{b}} \\ c, az, \frac{qa}{b}, \frac{ac}{b}, \frac{qc}{bz} \end{matrix}; q, q\right)
$$

$$
+ \frac{(a, az, bz, \frac{c}{b}, \frac{qaz}{c}; q)_\infty}{(z, z, c, \frac{qa}{b}, \frac{c}{bz}; q)_\infty} {}_6\phi_5\left(\begin{matrix} z, \frac{abz}{c}, \pm z\sqrt{\frac{ab}{c}}, \pm z\sqrt{\frac{qab}{c}} \\ az, bz, \frac{qaz}{c}, \frac{abz^2}{c}, \frac{qbz}{c} \end{matrix}; q, q\right) \tag{8.1}
$$

and [9, (8.8.12)]

$$
\left({}_2\phi_1\left(\begin{matrix} a, b \\ \frac{qa}{b} \end{matrix}; q, z\right)\right)^2 = \frac{(az, \frac{b^2z}{q}; q)_\infty}{(z, \frac{b^2z}{qa}; q)_\infty} {}_5\phi_4\left(\begin{matrix} a, \frac{qa}{b^2}, \pm\frac{\sqrt{q}a}{b}, -\frac{qa}{b} \\ az, \frac{qa}{b}, \frac{qa^2}{b^2}, \frac{q^2a}{b^2z} \end{matrix}; q, q\right)
$$

$$
+ \frac{(a, az, bz, bz, \frac{qa}{b^2}; q)_\infty}{(z, z, \frac{qa}{b}, \frac{qa}{b}, \frac{qa}{b^2z}; q)_\infty} {}_5\phi_4\left(\begin{matrix} z, -bz, \pm\frac{bz}{\sqrt{q}}, \frac{b^2z}{q} \\ az, bz, \frac{b^2z^2}{q}, \frac{b^2z}{a} \end{matrix}; q, q\right). \tag{8.2}
$$

However, (8.2) follows directly from their product formula (8.1) using the substitution $c = qa/b$.

8.1. Gasper & Rahman's product of two nonterminating ${}_2\phi_1$'s

Using a product formula for a product of two nonterminating ${}_2\phi_1$'s with modulus of the argument less than unity, one can obtain a q-Mellin–Barnes integral as its representation.

Theorem 8.1. *Let $q, z \in \mathbb{C}^\dagger$, $\sigma \in (0, \infty)$ $a, b, c, h \in \mathbb{C}^*$, $h\sqrt{\frac{c}{bz}} \notin \Upsilon_q$, and we assume there are no vanishing denominator factors (see Remark 4.1). Then, one has the following q-Mellin–Barnes integral for a product of two nonterminating ${}_2\phi_1$'s with arbitrary argument z:*

$$\int_{-\pi}^{\pi} \frac{((h\sqrt{\frac{c}{bz}}, \frac{q}{h}\sqrt{\frac{bz}{c}})\frac{\sigma}{w}, (h\sqrt{\frac{c}{bz}}, \frac{q}{h}\sqrt{\frac{bz}{c}}, \sqrt{bcz}, a\sqrt{\frac{cz}{b}}, qa\sqrt{\frac{z}{bc}}, a\sqrt{\frac{bz^3}{c}})\frac{w}{\sigma}; q)_\infty}{((\sqrt{\frac{c}{bz}}, \sqrt{\frac{bz}{c}})\frac{\sigma}{w}, (a\sqrt{\frac{bz}{c}}, \sqrt{\frac{cz}{b}}, \pm\sqrt{az}, \pm\sqrt{qaz})\frac{w}{\sigma}; q)_\infty} \mathrm{d}\psi$$
$$= \frac{2\pi\vartheta(h, h\frac{c}{bz}; q)(z, c, \frac{qa}{b}; q)_\infty}{(q, a, \frac{c}{b}, \frac{abz}{c}; q)_\infty} {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) {}_2\phi_1\left(\begin{matrix} a, \frac{qa}{c} \\ \frac{qa}{b} \end{matrix}; q, z\right), \tag{8.3}$$

where $w = \mathrm{e}^{i\psi}$ and the maximum modulus of the denominator factors in the integrand is less than unity.

Proof. Start with the formula for a product of two nonterminating ${}_2\phi_1$, namely (8.1). Now use Theorem 1.9 with the following sets of parameters with cardinalities $(A, C, D) = (4, 6, 2)$, given by

$$\mathbf{a} := \left\{\sqrt{bcz}, a\sqrt{\frac{bz^3}{c}}, qa\sqrt{\frac{z}{bc}}, a\sqrt{\frac{cz}{b}}\right\}, \ \mathbf{c} := \left\{a\sqrt{\frac{bz}{c}}, \sqrt{\frac{cz}{b}}, \pm\sqrt{az}, \pm\sqrt{qaz}\right\}, \tag{8.4}$$

$$\mathbf{d} := \left\{\sqrt{\frac{c}{bz}}, \sqrt{\frac{bz}{c}}\right\}. \tag{8.5}$$

This completes the proof. □

Now we take advantage of the q-Mellin–Barnes integral for a product of two nonterminating ${}_2\phi_1$'s with arbitrary argument z with modulus less than unity to obtain a seven-term transformation for the product of two nonterminating ${}_2\phi_1$'s with modulus of the argument less than unity.

Theorem 8.2. *Let $q, z \in \mathbb{C}^\dagger$, $a, b, c, h \in \mathbb{C}^*$, $h, h\frac{c}{bz} \notin \Upsilon_q$, and we assume there are no vanishing denominator factors (see Remark 4.1). Then, one has the following seven-term representation for a product of two nonterminating ${}_2\phi_1$'s with arbitrary argument z:*

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) {}_2\phi_1\left(\begin{matrix} a, \frac{qa}{c} \\ \frac{qa}{b} \end{matrix}; q, z\right) = \frac{1}{\vartheta(h, h\frac{c}{bz}; q)}$$

$$
\times\Bigg(\frac{\vartheta(ha,h\frac{c}{abz};q)(\frac{q}{b},\frac{c}{a},\frac{c}{b},\frac{c}{b};q)_\infty}{(c,\frac{c}{ab},\frac{c}{ab},\frac{qa}{b};q)_\infty}\,{}_6\phi_5\left(\begin{matrix}a,b,\frac{qa}{c},\frac{qb}{c},\frac{abz}{c},\frac{q}{z}\\ \frac{qab}{c},\pm\sqrt{\frac{qab}{c}},\pm q\sqrt{\frac{ab}{c}}\end{matrix};q,q\right)
$$

$$
+\frac{\vartheta(h\frac{c}{b},\frac{h}{z};q)(a,a,b,\frac{qa}{c},\frac{abz}{c},\frac{abz}{c};q)_\infty}{(z,z,c,\frac{qa}{b},\frac{ab}{c},\frac{ab}{c};q)_\infty}\,{}_6\phi_5\left(\begin{matrix}\frac{q}{a},\frac{q}{b},\frac{c}{a},\frac{c}{b},z,\frac{qc}{abz}\\ \frac{qc}{ab},\pm\sqrt{\frac{qc}{ab}},\pm q\sqrt{\frac{c}{ab}}\end{matrix};q,q\right)
$$

$$
+\frac{\vartheta(h\sqrt{\frac{ac}{b}},\frac{h}{z}\sqrt{\frac{c}{ab}},\sqrt{\frac{bc}{a}};q)(a,\frac{c}{b},\frac{abz}{c};q)_\infty}{(-1,\pm\sqrt{q},z,c,\frac{qa}{b},\sqrt{\frac{c}{ab}},\sqrt{\frac{ab}{c}};q)_\infty}
$$

$$
\times\,{}_6\phi_5\left(\begin{matrix}\sqrt{\frac{ac}{b}},\sqrt{\frac{bc}{a}},q\sqrt{\frac{a}{bc}},q\sqrt{\frac{b}{ac}},z\sqrt{\frac{ab}{c}},\frac{q}{z}\sqrt{\frac{c}{ab}}\\ -q,\pm\sqrt{q},q\sqrt{\frac{ab}{c}},q\sqrt{\frac{c}{ab}}\end{matrix};q,q\right)
$$

$$
+\frac{\vartheta(-h\sqrt{\frac{ac}{b}},-\frac{h}{z}\sqrt{\frac{c}{ab}},-\sqrt{\frac{bc}{a}};q)(a,\frac{c}{b},\frac{abz}{c};q)_\infty}{(-1,\pm\sqrt{q},z,c,\frac{qa}{b},-\sqrt{\frac{c}{ab}},-\sqrt{\frac{ab}{c}};q)_\infty}
$$

