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PEANUT HARMONIC EXPANSION FOR A FUNDAMENTAL SOLUTION OF LAPLACE'S EQUATION IN FLAT-RING COORDINATES

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Abstract. We derive an expansion for the fundamental solution of Laplace's equation in flat-ring cyclide coordinates in three-dimensional Euclidean space. This expansion is a double series of products of functions that are harmonic in the interior and exterior of coordinate surfaces which are peanut shaped and orthogonal to surfaces which are flat-rings. These internal and external peanut harmonic functions are expressed in terms of Lamé–Wangerin functions. Using the expansion for the fundamental solution, we derive an addition theorem for the azimuthal Fourier component in terms of the odd-half-integer degree Legendre function of the second kind as an infinite series in Lamé–Wangerin functions. We also derive integral identities over the Legendre function of the second kind for a product of three Lamé–Wangerin functions. In a limiting case we obtain the expansion of the fundamental solution in spherical coordinates.

1. Introduction

There are 17 conformally inequivalent triply-orthogonal curvilinear coordinate systems (ξ_1, ξ_2, ξ_3) which parametrize points $\mathbf{r} := (x, y, z) \in \mathbb{R}^3$ such that

$$\mathbf{r} = \big(x(\xi_1, \xi_2, \xi_3), y(\xi_1, \xi_2, \xi_3), z(\xi_1, \xi_2, \xi_3)\big),$$

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which yield solution by separation of variables for the three variable Laplace equation [10, Section 3.6] (see also [2]). These 17 coordinate systems can be divided into several groups depending on the properties of the two-dimensional surfaces which are obtained by setting one of the coordinates to a constant in its range.

Nine of the 17 coordinate systems are rotationally-invariant, that is they can be written as a coordinate transformation to Cartesian coordinates of a form

$$\mathbf{r} = (R(\xi_1, \xi_2) \cos \phi, R(\xi_1, \xi_2) \sin \phi, z(\xi_1, \xi_2)),$$

where $\phi \in \mathbb{R}$ or specifically $\phi \in [-\pi, \pi)$ to cover all of $\mathbb{R}^3$. Of the nine rotationally-invariant coordinate systems, five of them are represented by coordinate surfaces which are quadric (cylindrical, spherical, parabolic, oblate spheroidal and prolate spheroidal) and the other four are represented by coordinate surfaces which are cyclidic (toroidal, flat-ring, flat-disk and bi-cyclide). The study of the harmonics for the quadric coordinate systems is classical (yet not fully explored), however for the cyclidic coordinate systems much remains to be learned. In toroidal coordinates the separated solutions are given in terms of associated Legendre functions. However, in the three remaining rotationally-invariant cyclidic coordinate systems (flat-ring, flat-disk, bi-cyclide) the harmonic solutions are given in terms of second-order ordinary differential equations with four regular singularities, namely those of the Heun-type [6, Chapter 31] which specialize in the cyclidic case to ordinary differential equations of Lamé-type [6, Chapter 29] (Lamé functions, modified Lamé functions and Lamé–Wangerin functions). This paper is the second in a series of papers which will focus on the expansion of the $1/r$ potential in the rotationally invariant cyclide coordinate systems: flat-ring cyclide, flat-disk cyclide and bi-cyclide coordinates.

In a previous paper [1] we studied flat-ring coordinates (see §3.1 for their definition) originally introduced by Wangerin [15]. We introduced internal and external flat-ring harmonics. These are harmonic functions which are harmonic inside and outside of coordinate surfaces which are flat-ring cyclides. We found the expansion of the $1/r$ potential in terms of products of internal and external flat-ring harmonics. We also showed that flat-ring coordinates become toroidal coordinates in the limit $k \to 0$ and the expansion of $1/r$ approaches its known expansion in products of internal and external toroidal harmonics as $k \to 0$.

