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Dealing with counts and other quantal quantities in quantity calculus☆

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A R T I C L E I N F O

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A B S T R A C T

Continuity is usually assumed as a defining feature of measured quantities. This premise is false for counted quantities, amount of substance, electric charge, and others that are constrained to exist in integral multiples of a quantum. A software application that treats these quantities as continuous can predict outcomes that are physically impossible, such as the production of half a photon. Thus far, formalizations of quantity calculus have not addressed how quantities that are structured like the integers should interoperate with continuous quantities. This article introduces the extension of quantity calculus to include quantal quantities, which vary in steps rather than continuously, and discusses the consequences of including them.

1. Introduction

Quantity calculus emerged as the algebra of physical quantities a century ago [1–4]. Since then, it has been the subject of various approaches to mathematical formalization and much theoretical and philosophical debate over its significance and accuracy, in tandem with the International System of Units (SI) [5], as a representation of the world in general. The treatment of unit symbols as ordinary algebraic entities, a concept that remained controversial for a surprisingly long time [6], is now codified in the SI brochure [5, §5.4.1] and accepted by most theoreticians and practitioners without argument. Moreover, the use of quantity calculus has expanded beyond physics into other scientific disciplines.

In recent history, there has been discussion about the treatment of counted quantities in quantity calculus and the SI, including the possible indication of units of counting [7–10]. The SI brochure currently does not permit such units to be meaningfully distinguished from the dimensionless, mathematical number 1 or from each other. The proposal of a type system for counting units in [10] gave rise to some follow-up questions: Brown asked 'Would these units allow the same rules of algebraic manipulation as other units, within types and between types: for instance, may we use "atm/atm", "atm/mcl", "pcl/m^3" and "mcl/ent"?' [11] (in which the symbols atm, mcl, pcl, and ent refer to counting units atom, molecule, particle, and entity, respectively, and m is the symbol for the meter).

The answers to questions of that kind depend on interpretation details of the underlying algebra of quantity calculus. We soon encounter the more general issue that continuity is usually assumed as a defining feature of measured quantities [4, §A][12, §21][13–15]. All quantities are algebraically treated as if they were continuous in nature. For example, if x is a positive quantity, it is assumed that $x/2$ exists and is a positive quantity that is smaller than x. Thus far, formalizations of quantity calculus have not addressed how quantities that are instead structured like the integers, for which a half-increment does not always exist, should interoperate with continuous quantities [16,17].

Counting aside, even among quantities that are measured and treated in practice as if they were continuous, there are those that factually are not, including amount of substance and electric charge. The error introduced by modeling them as if they were continuous has usually been insignificant in context, but this is true only as long as the quantum is extremely small relative to the quantity (c.f. Section 2.2). For most counted quantities, this was never the case. For the others, the quantities are steadily shrinking as measurement technology improves.

These issues combine to undermine attempts to develop automated systems for reasoning about quantities. One cannot establish that software is doing the right thing if there is no benchmark of correctness. To take a formal model of continuous quantities, implement it in software, and just assume that it will do something reasonable when the quantities are not actually continuous would not be safe. Such needs motivate a close look at counted quantities and a better understanding of the hazards of integrating them with quantity calculus.

Readers should not misconstrue this as an attempt to complicate long-established practices in physics. When the error resulting from discrete–continuous mismatch is insignificant in context or has been avoided with a customary workaround, there is no crisis for those who have internalized those practices. Nevertheless, the issue is a general

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one, and a general approach to understanding it enables more robust automation of the physical reasoning.

A complete guide to working with counted quantities comes in two parts and solves a more general problem than how to deal with counts. First, in Section 2, I introduce the extension of quantity calculus to include quantal quantities in general. Then, in Section 3, I describe the answers that this implies for counted quantities specifically, including invariance to a change of unit. Section 4 gives a summary and concluding remarks.

2. Quantal algebra

2.1. Concepts and notation

In this discussion, it is important to recognize the distinction between a quantitative expression and the algebraic entity that it denotes. For example, '3 km' and '3000 m' are two different expressions that denote the same definite length. Unfortunately, the term *quantity value* has become problematic in this regard: internationally approved guidelines are not harmonious on whether '3 km' and '3000 m' together provide one value, two values, or 'it depends' [18]. For the purposes of this article, I will say that the answer is one: '3 km' and '3000 m' are two different expressions of a single value that may be assigned or attributed to a quantity. This choice is most appropriate in an algebraic context and is in harmony with the SI brochure's text, 'The value of a quantity is generally expressed as the product of a number and a unit' [5, §2.1].