$$
\times\,{}_6\phi_5\left(\begin{matrix}-\sqrt{\frac{ac}{b}},-\sqrt{\frac{bc}{a}},-q\sqrt{\frac{a}{bc}},-q\sqrt{\frac{b}{ac}},-z\sqrt{\frac{ab}{c}},-\frac{q}{z}\sqrt{\frac{c}{ab}}\\ -q,\pm\sqrt{q},-q\sqrt{\frac{ab}{c}},-q\sqrt{\frac{c}{ab}}\end{matrix};q,q\right)
$$

$$
+\frac{\vartheta(h\sqrt{\frac{qac}{b}},\frac{h}{z}\sqrt{\frac{c}{qab}};q)(a,\frac{c}{b},\frac{abz}{c},\sqrt{\frac{ac}{qb}},\sqrt{\frac{bc}{qa}},\sqrt{\frac{qa}{bc}},z\sqrt{\frac{ab}{qc}};q)_\infty}{(-1,\pm\frac{1}{\sqrt{q}},z,c,\frac{qa}{b},\sqrt{\frac{qac}{b}},\sqrt{\frac{c}{qab}},\sqrt{\frac{ab}{qc}},z\sqrt{\frac{qab}{c}};q)_\infty}
$$

$$
\times\,{}_6\phi_5\left(\begin{matrix}\sqrt{\frac{qac}{b}},\sqrt{\frac{qbc}{a}},\sqrt{\frac{q^3a}{bc}},\sqrt{\frac{q^3b}{ac}},z\sqrt{\frac{qab}{c}},\frac{1}{z}\sqrt{\frac{q^3c}{ab}}\\ -q,\pm q^{\frac{3}{2}},\sqrt{\frac{q^3ab}{c}},\sqrt{\frac{q^3c}{ab}}\end{matrix};q,q\right)
$$

$$
+\frac{\vartheta(-h\sqrt{\frac{qac}{b}},-\frac{h}{z}\sqrt{\frac{c}{qab}};q)(a,\frac{c}{b},\frac{abz}{c},-\sqrt{\frac{ac}{qb}},-\sqrt{\frac{bc}{qa}},-\sqrt{\frac{qa}{bc}},-z\sqrt{\frac{ab}{qc}};q)_\infty}{(-1,\pm\frac{1}{\sqrt{q}},z,c,\frac{qa}{b},-\sqrt{\frac{qac}{b}},-\sqrt{\frac{c}{qab}},-\sqrt{\frac{ab}{qc}},-z\sqrt{\frac{qab}{c}};q)_\infty}
$$

$$
\times\,{}_6\phi_5\left(\begin{matrix}-\sqrt{\frac{qac}{b}},-\sqrt{\frac{qbc}{a}},-\sqrt{\frac{q^3a}{bc}},-\sqrt{\frac{q^3b}{ac}},-z\sqrt{\frac{qab}{c}},-\frac{1}{z}\sqrt{\frac{q^3c}{ab}},\\ -q,\pm q^{\frac{3}{2}},-\sqrt{\frac{q^3ab}{c}},-\sqrt{\frac{q^3c}{ab}}\end{matrix};q,q\right)\Bigg). \tag{8.6}
$$

Proof. Applying **a**, **c** and **d** in (8.4), (8.5) to (1.50) and connecting with Theorem 8.1 completes the proof. □

8.2. Gasper & Rahman's formula for the square of a nonterminating well-poised ${}_2\phi_1$

Using the formula for the square of a nonterminating well-poised ${}_2\phi_1$ with modulus of the argument less than unity one can obtain a q-Mellin–Barnes integral as its representation.

Theorem 8.3. *Let $q, z \in \mathbb{C}^\dagger$, $a, b, h \in \mathbb{C}^*$, $\sigma \in (0, \infty)$, $\frac{h}{b}\sqrt{\frac{qa}{z}} \notin \Upsilon_q$. Then, one has the following q-Mellin–Barnes integral for the square of a nonterminating well-poised ${}_2\phi_1$ with arbitrary argument z:*

$$\int_{-\pi}^{\pi} \frac{((\frac{h}{b}\sqrt{\frac{qa}{z}}, \frac{qb}{h}\sqrt{\frac{z}{qa}})\frac{\sigma}{w}, (\frac{h}{b}\sqrt{\frac{qa}{z}}, \frac{qb}{h}\sqrt{\frac{z}{qa}}, \sqrt{qaz}, b\sqrt{\frac{az^3}{q}}, \frac{1}{b}\sqrt{qa^3z})\frac{w}{\sigma}; q)_\infty}{((b\sqrt{\frac{z}{qa}}, \frac{1}{b}\sqrt{\frac{qa}{z}})\frac{\sigma}{w}, (b\sqrt{\frac{az}{q}}, \pm\sqrt{az}, \frac{1}{b}\sqrt{qaz}, -\sqrt{qaz})\frac{w}{\sigma}; q)_\infty} \mathrm{d}\psi$$
$$= \frac{2\pi\vartheta(h, h\frac{qa}{b^2z}; q)(z, \frac{qa}{b}, \frac{qa}{b}; q)_\infty}{(q, a, \frac{qa}{b^2}, \frac{b^2z}{q}; q)_\infty} \left({}_2\phi_1\left(\begin{matrix} a, b \\ \frac{qa}{b} \end{matrix}; q, z\right)\right)^2, \tag{8.7}$$

where $w = \mathrm{e}^{i\psi}$ and the maximum modulus of the denominator factors in the integrand is less than unity and we assume there are no vanishing denominator factors (see Remark 4.1).

Proof. Start with the formula for the square of a nonterminating well-poised ${}_2\phi_1$, namely (8.2). Now use Theorem 1.9 with the following sets of parameters with cardinalities $(A, C, D) = (3, 5, 2)$, given by

$$\mathbf{a} := \left\{\sqrt{qaz}, b\sqrt{\frac{az^3}{q}}, \frac{\sqrt{qa^3z}}{b}\right\}, \mathbf{c} := \left\{b\sqrt{\frac{az}{q}}, \frac{\sqrt{qaz}}{b}, \pm\sqrt{az}, -\sqrt{qaz}\right\}, \tag{8.8}$$

$$\mathbf{d} := \left\{\frac{1}{b}\sqrt{\frac{qa}{z}}, b\sqrt{\frac{z}{qa}}\right\}. \tag{8.9}$$

This completes the proof. □

Now we take advantage of the q-Mellin–Barnes integral for the square of a nonterminating well-poised ${}_2\phi_1$ with arbitrary argument z with modulus less than unity to obtain a six-term transformation for the square.

Theorem 8.4. *Let $q, z \in \mathbb{C}^\dagger$, $a, b, h \in \mathbb{C}^*$, and we assume there are no vanishing denominator factors (see Remark 4.1). Then, one has the following six-term representation for a square of a nonterminating well-poised ${}_2\phi_1$ with arbitrary argument z:*

$$\left({}_2\phi_1\left(\begin{matrix} a, b \\ \frac{qa}{b} \end{matrix}; q, z\right)\right)^2 = \frac{1}{\vartheta(h, h\frac{qa}{b^2z}; q)}$$

$$\times\left(\frac{\vartheta(ha,h\frac{q}{b^2z};q)(\frac{q}{b},\frac{q}{b},\frac{qa}{b^2},\frac{qa}{b^2};q)_\infty}{(\frac{qa}{b},\frac{qa}{b},\frac{q}{b^2},\frac{q}{b^2};q)_\infty}{}_5\phi_4\left(\begin{matrix}a,b,\frac{b^2}{a},\frac{b^2z}{q},\frac{q}{z}\\ b^2,\pm b\sqrt{q},-b\end{matrix};q,q\right)\right.$$

$$+\frac{\vartheta(h\frac{qa}{b^2},\frac{h}{z};q)(a,a,b,b,\frac{b^2z}{q},\frac{b^2z}{q};q)_\infty}{(z,z,\frac{b^2}{q},\frac{b^2}{q},\frac{qa}{b},\frac{qa}{b};q)_\infty}{}_5\phi_4\left(\begin{matrix}\frac{q}{a},\frac{q}{b},\frac{qa}{b^2},z,\frac{q^2}{b^2z}\\ \frac{q^2}{b^2},\pm\frac{q^{\frac{3}{2}}}{b},-\frac{q}{b}\end{matrix};q,q\right)$$

$$+\frac{\vartheta(h\frac{a\sqrt{q}}{b},h\frac{\sqrt{q}}{bz};q)(\sqrt{q},a,\frac{qa}{b^2},\frac{b^2z}{q};q)_\infty}{(-1,-\sqrt{q},z,\frac{qa}{b},\frac{qa}{b},\frac{b}{\sqrt{q}},\frac{\sqrt{q}}{b};q)_\infty}{}_5\phi_4\left(\begin{matrix}\sqrt{q},\frac{a\sqrt{q}}{b},\frac{b\sqrt{q}}{a},\frac{bz}{\sqrt{q}},\frac{q^{\frac{3}{2}}}{bz}\\ -q,-\sqrt{q},b\sqrt{q},\frac{q^{\frac{3}{2}}}{b}\end{matrix};q,q\right)$$