In this paper we continue our work in flat-ring coordinates, however, we now consider a second family of compact coordinate surfaces which are given by rotationally-invariant "peanut" shaped cyclides (or simply peanut cyclides). In Section 3 of this paper we introduce corresponding internal and external peanut harmonics, and find the expansion of $1/r$ in a series of products of internal and external peanut harmonics (Theorem 3.6). The major difference between the expansion over flat-ring surfaces as opposed to peanut surfaces is that in the peanut case we require Lamé–Wangerin functions in place of periodic Lamé functions. Lamé–Wangerin functions are not as well-known as periodic Lamé functions. Therefore, in Section 2 we start with collecting the properties of Lamé–Wangerin functions that we will require in our analysis. In section 4 we show that flat-ring coordinates become spherical coordinates in the limit $k \to 1$ and the expansion of $1/r$ approaches its well-known expansion in products of internal and external spherical harmonics as $k \to 1$.

We believe that our results from Sections 3 and 4 (and partially also from Section 2) are new.

2. Lamé–Wangerin functions

Let $K = K(k)$ and $K' = K'(k) = K(k')$, $k' = \sqrt{1-k^2}$, denote the complete elliptic integral of the first kind and its corresponding complementary elliptic integral, respectively [6, (19.2.8-9)]. The Lamé differential equation [6, (29.2.1)] is

$$\frac{\mathrm{d}^2 w}{\mathrm{d}s^2} + (h - \nu(\nu+1)k^2 \operatorname{sn}^2(s,k))w = 0, \tag{2.1}$$

where $0 < k < 1$, $\nu \geq \frac{1}{2}$, and h is the eigenvalue parameter. This equation has regular singular points at $s = \pm iK'$ with exponents $\{-\nu, \nu+1\}$ at both points. In the application to flat-ring coordinates we require solutions of (2.1) that are Fuchs-Frobenius solutions [8, Chapter XVI] at $s = \pm iK'$ belonging to the exponent $\nu+1$ at both points simultaneously. This leads to an eigenvalue problem for the Lamé equation.

To simplify notation we set $s = it$ and using Jacobi's imaginary transformation [6, §22.6(iv)] we obtain the *modified Lamé equation*

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} + \big(\lambda - \nu(\nu+1)\operatorname{dc}^2(t,k')\big)w = 0, \quad \lambda = \nu(\nu+1) - h, \tag{2.2}$$

where we used Glaisher's notation [6, §22.2] for Jacobian elliptic functions. Again to simplify notation, we change k' back to k and consider the equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} + \big(\lambda - \nu(\nu+1)\operatorname{dc}^2(t,k)\big)w = 0. \tag{2.3}$$

Equation (2.3) has regular singularities at the points $t = \pm K$ with exponents $\{-\nu, \nu+1\}$. We impose the boundary conditions that the solution $w(t)$,

$-K < t < K$, belongs to the exponent $\nu + 1$ at both singular end points, that is, $w(t)$ can be written in the form

$$w(t) = \sum_{j=0}^{\infty} a_j (K-t)^{j+\nu+1} \quad \text{for } t \text{ close to } K, \tag{2.4}$$

and in the form

$$w(t) = \sum_{j=0}^{\infty} b_j (t+K)^{j+\nu+1} \quad \text{for } t \text{ close to } -K. \tag{2.5}$$

Conditions (2.4), (2.5) are equivalent to the conditions that $|t \mp K|^{-1/2} w(t)$ stays bounded as $t \to \pm K$, respectively. We call λ an eigenvalue if there exists a nontrivial solution $w(t)$, $-K < t < K$, of (2.3) which satisfies both boundary conditions (2.4), (2.5). An eigenfunction $w(t)$ is called a Lamé–Wangerin function [7, (15.6)] .

In [14, §3] it is shown that the eigenvalues λ are real and they form an increasing sequence indexed by $n \in \mathbb{N}_0$ with

$$\lambda := \Lambda_\nu^n(k),$$

such that $\Lambda_\nu^n(k) \to \infty$ as $n \to \infty$. Actually, in [14] the differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + (\lambda - \nu(\nu+1)\,\mathrm{ns}^2(r,k))w = 0 \tag{2.6}$$

is treated which agrees with (2.3) substituting $r = t + K$. The (real-valued) eigenfunction corresponding to $\Lambda_\nu^n(k)$ is denoted by $w(t) = W_\nu^n(t,k)$. The eigenfunctions are normalized according to

$$\int_{-K}^{K} \{W_\nu^n(t,k)\}^2 \,\mathrm{d}t = 1. \tag{2.7}$$

The function $W_\nu^n(t,k)$ has exactly n zeros in $(-K,K)$ [14, §8], and it is an even function for even n and an odd function for odd n. Clearly, $W_\nu^n(t,k)$ converges to 0 as $t \to \pm K$. We now state the completeness of the eigenfunctions [14, §3].