A quantal quantity (a *quantal* for short) is a quantity that is constrained to exist in integral multiples of a quantum. The relevant dictionary definition of quantum is 'a minimum amount of a physical quantity which can exist, and in multiples of which it can vary' [19, defn. 5]. I merely generalize this to apply outside of physics.

Although it is commonly used with the sense just described, the term *discrete quantity* is more general: strictly speaking, it says only that the values are countable, not necessarily that they are evenly spaced.

While the unit to express a value of a quantity may be chosen by convenience or convention, the quantum of a quantity is not a choice. The quantum is inherent in the nature of a quantity: it is determined for us by the actual behavior of the property that we are modeling algebraically. By analogy, though the size of a collection of eggs may be expressed in dozens when they are sold, the quantum still is the single egg.

If we choose to express a value of a quantal in a unit other than its quantum, the numerical value will not necessarily be an integer.

In current practice, the quantal nature of a quantity is sometimes suggested by involving a numeric variable that is understood to be an integer:

1. When the electric potential around a nucleus is expressed as $Ze/4\pi\epsilon_0 r$, the quantal charge Ze is the product of an integer Z and the elementary charge e. This convention is lost when the quantity is expressed in another form, such as a value expressed in coulomb.
2. Ideal gas equations often use N for an integral number of particles.

In both examples, the quantal nature of one of the inputs is disregarded when computing a continuous quantity as the result. This is no problem. But if one is instead computing the value of Z or N beginning from continuous inputs, one cannot ignore the quantal nature of the output, lest one assign a value that is not an integer. This can arise from imprecise measurement of the inputs or from a simple synthetic question such as find the value of Z where $Ze = 1$ C.

In reality, a quantal has its quantum regardless of how it or its value is expressed. We should be able to work with quantals at any scale without losing the information that they are constrained to be integral multiples of their quanta. The notation introduced below allows this to happen without limiting us to the Ze form of expression, but any notation that preserves the information would do.

In terms of the SI brochure, the quantum is 'extra information on the nature of the quantity', which may be attached to the quantity symbol in subscripts, superscripts, or brackets [5, §5.4]. The customary ordering of the $m \mid n$ notation for divisibility and the already-heavy utilization of right-side subscripts in the International System of Quantities [20] both suggest the following use of a left subscript to show that a quantity is evenly divisible by a quantum:

- ${}_{e|}Q$ denotes an electric charge Q whose quantum is the elementary charge e. This imposes no constraint on the unit chosen to express values of the quantity, which in SI would be the coulomb.
  In the context of quark physics, one would instead have a quantum of $e/3$. The choice between e and $e/3$ is made for us depending on whether we are modeling macroscopic charge or quark charge. (For modeling the transition between these two realms, see Section 2.4.)
- ${}_{B|}M$ denotes an amount of data M whose quantum is the byte B. The italic font of B indicates that we are using the byte as a quantity, not as a unit. The unit chosen to express values of the quantity could also be byte (B), or bit (b), or some multiple like terabyte (TB) or megabit (Mb).[1]
  Although the unit can be changed at will, the quantum is determined by the granularity of addressing of the data storage medium. The amount of data on a byte-addressable medium is constrained to be an integral number of bytes.
- ${}_{¢|}¤$ denotes an amount of money ¤ whose quantum is the cent. The unit conventionally chosen to express values of the quantity is equal to 100 ¢.
  Not every kind of money is quantal, but usually it is problematic to transfer amounts that cannot be realized using the legally minted units of the underlying currency. Thus, it is an error if payroll software treats these quantities as continuous, and such errors propagate and accumulate until they become significant.

If we express a quantity value as the product of a numerical value and a unit, adopting the $x = \{x\}[x]$ notation of [3, p. 12] that is now ubiquitous, $\{x\}$ is the numerical value and $[x]$ is the unit. If x is quantal, this is a property of the quantity, not of either the numerical value or the unit of a particular quantitative expression, so we should read ${}_{q|}\{x\}[x]$ as ${}_{q|}(\{x\}[x])$.

2.2. General operation

Any given quantal, by itself, is mathematically structured like the integers, $\mathbb{Z}$. This contrasts with continuous quantities, which are mathematically structured like the real numbers, $\mathbb{R}$. Counts and amounts that are constrained to be positive are structured like $\mathbb{Z}+$ and $\mathbb{R}+$ respectively. Like other quantities, quantals can be negative. Negative quantities are produced by the ordinary rules of algebra (subtraction) even if all input quantities are constrained to be positive.