$$+\frac{\vartheta(-h\frac{a\sqrt{q}}{b},-h\frac{\sqrt{q}}{bz};q)(-\sqrt{q},a,\frac{qa}{b^2},\frac{b^2z}{q};q)_\infty}{(-1,\sqrt{q},z,\frac{qa}{b},\frac{qa}{b},-\frac{b}{\sqrt{q}},-\frac{\sqrt{q}}{b};q)_\infty}$$

$$\times{}_5\phi_4\left(\begin{matrix}-\sqrt{q},-\frac{a\sqrt{q}}{b},-\frac{b\sqrt{q}}{a},-\frac{bz}{\sqrt{q}},-\frac{q^{\frac{3}{2}}}{bz}\\ -q,\sqrt{q},-b\sqrt{q},-\frac{q^{\frac{3}{2}}}{b}\end{matrix};q,q\right)$$

$$+\frac{\vartheta(-h\frac{qa}{b},-\frac{h}{bz};q)(-1,a,-\frac{a}{b},\frac{qa}{b^2},-\frac{bz}{q},\frac{b^2z}{q};q)_\infty}{(z,-bz,\pm\frac{1}{\sqrt{q}},\frac{qa}{b},\frac{qa}{b},-\frac{qa}{b},-\frac{1}{b},-\frac{b}{q};q)_\infty}$$

$$\left.\times{}_5\phi_4\left(\begin{matrix}-q,-bz,-\frac{qa}{b},-\frac{qb}{a},-\frac{q^2}{bz}\\ -qb,\pm q^{\frac{3}{2}},-\frac{q^2}{b}\end{matrix};q,q\right)\right). \tag{8.10}$$

Proof. Applying **a**, **c** and **d** in (8.8), (8.9) to (1.50) and connecting with Theorem 8.1 completes the proof. □

Remark 8.5. If you choose for some $n\in\mathbb{Z}$, $h\in q^n\left\{\frac{1}{a},\frac{b^2z}{q},\frac{b^2}{qa},z,\pm\frac{b}{a\sqrt{q}},\pm\frac{bz}{\sqrt{q}},-\frac{b}{qa},-bz\right\}$, then the six-term transformation formula for the square of a nonterminating well-poised ${}_2\phi_1$ with arbitrary argument z in (8.10) reduces to a five-term transformation formula. However we leave the representation of these transformation formulas to the reader.

An interesting application of these expansions (8.2), (8.10) is given in the following corollary which gives nonterminating three-term and five-term summation theorems for sums of nonterminating ${}_4\phi_3$'s.

Corollary 8.6. *Let $q\in\mathbb{C}^\dagger$, $a,b,h\in\mathbb{C}^*$, and we assume there are no vanishing denominator factors (see Remark 4.1). Then, one has the following analogues of the Bailey–Daum q-Kummer sum in the specialization as $z=-q/b$ for the square of the nonterminating well-poised ${}_2\phi_1$:*

$$\frac{(-q,-q;q)_\infty(qa,qa,\frac{q^2a}{b^2}\frac{q^2a}{b^2};q^2)_\infty}{(-\frac{q}{b},-\frac{q}{b},\frac{qa}{b},\frac{qa}{b};q)_\infty}$$

$$
\begin{aligned}
&= \frac{(-b,-\frac{qa}{b};q)_\infty}{(-\frac{q}{b},-\frac{b}{a};q)_\infty}\,{}_4\phi_3\left(\begin{matrix}\pm\frac{a\sqrt{q}}{b},\frac{qa}{b^2},a\\ \pm\frac{qa}{b},\frac{qa^2}{b^2}\end{matrix};q,q\right)\\
&\quad+\frac{(-q,-q,a,-\frac{qa}{b},\frac{qa}{b^2};q)_\infty}{(\frac{qa}{b},\frac{qa}{b},-\frac{q}{b},-\frac{q}{b},-\frac{a}{b};q)_\infty}\,{}_4\phi_3\left(\begin{matrix}\pm\sqrt{q},-b,-\frac{q}{b}\\ -q,-\frac{qa}{b},-\frac{qb}{a}\end{matrix};q,q\right) \qquad (8.11)\\
&= \frac{1}{\vartheta(h,-h\frac{a}{b};q)}\left(\frac{\vartheta(ha,-\frac{h}{b};q)(\frac{q}{b},\frac{q}{b},\frac{qa}{b^2},\frac{qa}{b^2};q)_\infty}{(\frac{qa}{b},\frac{qa}{b},\frac{q}{b^2},\frac{q}{b^2};q)_\infty}\,{}_4\phi_3\left(\begin{matrix}a,\pm b,\frac{b^2}{a}\\ b^2,\pm b\sqrt{q}\end{matrix};q,q\right)\right.\\
&\quad+\frac{\vartheta(h\frac{qa}{b^2},-h\frac{b}{q};q)(a,a,\pm b,\pm b;q)_\infty}{(-\frac{q}{b},-\frac{q}{b},\frac{b^2}{q},\frac{b^2}{q},\frac{qa}{b},\frac{qa}{b};q)_\infty}\,{}_4\phi_3\left(\begin{matrix}\frac{qa}{b^2},\pm\frac{q}{b},\frac{q}{a}\\ \frac{q^2}{b^2},\pm\frac{q^{\frac32}}{b}\end{matrix};q,q\right)\\
&\quad+\frac{\vartheta(h\frac{a\sqrt{q}}{b},-\frac{h}{\sqrt{q}};q)(\sqrt{q},a,-b,\frac{qa}{b^2};q)_\infty}{(-1,-\sqrt{q},-\frac{q}{b},\frac{qa}{b},\frac{qa}{b},\frac{b}{\sqrt{q}},\frac{\sqrt{q}}{b};q)_\infty}\,{}_4\phi_3\left(\begin{matrix}\frac{a\sqrt{q}}{b},\frac{b\sqrt{q}}{a},\pm\sqrt{q}\\ -q,b\sqrt{q},\frac{q^{\frac32}}{b}\end{matrix};q,q\right)\\
&\quad\left.+\frac{\vartheta(-h\frac{a\sqrt{q}}{b},\frac{h}{\sqrt{q}};q)(-\sqrt{q},a,-b,\frac{qa}{b^2};q)_\infty}{(-1,\sqrt{q},-\frac{q}{b},\frac{qa}{b},\frac{qa}{b},-\frac{b}{\sqrt{q}},-\frac{\sqrt{q}}{b};q)_\infty}\,{}_4\phi_3\left(\begin{matrix}-\frac{a\sqrt{q}}{b},-\frac{b\sqrt{q}}{a},\pm\sqrt{q}\\ -q,-b\sqrt{q},-\frac{q^{\frac32}}{b}\end{matrix};q,q\right)\right). \qquad (8.12)
\end{aligned}
$$

Proof. Simply start with (8.2), (8.10), let $z = -q/b$ and then compare the resulting expressions to the Bailey–Daum q-Kummer sum [15, (17.6.5)]

$$
{}_2\phi_1\left(\begin{matrix}a,b\\ \frac{qa}{b}\end{matrix};q,-\frac{q}{b}\right)=\frac{(-q;q)_\infty(qa,\frac{q^2a}{b^2};q^2)_\infty}{(-\frac{q}{b},\frac{qa}{b};q)_\infty}, \qquad (8.13)
$$

where $|q| < b$. For the expression which arises from (8.10) one of the terms vanishes due to the appearance of a unity factor in one of the numerator infinity q-shifted factorials. This completes the proof. □

Remark 8.7. If you choose for some $n \in \mathbb{Z}$, $h \in q^n\left\{\frac{1}{a},-b,\frac{b^2}{qa},-\frac{q}{b},\pm\frac{b}{a\sqrt{q}},\pm\sqrt{q}\right\}$, then the five-term summation theorem in (8.12) reduces to a four-term summation theorem. However we leave the representation of these summation theorems to the reader.