Theorem 2.1. *The system $\{W_\nu^n(t,k)\}_{n=0}^{\infty}$ forms an orthonormal basis in $L^2(-K,K)$.*

It is useful to consider equation (2.6) in the limit $k \to 0$:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + (\lambda - \nu(\nu+1)\csc^2 r)w = 0, \tag{2.8}$$

for $0 < r < \pi$. In this case the eigenvalues are $\lambda = (n+\nu+1)^2$, $n \in \mathbb{N}_0$, with corresponding (not normalized) eigenfunctions

$$w(r) = W_\nu^n(r) = (\sin r)^{\nu+1} P_n^{(\nu+\frac{1}{2},\nu+\frac{1}{2})}(\cos r) \simeq (\sin r)^{\nu+1} C_n^{\nu+1}(\cos r),$$

where $0 < r < \pi$, employing Jacobi polynomials $P_n^{(\alpha,\beta)}$ [6, Table 18.3.1] in the ultraspherical case $\alpha = \beta$ [12, (4.24.2)] and Gegenbauer (ultraspherical) polynomials C_n^μ [6, (18.7.1)].

We can compare (2.6) with (2.8) using the following lemma.

Lemma 2.2. *Let $k \in (0,1)$, $\omega := \frac{\pi}{2K}$. Then, for all $r \in (0,K]$, we have*

$$\mathrm{cs}(r,k) \le \omega \cot(\omega r), \tag{2.9}$$

$$\mathrm{sn}(r,k) \le \omega^{-1} \sin(\omega r). \tag{2.10}$$

Proof. (a) The inverse function of $\mathrm{sc}(r,k)$, $0 \le r < K$, is [6, (22.15.20)]

$$\mathrm{arcsc}(x,k) := \int_0^x \frac{\mathrm{d}t}{\sqrt{1+t^2}\sqrt{1+k'^2t^2}},$$

for $x \ge 0$. Now (2.9) is equivalent to

$$g(x) := \omega^{-1}\arctan(\omega x) - \mathrm{arcsc}(x,k) \ge 0, \tag{2.11}$$

for $x \ge 0$. We have $g(0) = \lim_{x\to\infty} g(x) = 0$. A calculation shows that $g'(x) > 0$ is equivalent to $1 + k'^2 - 2\omega^2 + x^2(k'^2 - \omega^4) > 0$. Now $k'^2 - \omega^4 < 0$ so g first increases and then decreases, establishing (2.11). Note that the inequality $\omega > \sqrt{k'}$ follows from the fact that ω equals the arithmetic-geometric mean of 1 and k', $M(1,k') = \frac{\pi}{2K} = \omega$ [6, (22.20.6)].

(b) By squaring both sides of inequality (2.9) we find

$$\mathrm{ns}^2(r,k) - 1 \le \omega^2(\csc^2(\omega r) - 1) \le \csc^2(\omega r) - 1,$$

so $\sin \omega r \le \mathrm{sn}(r,k)$ for $0 \le r \le K$. Since $\mathrm{sn}^2(x,k) + \mathrm{cn}^2(x,k) = 1$, this implies

$$\mathrm{cn}(r,k) \le \cos(\omega r)$$

for $0 \le r \le K$, so

$$\mathrm{dn}(r,k)\,\mathrm{cn}(r,k) \le \cos(\omega r),$$

for $0 \le r \le K$. Integrating from 0 to r gives (2.10). $\square$

Lemma 2.3. *Let $\omega := \frac{\pi}{2K}$. If $\nu \ge 0$ then*

$$\omega^2(n+\nu+1)^2 \le \Lambda_\nu^n(k) \le \nu(\nu+1)(1-\omega^2) + \omega^2(n+\nu+1)^2.$$

If $-\frac{1}{2} \le \nu < 0$ *then*

$$\nu(\nu+1)(1-\omega^2)+\omega^2(n+\nu+1)^2 \le \Lambda_\nu^n(k) \le \omega^2(n+\nu+1)^2.$$