It is readily observed that continuous quantities model the behavior of quantals as the ratios of quantity values to their quanta approach infinity; or to say it another way, as the quanta become vanishingly small relative to the relevant quantity values. Thus it is that a continuous model for amount of substance is less problematic in general than a continuous model for amount of data would be. The error introduced by modeling amount of substance as if it were continuous has usually been insignificant in context. The same has never been true of amounts of data expressed in bits or bytes: being 'off by one' can be very bad [21].

[1] These symbols are also used for the barn, a non-SI unit of area, and the bel, a non-SI unit of logarithmic ratio quantities. The collision is unfortunate, but it is already established in common usage. Excluding historical uses, a byte is now conventionally defined to be comprised of 8 bits.

2.3. Sums and differences

If we express the values of two quantals of the same kind as the product of a numerical value and a shared unit $[x]$, their sum or difference is simply ${}_{q|}x_1 + {}_{q|}x_2 = {}_{q|}(\{x_1\}[x] + \{x_2\}[x]) = {}_{q|}(\{x_1\} + \{x_2\})[x]$.

In the event that quantals with different quanta are being added or subtracted, the correct resolution depends entirely on the actual behavior of the process that this operation is supposed to be modeling, and there is good reason to question what exactly is going on. For example, if two amounts of data are being added, ${}_{B|}M_1 + {}_{b|}M_2$, we cannot tell just from looking at these expressions what is happening or what the quantum of the result ought to be. If the context of the result is only byte-addressable, the storage is usually rounded up to a whole number of bytes, with some bits going to waste. In that case, M_2 is rounded up and re-quantized with a larger quantum, and the addition proceeds as before. If the context of the result is bit-addressable, M_1 is given a smaller quantum instead, and no bits are gained or lost in the process.

The same analysis applies if the addend is a continuous quantity. One has to determine whether the result exists in a continuous space or a quantized one, and that information is not in the input quantities.

2.4. Number times quantal

Multiplying a quantal by an integer is sure to produce a result in the same space as the input, with the same quantum: $n \cdot {}_{q|}x = {}_{q|}(n\{x\})[x]$ (for $n \in \mathbb{Z}$). This is consistent with the reduction of multiplication to repeated addition.

If the number is not an integer and the mathematical result is not coincidentally an integral multiple of the quantum, the correct resolution depends on what the context dictates regarding quantization of the result. The options are:

1. The result is a continuous quantity. For example, if the result is a statistical estimate or average, it need not conform to the quantization of the measurand. It is valid, though not always desirable, to produce an estimated value on a continuous scale even if the true value is quantized. For example, the estimate of a quantity having the nature of a count is not itself a count, but it provides information about a count. As Brown observed, 'Whilst a fraction of an entity is a physical impossibility, in the sense of expressing a measurement result it is still a valid concept' [22].
2. The result is a quantal with a different quantum. For example, in the context of quark physics, $\frac{2}{3}({}_{e|}Q) = {}_{e/3|}(\frac{2}{3}Q)$ is applicable.
3. The result is a quantal with the same quantum as the input. This is appropriate in contexts where the mathematical operation does not correctly model the actual process, i.e., where the supposed result is impossible to realize and the magnitude must either increase or decrease to satisfy the quantization constraint. For example, ${}_{B|}1$ kB of data that are transformed with a compression ratio of $\frac{1}{3}\frac{\mathrm{b}}{\mathrm{b}}$ will yield ${}_{B|}334$ B.

For the one who formalizes or implements quantity calculus, it is simpler to dictate that all quantities shall be continuous and tell those users for whom option 2 or 3 is more appropriate that it is their own problem. But regardless who makes the decision, whether the choice is mechanical or manual, imposed or voluntary, an incorrect choice among options 1–3 creates the possibility of error.

This hazard corresponds to a known, identified class of weaknesses for calculations in software [23]. A simple example follows from the data compression scenario given above. Suppose that memory is allocated in whole bytes, but the result of the multiplication is erroneously treated as a continuous quantity. Under typical numeric rules, the floating-point value approximating $333\frac{1}{3}$ would become 333 when supplied to a memory allocation function that expects an integer. This would lead to an insufficient amount of memory being allocated for the compressed data, potentially resulting in a buffer overflow with security implications.