Remark 8.8. We also note that using the following transformation of a nonterminating well-poised ${}_2\phi_1$ to a nonterminating very-well-poised ${}_8W_7$, cf. [9, (8.8.16)],

$$
{}_2\phi_1\left(\begin{matrix}a,b\\ \frac{qa}{b}\end{matrix};q,z\right)=\frac{(\pm z\sqrt{a},\pm zb\sqrt{\frac{a}{q}};q)_\infty}{(z,-az,\pm\frac{bz}{\sqrt{q}};q)_\infty}\,{}_8W_7\left(-\frac{az}{q};-\frac{bz}{q},\pm\sqrt{a},\pm\frac{\sqrt{qa}}{b};q,-bz\right), \qquad (8.14)
$$

where we assume there are no vanishing denominator factors (see Remark 4.1). Hence, the above formulas can also be expressed as a product of two nonterminating very-well-poised ${}_8W_7$'s. However, we leave the representations of these formulas to the reader. Furthermore since the nonterminating ${}_2\phi_1$ can also be expressed in terms of a nonterminating ${}_2\phi_2$ [15, (23.5.2)], and as well as a sum of two ${}_3\phi_2$'s with vanishing numerator parameter [15, (17.9.3)] or as a sum of two ${}_3\phi_2$'s with vanishing denominator parameter [15, (17.9.3_5)], there are many alternative ways to represent the above formulas as well as those formulas given in the previous section.

9. Verma & Jain's transformations for a very-well-poised nonterminating ${}_{12}W_{11}$ and ${}_{10}W_9$

In a paper by Verma & Jain (1982) [19], the authors present examples of transformation formulas for very-well-poised basic hypergeometric series. In this section we exploit several of these formulas to derive integral representations for these nonterminating very-well-poised basic hypergeometric series and then use the integral representations to derive new transformations for these nonterminating very-well-poised basic hypergeometric series. We will focus in particular on a formula they derived for very-well-poised ${}_{12}W_{11}$ and ${}_{10}W_9$.

9.1. The Verma–Jain ${}_{12}W_{11}$ transformation

Using a transformation for a nonterminating very-well-poised ${}_{12}W_{11}$ as a sum of two nonterminating balanced ${}_6\phi_5$ with argument q we derive the following integral representation.

Define the multiset notation $\boldsymbol{\omega}a := \{a, \omega a, \omega^2 a\}$ for $a \in \mathbb{C}$, $\omega = \mathrm{e}^{\frac{2}{3}\pi i}$, the cube root of unity. Note that $\omega^3 = 1$, $\omega^5 = \omega^{-1} = \omega^2$, $\omega^4 = \omega^{-2} = \omega$.

Theorem 9.1. *Let $q \in \mathbb{C}^\dagger$, $\omega = \mathrm{e}^{\frac{2}{3}\pi i}$, $h, a, x, y, z \in \mathbb{C}^*$, $\sigma \in (0, \infty)$. Then*

$$
\begin{aligned}
&{}_{12}W_{11}\left(a; x, qx, q^2x, y, qy, q^2y, z, qz, q^2z; q^3, \frac{q^3a^4}{(xyz)^2}\right) \\
&= \frac{(q, qa, \frac{qa}{xy}, \frac{qa}{xz}, \frac{qa}{yz}, x, y, z, \boldsymbol{\omega}a^{\frac{1}{3}}; q)_\infty}{2\pi\vartheta(h, h\frac{xyz}{qa}; q)(\frac{qa}{x}, \frac{qa}{y}, \frac{qa}{z}, a; q)_\infty} \\
&\quad\times \int_{-\pi}^{\pi} \frac{((h\sqrt{\frac{xyz}{qa}}, \frac{q}{h}\sqrt{\frac{qa}{xyz}})\frac{\sigma}{w}, (h\sqrt{\frac{xyz}{qa}}, \frac{q}{h}\sqrt{\frac{xya}{xyz}}, \pm a\sqrt{\frac{q}{xyz}}, \pm\frac{qa}{\sqrt{xyz}})\frac{w}{\sigma}; q)_\infty}{((\sqrt{\frac{xyz}{qa}}, \sqrt{\frac{qa}{xyz}})\frac{\sigma}{w}, (\sqrt{\frac{qax}{yz}}, \sqrt{\frac{qay}{xz}}, \sqrt{\frac{qaz}{xy}}, \boldsymbol{\omega}a^{\frac{5}{6}}\sqrt{\frac{q}{xyz}})\frac{w}{\sigma}; q)_\infty} \mathrm{d}\psi, \quad (9.1)
\end{aligned}
$$

where $w = \mathrm{e}^{i\psi}$ and the maximum modulus of the denominator factors in the integrand is less than unity and we assume there are no vanishing denominator factors (see Remark 4.1).

Proof. Start with the transformation formula for a nonterminating very-well-poised ${}_{12}W_{11}$ in terms of a sum of two balanced ${}_6\phi_5$ with argument q [19, 6.1]:

$$
{}_{12}W_{11}\left(a; x, qx, q^2x, y, qy, q^2y, z, qz, q^2z; q^3, \frac{q^3a^4}{(xyz)^3}\right)
$$

$$= \frac{(qa, \frac{qa}{xy}, \frac{qa}{xz}, \frac{qa}{yz}; q)_\infty}{(\frac{qa}{x}, \frac{qa}{y}, \frac{qa}{z}, \frac{qa}{xyz}; q)_\infty} {}_6\phi_5\left(\begin{matrix} x, y, z, \boldsymbol{\omega} a^{\frac{1}{3}} \\ \frac{xyz}{a}, \pm\sqrt{a}, \pm\sqrt{qa} \end{matrix}; q, q\right)$$

$$+ \frac{(x, y, z, \frac{q^2a^3}{(xyz)^2}; q)_\infty}{(\frac{qa}{x}, \frac{qa}{y}, \frac{qa}{z}, \frac{xyz}{qa}; q)_\infty} \frac{(q^3a; q^3)_\infty}{(\frac{q^3a^4}{(xyz)^3}; q^3)_\infty} {}_6\phi_5\left(\begin{matrix} \frac{qa}{xy}, \frac{qa}{xz}, \frac{qa}{yz}, \boldsymbol{\omega}\frac{qa^{\frac{4}{3}}}{xyz} \\ \frac{q^2a}{xyz}, \pm\frac{qa^{\frac{3}{2}}}{xyz}, \pm\frac{(qa)^{\frac{3}{2}}}{xyz} \end{matrix}; q, q\right). \quad (9.2)$$

Now use Theorem 1.9 with the following sets of parameters with cardinalities $(A, C, D) = (3, 5, 2)$, given by

$$\mathbf{a} := \left\{\pm\frac{q^{\frac{1}{2}}a}{\sqrt{xyz}}, \pm\frac{qa}{\sqrt{xyz}}\right\}, \mathbf{c} := \left\{\sqrt{\frac{qax}{yz}}, \sqrt{\frac{qay}{xz}}, \sqrt{\frac{qaz}{xy}}, \boldsymbol{\omega}\frac{q^{\frac{1}{2}}a^{\frac{5}{6}}}{\sqrt{xyz}}\right\}, \quad (9.3)$$

$$\mathbf{d} := \left\{\sqrt{\frac{xyz}{qa}}, \sqrt{\frac{qa}{xyz}}\right\}. \quad (9.4)$$

This completes the proof. □

Now we compute a seven-term transformation for the ${}_{12}W_{11}$.

Theorem 9.2. *Let $q \in \mathbb{C}^\dagger$, $\omega = \mathrm{e}^{\frac{2}{3}\pi i}$, $a, x, y, z \in \mathbb{C}^*$, and we assume there are no vanishing denominator factors (see Remark 4.1). Then*

$${}_{12}W_{11}\left(a; x, qx, q^2x, y, qy, q^2y, z, qz, q^2z; q^3, \frac{q^3a^4}{(xyz)^3}\right) = \frac{(qa, x, y, z, \frac{qa}{xy}, \frac{qa}{xz}, \frac{qa}{yz}, \boldsymbol{\omega} a^{\frac{1}{3}}; q)_\infty}{\vartheta(h, h\frac{xyz}{qa}; q)(\frac{qa}{x}, \frac{qa}{y}, \frac{qa}{z}, a; q)_\infty}$$

$$\times \left(\coprod^{x;y,z} \frac{\vartheta(hx, h\frac{yz}{qa}; q)(\frac{a}{x^2}, y, z; q)_\infty}{(x, \frac{y}{x}, \frac{z}{x}, \frac{qa}{yz}, \boldsymbol{\omega}\frac{a^{1/3}}{x}; q)_\infty} {}_6\phi_5\left(\begin{matrix} x, \frac{qa}{yz}, \pm\frac{q^{\frac{1}{2}}x}{\sqrt{a}}, \pm\frac{qx}{\sqrt{a}} \\ \frac{qx}{y}, \frac{qx}{z}, \boldsymbol{\omega}\frac{qx}{a^{1/3}} \end{matrix}; q, q\right)\right.$$

$$+ \frac{1}{(\omega, \omega^2; q)_\infty} \coprod^{\omega;\omega^2,\omega^3} \frac{\vartheta(h\omega a^{\frac{1}{3}}, h\frac{\omega^2 xyz}{qa^{4/3}}; q)}{(\frac{q\omega a^{4/3}}{xyz}, \frac{\omega^2 x}{a^{1/3}}, \frac{\omega^2 y}{a^{1/3}}, \frac{\omega^2 z}{a^{1/3}}; q)_\infty}$$

$$\left.\times {}_6\phi_5\left(\begin{matrix} \omega a^{\frac{1}{3}}, \frac{q\omega a^{4/3}}{xyz}, \pm\frac{\omega q^{\frac{1}{2}}}{a^{1/6}}, \pm\frac{\omega q}{a^{1/6}} \\ q\omega, q\omega^2, \frac{q\omega a^{1/3}}{x}, \frac{q\omega a^{1/3}}{y}, \frac{q\omega a^{1/3}}{z} \end{matrix}; q, q\right)\right). \quad (9.5)$$