Proof. By (2.9) and (2.10),

$$\omega^2 \csc^2(\omega r) \le \operatorname{ns}^2(r,k) \le \omega^2 \csc^2(\omega r) + 1 - \omega^2$$

for $0 < r < 2K$. The bounds for $\Lambda_\nu^n(k)$ follow from the Sturm comparison theorem [8, §10.4] comparing (2.6) with (2.8). $\square$

If $\nu \ge 0$ we can also estimate $\nu(\nu+1)\operatorname{ns}^2(r,k) \ge \nu(\nu+1)$ which leads to

$$\nu(\nu+1)+\omega^2(n+1)^2 \le \Lambda_\nu^n(k). \tag{2.12}$$

Lemma 2.4. *If* $\nu \ge 0$, $n \in \mathbb{N}_0$, $-K < t < K$ *then*

$$\big\{W_\nu^n(t,k)\big\}^2 \le \frac{\pi}{2K}(n+\nu+1) \tag{2.13}$$

and

$$\operatorname{dc}(t,k)\big\{W_\nu^n(t,k)\big\}^2 \le \frac{\pi^2}{4K}(n+\nu+1)^2. \tag{2.14}$$

Proof. Note that the function $w(t) := W_\nu^n(t,k)$ is even or odd so it is enough to consider $t \in [0,K)$. Set

$$q(t) = \nu(\nu+1)k'^2 \operatorname{sc}^2(t,k) = \nu(\nu+1)(\operatorname{dc}^2(t,k)-1)$$

and $h = \nu(\nu+1) - \Lambda_\nu^n(k)$. By multiplying (2.3) by w and integrating from 0 to t with $0 \le t < K$, we get

$$w(t)w'(t) - \int_0^t w'(\tau)^2\,\mathrm{d}\tau - h\int_0^t w(\tau)^2\,\mathrm{d}\tau - \int_0^t q(\tau)w(\tau)^2\,\mathrm{d}\tau = 0.$$

Since $q \ge 0$ and $h \le 0$ by (2.12), this gives

$$\int_0^t w'(\tau)^2\,\mathrm{d}\tau \le \tfrac{1}{2}(-h) + w(t)w'(t). \tag{2.15}$$

Since $w(t)(K-t)^{-\nu-1}$ is analytic at $t=K$, $w(t)w'(t) \to 0$ as $t \to K$. Therefore, by Lemma 2.3,

$$\int_0^K w'(\tau)^2\,\mathrm{d}\tau \le \tfrac{1}{2}(-h) \le \frac{\pi^2}{8K^2}(n+\nu+1)^2.$$

Now (2.13) follows from

$$w(t)^2 = 2\int_K^t w(\tau)w'(\tau)\,\mathrm{d}\tau \le 2\left(\int_0^K w(\tau)^2\,\mathrm{d}\tau\right)^{1/2}\left(\int_0^K w'(\tau)^2\,\mathrm{d}\tau\right)^{1/2}.$$

To get (2.14) we infer from (2.15) that

$$-2w(t)w'(t) \le -h.$$

Integrating from t to K gives

$$w(t)^2 \le (-h)(K-t), \tag{2.16}$$

for $0 \le t < K$. Since $\operatorname{sn}(x,k)$ is a concave function of $x \in [0,K]$,

$$\operatorname{sn}(x,k) \ge \frac{x}{K} \tag{2.17}$$

for $x \in [0,K]$, so

$$\operatorname{dc}(t,k)w(t)^2 = \operatorname{ns}(K-t,k)w(t)^2 \le K(K-t)^{-1}w(t)^2$$

which together with (2.16) yields (2.14). $\square$

The case $\nu = -\frac{1}{2}$ (which is not covered by the preceding lemma) is treated in the following lemma.