2.5. Products, ratios, and quotients

When quantities are multiplied, it produces a result of a different dimension than the input quantities. It follows that when quantals are multiplied, the quantum of the result (if indeed the result is quantal) must belong to that dimension. But one cannot immediately conclude that it is simply the product of the quanta of the inputs, as:

$${}_{q_1|}x_1 \cdot {}_{q_2|}x_2 \text{ is not necessarily } {}_{q_1 q_2|}(\{x_1\}\{x_2\})([x_1][x_2])$$

Similarly, for ratios of quantals:

$$\frac{{}_{q_1|}x_1}{{}_{q_2|}x_2} \text{ is not necessarily } {}_{q_1/q_2|}(\{x_1\}/\{x_2\})([x_1]/[x_2])$$

As described in Section 2.1, the quantum of a quantity is not a choice. The quantum is inherent in the nature of a quantity: it is determined for us by the actual behavior of the property that we are modeling algebraically. It follows that if we are spending too much time deciding what the quantum should be, then we probably are not dealing with a quantal at all! The act of constructing a product or ratio does not dictate the nature of the quantity being modeled.

An example would be calculating the cost of data storage. The two input quantities are an amount of money quantized to cents and an amount of data quantized to bytes. The result may be expressed in the unit \$/TB (dollars per terabyte), but it is not quantized to cents per byte. It may or may not be rounded to the nearest cent or dollar per terabyte, but this is a matter of context, convention, and convenience that has nothing to do with the input quantities or the algebra. The result is not quantal.

An example of a ratio involving a quantal and a continuous quantity is particle count divided by duration. The particle count is quantal; the duration is continuous. Depending on the need, you can choose to keep the original ratio to preserve the information that the numerator is quantal, or you can express the value of a continuous quotient in becquerel.

Whether the multiplicand is quantal or continuous, the quantization of the result depends not on the quantization of the inputs but on the nature of the new quantity being derived and the context in which it exists.

2.6. Numerical powers

As with other quantities, integral powers of quantals are equivalent to expanded products or their reciprocals, or to the multiplicative identity element if the exponent is 0.

Non-integral powers have limited application in practice and are sometimes considered harmful in theory. There is no consensus that non-integral powers belong in quantity calculus at all or what their interpretation should be. Raposo holds that fractional exponents are unnecessary and undesirable in an algebraic structure for quantity calculus, and that in all cases where square root has been applied in practice, it 'acts on a quantity which is already a square' [17]. Other sources, however, imply that fractional powers of metrological dimensions have something to do with fractal dimensions: 'Recall that power functions $y = ax^b$ with non-integral b were awkward children for classical dimensional analysis… Come a theory of fractional (fractal) dimensions, and they find a welcoming home' [24, p. 480].

As Raposo did, I will address fractional powers for only a few special cases:

- The special case of a square quantal area, ${}_{a|}A$, for which we want to find the length of one side, requires the quantum of area, a, to itself be a square whose sides are the quantum of length.
- Quantal volume would be analogous, with a cubic quantum.
- Square root arises in the calculation of standard deviation. Standard deviation belongs to statistics rather than quantity calculus, and its definition tells us that it is not quantal (c.f. Section 2.4, option 1).

3. Counted quantities

3.1. Sums and differences

Quantities can be added to or subtracted from one another only if they are of the same kind, and, thus, mutually comparable. For counted quantities in particular, comparability is not a binary, yes-or-no decision. A previous work provides a model within which two counts that are initially of different kinds, such as a number of neutrons and a number of protons, can be comparable in a more general sense (a number of nucleons) [10]. To generalize them and add them together is to declare that the differences of kind are immaterial for the purpose at hand. Whether it is appropriate to do this is not an aspect of the individual quantities but, rather, depends on the context of use. But once the counts have been harmonized in terms of kind, the operation can be completed as previously described in Section 2.3.

3.2. Invariance to change of unit

Some previous writers have made strong assertions to the effect that the single entity or event is the only possible unit for a count. For example:

- Ellis thinks of a count as simply a number, so asking if there can be other units of counting is like asking 'Is there any 1 other than 1?': 'There appears to be no such thing as a scale or unit of number. … I cannot arbitrarily select a group of, say, apples and assign to it the numeral 1, saying "This group contains one apple". For this statement is already true or false. … I cannot arbitrarily select a group as an initial standard for number in the way that I can arbitrarily select an object as an initial standard for length' [25, Ch. 10].
- Roberts says that counting produces a measurement on an 'absolute scale' for which the change-of-unit transformation is disallowed [26, §2.3].[2]
- Similarly, Chrisman says 'Because zero is a fixed value, counts may seem ratios, but, being tied to the discrete unit counted, it cannot be rescaled by some arbitrary factor' [27].