Proof. Applying **a**, **c** and **d** in (9.3), (9.4) to (1.50) and connecting with Theorem 9.1 completes the proof. □

Remark 9.3. If you choose for some $n \in \mathbb{Z}$, $h \in q^n \left\{ \frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{qa}{xy}, \frac{qa}{xz}, \frac{qa}{yz}, \frac{\boldsymbol{\omega}}{a^{1/3}}, \frac{\boldsymbol{\omega} q a^{\frac{4}{3}}}{xyz} \right\}$, then the seven-term transformation in (9.5) reduces to a six-term transformation. However we leave the representation of these transformations to the reader.

9.2. The Verma–Jain ${}_{10}W_9$ transformation

There is a transformation for a ${}_{10}W_9$ represented as a sum of two balanced ${}_5\phi_4(q)$ which we originally located in [16, (7.11)], where it refers the reader to [19, (4.1)]. In this subsection we exploit this transformation to write the ${}_{10}W_9$ as a q-Mellin–Barnes integral and then from there to obtain six and five term transformation formulas for the ${}_{10}W_9$.

Theorem 9.4. *Let $q \in \mathbb{C}^\dagger$, $a, b, x, y, z \in \mathbb{C}^*$, $\sigma \in (0, \infty)$. Then*

$$
{}_{10}W_9\left(a^2; b^2, x, qx, y, qy, z, qz; q^2, \frac{q^3 a^6}{(bxyz)^2}\right) = \frac{(q, \pm qa, x, y, z, \frac{qa^2}{xy}, \frac{qa^2}{xz}, \frac{qa^2}{yz}; q)_\infty}{2\pi\vartheta(h, h\frac{xyz}{qa^2}; q)(\pm\frac{qa}{b}, \frac{qa^2}{x}, \frac{qa^2}{y}, \frac{qa^2}{z}; q)_\infty}
$$
$$
\times \int_{-\pi}^{\pi} \frac{((\frac{h}{a}\sqrt{\frac{xyz}{q}}, \frac{qa}{h}\sqrt{\frac{q}{xyz}})\frac{\sigma}{w}, (\frac{h}{a}\sqrt{\frac{xyz}{q}}, \frac{qa}{h}\sqrt{\frac{q}{xyz}}, \pm\frac{qa^2}{\sqrt{xyz}}, \frac{q^{\frac{3}{2}}a^3}{b^2\sqrt{xyz}})\frac{w}{\sigma}; q)_\infty}{((\frac{1}{a}\sqrt{\frac{xyz}{q}}, a\sqrt{\frac{q}{xyz}})\frac{\sigma}{w}, (a\sqrt{\frac{qx}{yz}}, a\sqrt{\frac{qy}{xz}}, a\sqrt{\frac{qz}{xy}}, \pm\frac{qa^2}{b\sqrt{xyz}})\frac{w}{\sigma} q)_\infty} \mathrm{d}\psi, \quad (9.6)
$$

where $w = \mathrm{e}^{i\psi}$ and the maximum modulus of the denominator factors in the integrand is less than unity and we assume there are no vanishing denominator factors (see Remark 4.1).

Proof. Start with [19, (4.1)]

$$
{}_{10}W_9\left(a^2; b^2, x, qx, y, qy, z, qz; q^2, \frac{q^3a^6}{(bxyz)^2}\right)
$$
$$
= \frac{(qa^2, \frac{qa^2}{xy}, \frac{qa^2}{xz}, \frac{qa^2}{yz}; q)_\infty}{(\frac{qa^2}{x}, \frac{qa^2}{y}, \frac{qa^2}{z}, \frac{qa^2}{xyz}; q)_\infty} {}_5\phi_4\left(\begin{matrix} x, y, z, \pm\sqrt{q}\frac{a}{b} \\ q\frac{a^2}{b^2}, \frac{xyz}{a^2}, \pm\sqrt{q}a \end{matrix}; q, q\right)
$$
$$
+ \frac{(qa^2, x, y, z, \frac{q^2a^4}{b^2xyz}, \pm\sqrt{q}\frac{a}{b}, \pm\frac{q^{\frac{3}{2}}a^2}{xyz}; q)_\infty}{(\frac{qa^2}{b^2}, \frac{qa^2}{x}, \frac{qa^2}{y}, \frac{qa^2}{z}, \frac{xyz}{qa^2}, \pm\sqrt{q}a, \pm\frac{q^{\frac{3}{2}}a^3}{bxyz}; q)_\infty} {}_5\phi_4\left(\begin{matrix} \frac{qa^2}{xy}, \frac{qa^2}{xz}, \frac{qa^2}{yz}, \pm\frac{q^{\frac{3}{2}}a^3}{bxyz} \\ \frac{q^2a^2}{xyz}, \frac{q^2a^4}{b^2xyz}, \pm\frac{q^{\frac{3}{2}}a^3}{xyz} \end{matrix}; q, q\right), \quad (9.7)
$$

where we have replaced $(a, b) \mapsto (a^2, b^2)$ in the original reference. Note that in [19, (4.1)] there is a typo in the second term, namely the numerator factor $q^2a^2/(bx^2yz)$ should be replaced with $q^2a^2/(bxyz)$. Furthermore note that this same formula appears also in [16, (7.11)], however there are several typos in the second term of their formula. Now use Theorem 1.9 with the following sets of parameters with cardinalities $(A, C, D) = (3, 5, 2)$, given by

$$\mathbf{a} := \left\{ \pm \frac{qa^2}{\sqrt{xyz}}, \frac{q^{\frac{3}{2}}a^3}{b^2\sqrt{xyz}} \right\}, \ \mathbf{c} := \left\{ a\sqrt{\frac{qx}{yz}}, a\sqrt{\frac{qy}{xz}}, a\sqrt{\frac{qz}{xy}}, \pm\frac{qa^2}{b\sqrt{xyz}} \right\}, \tag{9.8}$$

$$\mathbf{d} := \left\{ \frac{1}{a}\sqrt{\frac{xyz}{q}}, a\sqrt{\frac{q}{xyz}} \right\}. \tag{9.9}$$

This completes the proof. □

Now we compute a six-term transformation for the ${}_{10}W_9$.

Theorem 9.5. *Let $q \in \mathbb{C}^\dagger$, $a, b, x, y, z \in \mathbb{C}^*$, and we assume there are no vanishing denominator factors (see Remark 4.1). Then*

$$\begin{aligned}
&{}_{10}W_9\left(a^2; b^2, x, qx, y, qy, z, qz; q^2, \frac{q^3a^6}{(bxyz)^2}\right) = \frac{(\pm qa, x, y, z, \frac{qa^2}{xy}, \frac{qa^2}{xz}, \frac{qa^2}{yz}; q)_\infty}{\vartheta(h, h\frac{xyz}{qa^2}; q)(\pm\frac{qa}{b}, \frac{qa^2}{x}, \frac{qa^2}{y}, \frac{qa^2}{z}; q)_\infty} \\
&\times\left(\sum_{x;y,z} \frac{\vartheta(hx, h\frac{yz}{qa^2}; q)(\pm\frac{\sqrt{q}a}{x}, \frac{qa^2}{b^2x}; q)_\infty}{(x, \frac{y}{x}, \frac{z}{x}, \frac{qa^2}{yz}, \pm\frac{\sqrt{q}a}{bx}; q)_\infty} {}_5\phi_4\left(\begin{matrix} x, \frac{qa^2}{yz}, \pm\frac{q^{\frac{1}{2}}x}{a}, \frac{b^2x}{a^2} \\ \frac{qx}{y}, \frac{qx}{z}, \pm\frac{\sqrt{q}bx}{a} \end{matrix}; q, q \right) \right. \\
&+ \frac{(\pm b; q)_\infty}{2(-q; q)_\infty} \left(\frac{\vartheta(h\frac{\sqrt{q}a}{b}, h\frac{bxyz}{q^{3/2}a^3}; q)}{(\frac{q^{3/2}a^3}{bxyz}, \frac{bx}{\sqrt{q}a}, \frac{by}{\sqrt{q}a}, \frac{bz}{\sqrt{q}a}; q)_\infty} {}_5\phi_4\left(\begin{matrix} \pm\frac{q}{b}, \frac{\sqrt{q}a}{b}, \frac{\sqrt{q}b}{a}, \frac{q^{3/2}a^3}{bxyz} \\ -q, \frac{q^{3/2}a}{bx}, \frac{q^{3/2}a}{by}, \frac{q^{3/2}a}{bz} \end{matrix}; q, q \right) \right. \\
&\left.\left. + \frac{\vartheta(-h\frac{\sqrt{q}a}{b}, -h\frac{bxyz}{q^{3/2}a^3}; q)}{(-\frac{q^{3/2}a^3}{bxyz}, -\frac{bx}{\sqrt{q}a}, -\frac{by}{\sqrt{q}a}, -\frac{bz}{\sqrt{q}a}; q)_\infty} {}_5\phi_4\left(\begin{matrix} \pm\frac{q}{b}, -\frac{\sqrt{q}a}{b}, -\frac{\sqrt{q}b}{a}, -\frac{q^{3/2}a^3}{bxyz} \\ -q, -\frac{q^{3/2}a}{bx}, -\frac{q^{3/2}a}{by}, -\frac{q^{3/2}a}{bz} \end{matrix}; q, q \right) \right) \right).
\end{aligned} \tag{9.10}$$

Proof. Applying **a**, **c** and **d** in (9.8), (9.9) to (1.50) and connecting with Theorem 9.4 completes the proof. □

Remark 9.6. If you choose for some $n \in \mathbb{Z}$, $h \in q^n \left\{ \frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{qa^2}{xy}, \frac{qa^2}{xz}, \frac{qa^2}{yz}, \pm\frac{b}{\sqrt{q}a}, \pm\frac{q^{\frac{3}{2}}a^3}{bxyz} \right\}$, then the six-term transformation in (9.10) reduces to a five-term transformation. However we leave the representation of these transformations to the reader.

10. Guo & Schlosser's transformation for a very-well-poised nonterminating ${}_{12}W_{11}$

It was brought to our attention by one of the referees that in the appendix of a recent paper by Guo & Schlosser (2021) [11], the authors present a transformation formula for a very-well-poised nonterminating ${}_{12}W_{11}$. Furthermore, they indicated that it might be possible to exploit this transformation along the lines of the examples given above and those obtained in [6].

The nonterminating transformation for a ${}_{12}W_{11}$ given in [11, Theorem A.1] is represented as a sum of two ${}_4\phi_3$'s (see (10.2) below). The Guo–Schlosser transformation extends Gasper's terminating ${}_{12}W_{11}$ to ${}_4\phi_3$ transformation [9, Exercise 8.15] (see in particular [7, (3.2)]). In this section we exploit the Guo–Schlosser nonterminating transformation to write their ${}_{12}W_{11}$ as a q-Mellin–Barnes integral and then from there to obtain another three-term nonterminating ${}_4\phi_3$ transformation to complement Guo–Schlosser's transformation. Note that the relation between the Guo–Schlosser transformation and the new three-term transformation we derive (see (10.5) below) is encapsulated in [6, Corollary 2.17] with $\frac{ab}{ef}=\frac{cd}{gh}$ and $t=1$. Then we give several implications of this new nonterminating transformation and its relation with the Guo–Schlosser transformation.

First we give the q-Mellin–Barnes integral representation of the Guo–Schlosser ${}_{12}W_{11}$ which as one will notice is symmetric in the variables c,d. The proof of the integral representation from the Guo–Schlosser transformation becomes clear when one notices the existence of the factors qd_1/d_2 and qd_2/d_1 in the denominator factors of the ${}_4\phi_3$'s below. This is the typical start for all the transformations we have utilized above.

Theorem 10.1. *Let $q\in\mathbb{C}^\dagger$, $a,b,c,d,\mathbb{C}^*$, $\sigma\in(0,\infty)$. Then*

$$
{}_{12}W_{11}\left(a;b,c,d,\frac{ab}{c},\frac{ab}{d},\pm\sqrt{\frac{qa}{b}},\pm q\sqrt{\frac{a}{b}};q,\frac{q}{b}\right)=\frac{(q,qa,c,d,\frac{ab}{c},\frac{ab}{d};q)_\infty}{2\pi(ab,\frac{q}{b},\frac{qa}{c},\frac{qa}{d},\frac{qc}{b},\frac{qd}{b};q)_\infty}
\times\int_{-\pi}^{\pi}\frac{((q\sqrt{\frac{a}{bcd}},q\sqrt{\frac{cd}{ab^3}})\frac{\sigma}{w},(q\sqrt{\frac{ac}{bd}},q\sqrt{\frac{ad}{bc}})\frac{w}{\sigma};q)_\infty}{(\sqrt{\frac{ab}{cd}},\sqrt{\frac{cd}{ab}})\frac{\sigma}{w},(\sqrt{\frac{abd}{c}},\sqrt{\frac{abc}{d}})\frac{w}{\sigma};q)_\infty}\,\mathrm{d}\psi, \tag{10.1}
$$

where $w=\mathrm{e}^{i\psi}$ and the maximum modulus of the denominator factors in the integrand is less than unity and we assume there are no vanishing denominator factors (see Remark 4.1).

Proof. Start with [11, Theorem A.1]

$$
\begin{aligned}
&{}_{12}W_{11}\left(a;b,c,d,\frac{ab}{c},\frac{ab}{d},\pm\sqrt{\frac{qa}{b}},\pm q\sqrt{\frac{a}{b}};q,\frac{q}{b}\right)\\
&=\frac{(qa,c,d,\frac{qcd}{ab^2};q)_\infty}{(ab,\frac{qc}{b},\frac{qd}{b},\frac{cd}{ab};q)_\infty}\,{}_4\phi_3\left(\begin{matrix}b,\frac{ab}{c},\frac{ab}{d},\frac{ab^2}{cd}\\ \frac{qa}{c},\frac{qa}{d},\frac{qab}{cd}\end{matrix};q,\frac{q^2}{b^2}\right)\\
&\quad+\frac{(qa,\frac{ab}{c},\frac{ab}{d},\frac{qa}{cd};q)_\infty}{(ab,\frac{qa}{c},\frac{qa}{d},\frac{ab}{cd};q)_\infty}\,{}_4\phi_3\left(\begin{matrix}b,c,d,\frac{cd}{a}\\ \frac{qc}{b},\frac{qd}{b},\frac{qcd}{ab}\end{matrix};q,\frac{q^2}{b^2}\right).
\end{aligned}\tag{10.2}
$$

Now use Theorem 1.5 with the following sets of parameters with cardinalities $(A,B,C,D)=(2,2,2,2)$, given by

$$\mathbf{a} := \left\{ q\sqrt{\frac{ac}{bd}}, q\sqrt{\frac{ad}{bc}} \right\}, \ \mathbf{b} := \left\{ q\sqrt{\frac{a}{bcd}}, \frac{q}{b^{\frac{3}{2}}}\sqrt{\frac{cd}{a}} \right\}, \tag{10.3}$$

$$\mathbf{c} := \left\{ \sqrt{\frac{abc}{d}}, \sqrt{\frac{abd}{c}} \right\}, \ \mathbf{d} := \left\{ \sqrt{\frac{ab}{cd}}, \sqrt{\frac{cd}{ab}} \right\}. \tag{10.4}$$

This completes the proof. □

Now we exploit Theorem 10.1 in conjunction with (1.35) in Theorem 1.5 to obtain another three-term nonterminating ${}_4\phi_3$ transformation for the Guo–Schlosser ${}_{12}W_{11}$.