These special treatments of counts are detached from the reality of ongoing practice. For example, an amount of data may be expressed in terms of either the bit (symbol b) or the byte (symbol B) as a unit. It is as futile to argue that only the bit is permissible as it is to insist that eggs must not be sold by the dozen.

Furthermore, once you define a *unit* corresponding to a counted entity or event type, the special treatments of counts quoted above are incompatible with the metrological ground rule that a quantity's definition should be independent of the unit that is chosen to express its value.

With the infrastructure introduced in Section 2, the quantum of a count is distinguished from the unit in which its value is expressed. Nothing then remains to prevent a change-of-unit transformation from being applied to counts that is consistent with its application to other quantities. A separate scale to accommodate the lack of such a transformation is no longer necessary, as the transformation has now been defined.

Given $1\ \mathrm{B} = 8\ \mathrm{b}$, ${}_{b|}x\ \mathrm{b} = {}_{b|}\left(\frac{x}{8}\ \mathrm{B}\right)$ (exactly). The quantal value expressed in bits is transformed into a quantal value expressed in bytes. While the numerical value of the value expressed in bits is a whole number (0, 1, 2, …), the numerical value of the value expressed in bytes is a rational number that is discretized to eighths (0, $\frac{1}{8}$, $\frac{1}{4}$, …), but both are mathematically structured like the integers (or $\mathbb{Z}+$ as the case may be).

The change of unit does not change the definition of the quantal. Changing the unit in which the result is expressed does not change the quantum. The entities that were indivisible before the change-of-unit transformation remain so after the transformation.

3.3. Products, ratios, and quotients

When division in quantity calculus is interpreted as expressing a this-per-that relation, counted quantities are simply quantals behaving as described in Section 2.5. But the presence of counted quantities may further discourage the reduction to a quotient, such as in the previously used example of particle count divided by duration. The numerator of the ratio is quantal; reducing the ratio to a continuous quotient discards that information.

A human-scale example would be a ratio of parking spaces to residents of a community. Neither fractions of a parking space nor fractions of a resident are realizable, so if we are computing a quotient from observations, both numerator and denominator will be quantal. On the other hand, we may be provided with a continuous quotient and tasked with determining the number of parking spaces that should be provided for a given number of residents. In this case, the product of the ratio with a count of residents is quantal because parking spaces can be built or allocated only in whole numbers. This fact determines how the computation must proceed.

Simplifying a ratio involving counts through cancellation of units that appear in the expression of both the numerator and the denominator is just as valid, and as precarious [28,29], as the same operation on continuous quantities. But with counting units, one has the option to generalize units in order to make a cancellation possible that would not be possible at first. Returning to an example from Brown [11], the unit mcl/ent might be generalized to ent/ent if, in context, the fact that the entities in the numerator are molecules is immaterial. Similarly, the unit ent/ent might then be replaced by 1 if, in context, it does not matter that the ratio pertains to entities. We would not exercise this option for a ratio of parking spaces to residents of a community since the named counting units provide the only context we have.

3.4. Logarithmic functions

The logarithm function arises routinely in association with counted quantities. For example, suppose that an alphabet of n characters is used in digitized text, and we want to define a fixed-width character encoding for it. The width of that encoding, i.e., the number of bits per character of a text document using that encoding, is $\lceil \log_2 n \rceil$ b. It is reasonable to ask how this changes if the logarithm applies not to the plain number n but to the counted quantity, n char. It has long been held that to take the logarithm of a dimensioned quantity is nonsense [30, p. 346][31, p. 74], but let us look at this application with fresh eyes and ask what does make sense about it if we do not just equate counts to plain numbers.