Corollary 10.2. *Let $q \in \mathbb{C}^\dagger$, $a, b, c, d, \mathbb{C}^*$, $\sigma \in (0, \infty)$, $w = \mathrm{e}^{i\psi}$. Then*

$$\begin{aligned} &{}_{12}W_{11}\left(a; b, c, d, \frac{ab}{c}, \frac{ab}{d}, \pm\sqrt{\frac{qa}{b}}, \pm q\sqrt{\frac{a}{b}}; q, \frac{q}{b}\right) \\ &\quad = \frac{(qa, c, \frac{ab}{d}, \frac{qc}{bd}; q)_\infty}{(ab, \frac{c}{d}, \frac{qc}{b}, \frac{qa}{d}; q)_\infty} {}_4\phi_3\left(\begin{matrix} b, d, \frac{ab}{c}, \frac{bd}{c} \\ \frac{qa}{c}, \frac{qd}{b}, \frac{qd}{c} \end{matrix}; q, \frac{q^2}{b^2} \right) \\ &\quad\quad + \frac{(qa, d, \frac{ab}{c}, \frac{qd}{bc}; q)_\infty}{(ab, \frac{d}{c}, \frac{qa}{c}, \frac{qd}{b}; q)_\infty} {}_4\phi_3\left(\begin{matrix} b, c, \frac{ab}{d}, \frac{bc}{d} \\ \frac{qc}{b}, \frac{qc}{d}, \frac{qa}{d} \end{matrix}; q, \frac{q^2}{b^2} \right). \end{aligned} \tag{10.5}$$

Proof. Starting with Theorem 10.1 and utilizing (1.35) in Theorem 1.5 with the choice of parameters given by (10.3), (10.4) directly produces (10.5). This completes the proof. □

As is indicated near [11, Corollary A.5], by choosing $d = q^{-n}$ in (10.2), one obtains Gasper's terminating ${}_{12}W_{11}$ to ${}_4\phi_3$ transformation [9, Exercise 8.15] (see in particular [7, (3.2)]). Furthermore, if you make the same substitution in (10.2), one obtains a new terminating transformation between a terminating ${}_{12}W_{11}$ and a terminating ${}_4\phi_3$.

Corollary 10.3. *Let $n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $a, b, c \in \mathbb{C}^*$. Then*

$$\begin{aligned} &{}_{12}W_{11}\left(a; q^{-n}, b, c, q^n ab, \frac{ab}{c}, \pm\sqrt{\frac{qa}{b}}, \pm q\sqrt{\frac{a}{b}}; q, \frac{q}{b}\right) \\ &\quad = \frac{(qa, \frac{ab}{c}; q)_n}{(ab, \frac{qa}{c}; q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, b, c, \frac{q^{-n}c}{a} \\ \frac{qc}{b}, \frac{q^{1-n}}{b}, \frac{q^{1-n}c}{ab} \end{matrix}; q, \frac{q^2}{b^2} \right) \\ &\quad = \frac{(qa, c; q)_n}{(ab, \frac{qc}{b}; q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, b, \frac{ab}{c}, \frac{q^{-n}b}{c} \\ \frac{qa}{c}, \frac{q^{1-n}}{b}, \frac{q^{1-n}}{c} \end{matrix}; q, \frac{q^2}{b^2} \right). \end{aligned} \tag{10.6}$$

Proof. Setting $d = q^{-n}$ with $n \in \mathbb{N}_0$ in (10.5) completes the proof. □

As a consequence of Corollary 10.2, we obtain the following non-q result where we are using a generalization of Bailey's notation (6.4) for very-well-poised generalized hypergeometric series ${}_9F_8$ with argument unity, where

$$
{}_9\mathrm{W}_8(a;b,c,d,e,f,g,h) := {}_9F_8\left({a,\frac{a}{2}+1,b,c,d,e,f,g,h \atop \frac{a}{2},1+a-b,1+a-c,1+a-d,1+a-e,1+a-f,1+a-g,1+a-h};1\right). \tag{10.7}
$$

Corollary 10.4. *Let $a,b,c,d\in\mathbb{C}$ such that $\Re(a+2b)<\frac12$, $\Re b<\frac34$ so that the generalized hypergeometric series at unity are absolutely convergent. Then*

$$
\begin{aligned}
&{}_9\mathrm{W}_8\left(a;b,c,d,a+b-c,a+b-d,\tfrac12(a-b+1),\tfrac12(a-b+2)\right)\\
&=\frac{\Gamma(a+b,c-d,c-b+1,a-d+1)}{\Gamma(a+1,c,a+b-d,c-b-d+1)}\,{}_4F_3\left({b,d,a+b-c,b+d-c \atop a-c+1,d-b+1,d-c+1};1\right)\\
&\quad+\frac{\Gamma(a+b,d-c,a-c+1,d-b+1)}{\Gamma(a+1,d,a+b-c,d-b-c+1)}\,{}_4F_3\left({b,c,a+b-d,b+c-d \atop c-b+1,c-d+1,a-d+1};1\right).
\end{aligned} \tag{10.8}
$$

Proof. First start with (10.5), perform the map

$$
(a,b,c,d)\mapsto(q^a,q^b,q^c,q^d),
$$

and apply (1.13). Then taking the $q\to1^-$ limit while utilizing (1.14) and [15, (17.4.2)] completes the proof. □

Remark 10.5. It is noticed in [11, Corollary A.2] that if you take $d=a/c$ in the Guo–Schlosser ${}_{12}W_{11}$ transformation you obtain a new summation formula and that this is also equivalent to taking $d=ab^2/c$. We also notice that if you take $d=bc$ then the transformation (10.5) also produces a summation formula for an ${}_{12}W_{11}$. However this summation formula is identical to [11, Corollary A.2].

As remarked in [11, Corollary A.4], the substitution $d=ab^2/(qc)$ results in a non-terminating ${}_{10}W_9$ to ${}_3\phi_2$ transformation. The application of this same substitution in (10.5) results in a different ${}_3\phi_2$ transformation, which has some interesting side-effects. We will discuss that now.

Corollary 10.6. *Let $q\in\mathbb{C}^\dagger$, $a,b,c\in\mathbb{C}^*$. Then*

$$
{}_{10}W_9\left(a;b,c,\frac{ab^2}{qc},\pm\sqrt{\frac{qa}{b}},\pm q\sqrt{\frac{a}{b}};q,\frac{q}{b}\right)=\frac{(qa,\frac{ab}{c},\frac{qc}{b},\frac{q^2}{b^2};q)_\infty}{(ab,\frac{q}{b},\frac{qa}{c},\frac{q^2c}{b^2};q)_\infty}\,{}_3\phi_2\left({c,\frac{b^2}{q},\frac{ab^2}{qc} \atop \frac{ab}{c},\frac{qc}{b}};q,\frac{q^2}{b^2}\right) \tag{10.9}
$$

$$
=\frac{(qa,c,\frac{q^2c^2}{ab^3};q)_\infty}{(ab,\frac{q^2c}{b^2},\frac{qc^2}{ab^2};q)_\infty}\,{}_3\phi_2\left({b,\frac{ab^2}{qc},\frac{ab^3}{qc^2} \atop \frac{qa}{c},\frac{ab^2}{c^2}};q,\frac{q^2}{b^2}\right)+\frac{(qa,\frac{ab}{c^2},\frac{ab^2}{qc};q)_\infty}{(ab,\frac{qa}{c},\frac{ab^2}{qc^2};q)_\infty}\,{}_3\phi_2\left({b,c,\frac{qc^2}{ab} \atop \frac{q^2c}{b^2},\frac{q^2c^2}{ab^2}};q,\frac{q^2}{b^2}\right). \tag{10.10}
$$

Proof. Starting with (10.2) and setting $d = \frac{ab^2}{qc}$ produces the ${}_{10}W_9$ to ${}_3\phi_2$ transformation (10.9) and the same substitution in (10.5) produces (10.10). This completes the proof. □

One of the interesting facts about Corollary 10.6 is that if you compare only the terms which are ${}_3\phi_2$'s you obtain a three-term transformation of nonterminating ${}_3\phi_2$'s where they all have the same argument. Furthermore, that argument is not necessarily q. If you set $b^2 = q$ then the transformation becomes the nonterminating q-Saalschütz sum [15, (17.7.5)].