If one has a document of length l char and an encoding of width w b/char, the length of the encoded document can be computed as lw b through normal quantity calculus. To achieve this dimensional consistency, we need a function of two separate arguments, not merely a numerical function of a single, dimensionless quotient. For a counted quantity c and a reference quantity r, define

$$\mathcal{L}(c, r) = \left\lceil \log_2 \frac{c}{r} \right\rceil \frac{\mathrm{b}}{r}$$

The required width of the encoding depends on the number of discrete individuals that must be representable, not on the units chosen to express the values of the count or the reference quantity. If we

[2] *Absolute scale* has two other, conflicting definitions. Ellis used the term *absolute scale* in the sense of temperature scales having an 'absolute zero' [25]. Chrisman used it for items whose numerical value is constrained to the range 0 to 1, notably probabilities and proportions, but confusingly cited Ellis [25] in the definition [27].

introduce a unit corresponding to a pair of characters, 1 dichar = 2 char, the following computations are all equivalent:

$$\mathcal{L}\,(n \text{ char}, 1 \text{ char}) = \left\lceil \log_2 \frac{n \text{ char}}{1 \text{ char}} \right\rceil \frac{\text{b}}{1 \text{ char}} = \lceil \log_2 n \rceil \frac{\text{b}}{\text{char}}$$

$$\mathcal{L}\,(n \text{ char}, 1/2 \text{ dichar}) = \left\lceil \log_2 \frac{n \text{ char}}{1/2 \text{ dichar}} \right\rceil \frac{\text{b}}{1/2 \text{ dichar}} = \lceil \log_2 n \rceil \frac{\text{b}}{\text{char}}$$

$$\mathcal{L}\,(n/2 \text{ dichar}, 1 \text{ char}) = \left\lceil \log_2 \frac{n/2 \text{ dichar}}{1 \text{ char}} \right\rceil \frac{\text{b}}{1 \text{ char}} = \lceil \log_2 n \rceil \frac{\text{b}}{\text{char}}$$

$$\mathcal{L}\,(n/2 \text{ dichar}, 1/2 \text{ dichar}) = \left\lceil \log_2 \frac{n/2 \text{ dichar}}{1/2 \text{ dichar}} \right\rceil \frac{\text{b}}{1/2 \text{ dichar}} = \lceil \log_2 n \rceil \frac{\text{b}}{\text{char}}$$

In contrast, if we now see our alphabet as consisting of discrete dichars, then the reference quantity has changed, and the result is different. To represent every combination of 2 characters from an alphabet of n characters, you need n^2 dichars. So, the computation becomes

$$\mathcal{L}\left(n^2 \text{ dichar}, 1 \text{ dichar}\right) = \left\lceil \log_2 \frac{n^2 \text{ dichar}}{1 \text{ dichar}} \right\rceil \frac{\text{b}}{1 \text{ dichar}} = \left\lceil \log_2 n^2 \right\rceil \frac{\text{b}}{\text{dichar}}$$

Using logarithmic identity, we know

$$\log_2 \frac{n^2 \text{ dichar}}{1 \text{ dichar}} = 2 \log_2 \frac{n \text{ char}}{1 \text{ char}}$$

But the ceiling function, which enforces the quantization, is not distributive, so the function $\mathcal{L}$ is not in this sense invariant to a change in the reference quantity. The reference quantity in these computations is none other than the quantum of length for digitized text. The changed computation models a re-quantization to a coarser granularity.

4. Conclusion

This article has answered questions pertaining to the role of counts and other quantal quantities in quantity calculus by introducing quantal algebra, the extension of quantity calculus to include quantities that are constrained to exist in integral multiples of a quantum.

Some of the misgivings about counted quantities have as a premise that the existing system of quantity calculus, with only continuous quantities in scope, can be used in a context-free manner, mechanically, as a formal system that is both complete and consistent with the structure of the world. The worry is that introducing anything that might break that completeness and consistency is not worth it. But correct application of quantity calculus has always required context beyond the mechanical manipulation of numbers and units. To begin with, it requires the context of kind-of-quantity, which arose in the earliest description of the topic within the canon of modern metrology [1]. Substitution of dimensional equivalence for actual comparability (being of the same kind) was an enormous compromise to make in exchange for the 'rigor' of formal models. In comparison to kind-of-quantity, the context needed to work with quantals is better characterized and less subject to conflicting conceptual models.

Although excluding quantals from the domain of discourse is a simplifying assumption for those who can make that choice, those who are required to work with quantals will find that the concomitant issues are neither new nor surprising and are already being lived with in practice. The challenge is not that they have not been dealt with somewhere, but that they have been dealt with in different ways, to different degrees, with different levels of success in different guidance documents and in different implementations of quantity calculus [32]. The opportunity exists, therefore, for international standards to define a path to convergence.

CRediT authorship contribution statement

David Flater: Conceptualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

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