11. Applications of the q-Gauss sum and a generalization of the nonterminating q-Saalschütz sum from the Askey–Roy and Gasper integrals

By starting with the Askey–Roy integral [3, (2.8)]

$$\int_{-\pi}^{\pi} \frac{((fc, \frac{q}{f}d)\frac{\sigma}{z}, (\frac{f}{d}, \frac{q}{fc})\frac{z}{\sigma}; q)_\infty}{\left((c,d)\frac{\sigma}{z}, (a,b)\frac{z}{\sigma}; q\right)_\infty} \mathrm{d}\psi = \frac{2\pi\, \vartheta(f, f\frac{c}{d}; q)(abcd; q)_\infty}{(q, ac, ad, bc, bd; q)_\infty}, \tag{11.1}$$

and its generalization, the Gasper integral [8, (1.8)]

$$\int_{-\pi}^{\pi} \frac{((fc, \frac{q}{f}d)\frac{\sigma}{z}, (\frac{f}{d}, \frac{q}{fc}, abcde)\frac{z}{\sigma}; q)_\infty}{\left((c,d)\frac{\sigma}{z}, (a,b,e)\frac{z}{\sigma}; q\right)_\infty} \mathrm{d}\psi = \frac{2\pi\, \vartheta(f, f\frac{c}{d}; q)(abcd, bcde, acde; q)_\infty}{(q, ac, ad, bc, bd, ce, de; q)_\infty}, \tag{11.2}$$

and applying its alternate realization (1.50), we are able to derive two new summation theorems which involve the q-Gauss sum and the nonterminating q-Saalschütz sums. The straightforward application of (1.50) to these integrals produces the following interesting results. Starting with the left-hand side of (11.1) and applying (1.50) with cardinalities $(A, C, D) = (0, 2, 2)$, given by $\mathbf{c} := \{a, b\}$, $\mathbf{d} := \{c, d\}$, and then comparing with the right-hand side of (11.1) produces the following result:

$$\begin{aligned} &\frac{\vartheta(acf, \frac{f}{ad}; q)}{(ac, ad, \frac{b}{a}; q)_\infty} {}_2\phi_1\left(\begin{matrix} ac, ad \\ \frac{qa}{b} \end{matrix}; q, \frac{q}{abcd}\right) + \frac{\vartheta(bcf, \frac{f}{bd}; q)}{(bc, bd, \frac{a}{b}; q)_\infty} {}_2\phi_1\left(\begin{matrix} bc, bd \\ \frac{qb}{a} \end{matrix}; q, \frac{q}{abcd}\right) \\ &= \mathop{\Big[\!\!\Big]}^{a;b} \frac{\vartheta(acf, \frac{f}{ad}; q)}{(ac, ad, \frac{b}{a}; q)_\infty} {}_2\phi_1\left(\begin{matrix} ac, ad \\ \frac{qa}{b} \end{matrix}; q, \frac{q}{abcd}\right) = \frac{\vartheta(f, \frac{fc}{d}; q)(abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty}. \end{aligned} \tag{11.3}$$

If you choose for some $n \in \mathbb{Z}$, $f \in q^n\{ad, \frac{1}{ac}, bc, \frac{1}{bd}, ec, \frac{1}{ed}\}$ in (11.3), then the nonterminating ${}_2\phi_1$ summation becomes a one term summation which is equivalent to the (nonterminating) q-Gauss sum [15, (17.6.1)]. In fact both ${}_2\phi_1$'s in (11.3) can be summed using the q-Gauss sum. Performing a particular replacement converts the above expression to something close to the form of a q-Gauss sum. However, this replacement eliminates one the four variables $\{a,b,c,d\}$ (in this case d). Starting with (11.3) and replacing $(a,b,c,d)\mapsto(d,\frac{qd}{c},\frac{a}{d},\frac{b}{d})$ produces:

$$
{}_2\phi_1\left(\begin{matrix}a,b\\ c\end{matrix};q,\frac{c}{ab}\right)+\frac{\vartheta(\frac{fc}{qb},\frac{qfa}{c};q)(a,b,\frac{q}{c};q)_\infty}{\vartheta(fa,\frac{f}{b};q)(\frac{qa}{c},\frac{qb}{c},\frac{c}{q};q)_\infty}\,{}_2\phi_1\left(\begin{matrix}\frac{qa}{c},\frac{qb}{c}\\ \frac{q^2}{c}\end{matrix};q,\frac{c}{ab}\right)
=\frac{\vartheta(f,\frac{fa}{b};q)(\frac{q}{c},\frac{qab}{c};q)_\infty}{\vartheta(fa,\frac{f}{b};q)(\frac{qa}{c},\frac{qb}{c};q)_\infty}. \tag{11.4}
$$

Evaluating these summations produces the following interesting relations for modified theta functions.

Corollary 11.1. *Let $q\in\mathbb{C}^\dagger$, $a,b,c,d,f\in\mathbb{C}^*$. Then*

$$
\vartheta(f,\tfrac{fa}{b},abcd;q)=\frac{\vartheta(fac,\frac{f}{bc},ad,bd;q)}{\vartheta(\frac{d}{c};q)}+\frac{\vartheta(fad,\frac{f}{bd},ac,bc;q)}{\vartheta(\frac{c}{d};q)}, \tag{11.5}
$$

$$
\vartheta(f,\tfrac{fa}{b},\tfrac{c}{ab};q)=\frac{\vartheta(fa,\frac{f}{b},\frac{c}{a},\frac{c}{b};q)}{\vartheta(c;q)}+\frac{\vartheta(\frac{c}{fa},\frac{fc}{qb},a,b;q)}{\vartheta(\frac{c}{q};q)}. \tag{11.6}
$$

Proof. Using the q-Gauss sum in (11.3), (11.4) completes the proof. □

By utilizing (11.2), we are also able to derive a nonterminating summation formula for a balanced ${}_3\phi_2$ which is a generalization of the nonterminating q-Saalschütz sum [15, (17.7.5)].

Theorem 11.2. *Let $q\in\mathbb{C}^\dagger$, $a,b,c,e,f,h\in\mathbb{C}^*$, $h\neq q^n$ for some $n\in\mathbb{Z}$. Then*

$$
\begin{aligned}
&{}_3\phi_2\left(\begin{matrix}a,b,c\\ e,f\end{matrix};q,q\right)+\frac{\vartheta(\frac{hqa}{e},\frac{he}{qb};q)(\frac{q}{e},a,b,\frac{e}{c},\frac{q}{f};q)_\infty}{\vartheta(ha,\frac{h}{b};q)(\frac{e}{q},\frac{e}{f},\frac{q}{c},\frac{qa}{e},\frac{qb}{e};q)_\infty}\,{}_3\phi_2\left(\begin{matrix}\frac{qa}{e},\frac{qb}{e},\frac{qc}{e}\\ \frac{q^2}{e},\frac{qf}{e}\end{matrix};q,q\right)\\
&\quad+\frac{\vartheta(\frac{he}{bc},\frac{hac}{e};q)(a,b,\frac{q}{e},\frac{q}{f},\frac{qab}{e};q)_\infty}{\vartheta(ha,\frac{h}{b};q)(\frac{f}{e},\frac{f}{q},\frac{q}{c},\frac{e}{ac},\frac{e}{bc};q)_\infty}\,{}_3\phi_2\left(\begin{matrix}\frac{e}{ab},\frac{e}{ac},\frac{e}{bc}\\ \frac{q^2}{f},\frac{qe}{f}\end{matrix};q,q\right)\\
&=\frac{\vartheta(h,\frac{ha}{b};q)(\frac{q}{e},\frac{q}{f},\frac{e}{c},\frac{qab}{e};q)_\infty}{\vartheta(ha,\frac{h}{b};q)(\frac{qa}{e},\frac{qb}{e},\frac{e}{ac},\frac{e}{bc};q)_\infty},
\end{aligned} \tag{11.7}
$$

where $ef=qabc$.

Proof. Starting with the left-hand side of (11.2) and applying (1.50) with cardinalities $(A,C,D)=(1,3,2)$, given by $\mathbf{a}:=\{abcde\}$, $\mathbf{c}:=\{a,b,e\}$, $\mathbf{d}:=\{c,d\}$, and then comparing with the right-hand side of (11.2) produces

$$\mathop{\Big[\!\!\Big]}^{a;b,e} \frac{\vartheta(acf,\frac{f}{ad};q)(bcde;q)_\infty}{(ac,ad,\frac{b}{a},\frac{e}{a};q)_\infty}\,{}_3\phi_2\left(\begin{matrix}ac,ad,\frac{q}{bcde}\\ \frac{qa}{b},\frac{qa}{e}\end{matrix};q,q\right)=\frac{\vartheta(f,\frac{fc}{d};q)(abcd,bcde,acde;q)_\infty}{(ac,ad,bc,bd,ce,de;q)_\infty}. \tag{11.8}$$

Then replacing $(a,b,c,d,f)\mapsto(ab/q,qbcde/a^2,c/a,d/a,h)$, followed by $(a,b,c,d)\mapsto(d,c,a,b)$ with the restriction $ef=qabc$, completes the proof. □

Remark 11.3. If you choose for some $n\in\mathbb{Z}$, $f\in q^n\{ad,\frac{1}{ac},bc,\frac{1}{bd},ec,\frac{1}{ed}\}$, then the nonterminating ${}_3\phi_2$ (11.8) summation becomes a three-term summation. This three-term summation is equivalent to the nonterminating q-Saalschütz sum. Therefore (11.7) is a generalization of the nonterminating q-Saalschütz sum (similarly for Theorem 11.2). Note also that if one makes a judicious replacement in (11.7), such as $e=\frac{q^{1+n}}{bcd}$ (similarly for Theorem 11.2), then one of the sums will become terminating and one may therefore obtain a terminating summation. However, these terminating summations are a bit strange because they also will involve two nonterminating ${}_3\phi_2$'s as well.

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