LEMMA 2.5. *There is a positive constant C independent of n and t such that, for all $n \in \mathbb{N}_0$ and $-K < t < K$,*

$$\left\{W^n_{-\frac{1}{2}}(t,k)\right\}^2 \le C \tag{2.18}$$

and

$$\operatorname{dc}(t,k)\left\{W^n_{-\frac{1}{2}}(t,k)\right\}^2 \le C(1+n)^2. \tag{2.19}$$

PROOF. The function $w(t) := W^n_{-\frac{1}{2}}(t,k)$, $t \in (-K,K)$, satisfies the differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} + p(t)w = 0, \tag{2.20}$$

where

$$p(t) = \Lambda^n_{-\frac{1}{2}}(k) + \tfrac{1}{4} + \tfrac{1}{4}k'^2\operatorname{sc}^2(t,k).$$

By Lemma 2.3, $\Lambda^n_{-\frac12}(k) > -\frac14$ so $p(t) > 0$ and $p'(t) > 0$ for all $t \in [0, K)$. It follows from (2.20) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[w(t)^2 + \frac{1}{p(t)}w'(t)^2\right] = -\frac{p'(t)}{p(t)^2}w'(t)^2 \le 0,$$

for $t \in [0, K)$. This implies that the amplitude values of $w(t)^2$ (when $w'(t) = 0$) decrease on $[0, K)$. Since $w(t) \to 0$ as $t \to \pm K$ the maximum value of $w(t)^2$ is one of the amplitude values. If n is even the maximum value of $w(t)^2$ is $w(0)^2$. If n is sufficiently large, there is $t_0 \in [\frac13 K, \frac23 K]$ such that $w(t_0) = 0$. Then (2.18) follows from [1, Lemma 4.3] applied to the interval $[0, t_0]$. If n is odd the proof is similar.

To prove (2.19), we set

$$u(r) := \mathrm{sn}^{-1/2}(r, k)\, w(r - K),$$

for $0 < r < 2K$. Then u satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\mathrm{sn}(r, k)\,\frac{\mathrm{d}u}{\mathrm{d}r}\right) + q(r)\,\mathrm{sn}(r, k)u = 0, \tag{2.21}$$

where

$$q(r) = -h + \tfrac14 k'^2 - \tfrac34\,\mathrm{dn}^2(r, k) \le -h, \quad -h = \tfrac14 + \Lambda^n_{-\frac12}(k) > 0.$$

This differential equation has regular singular points at $r = 0, 2K$ with exponents $0, 0$. This shows that $u(r)$ is analytic at the points $r = 0, 2K$. We multiply (2.21) by $u(r)$ and integrate from 0 to r to find

$$-\,\mathrm{sn}(r, k)u(r)u'(r) + \int_0^r \mathrm{sn}(\sigma, k)u'(\sigma)^2\,\mathrm{d}\sigma = \int_0^r q(\sigma)w(\sigma - K)^2\,\mathrm{d}\sigma.$$

Let $\|w\|_\infty$ be the maximum norm of w on the interval $[-K, K]$. Then we obtain

$$-\,\mathrm{sn}(r, k)u(r)u'(r) \le (-h)r\|w\|_\infty^2,$$

for $0 \le r \le 2K$. Using (2.17) we find

$$-u(r)u'(r) \le (-h)K\|w\|_\infty^2,$$

for $0 \le r \le K$. Upon integrating the last inequality from r to K we get

$$u(r)^2 - u(K)^2 \le 2(-h)K^2\|w\|_\infty^2, \tag{2.22}$$

for $0 \le r \le K$. Since $u(K) = w(0)$ we obtain

$$u(r)^2 \le (1 + 2(-h)K^2)\|w\|_\infty.$$

Now Lemma 2.3 and (2.18) give (2.19). $\square$

We will need the following lemma.

LEMMA 2.6. *Let $u\colon [0,b]\to\mathbb{R}$ be a solution of the differential equation*

$$u''(t)=q(t)u(t),\quad t\in[0,b],$$

determined by the initial conditions $u(0)=1, u'(0)=0$ or $u(0)=0, u'(0)=1$, where $q\colon [0,b]\to\mathbb{R}$ is a continuous function. Suppose that $q(t)\ge 0$ on $[0,b]$ and $q(t)\ge\lambda^2$ on $[c,b]$ for some $\lambda>0$ and $c\in[0,b)$. Then $u(t)>0$ and $u'(t)\ge 0$ for $t\in(0,b]$, and

$$\frac{u(t)}{u(b)}\le 2\mathrm{e}^{-\lambda(b-c)},$$

for all $t\in[0,c]$.

PROOF. For the proof see the proof of [1, Lemma 4.5]. $\square$

So far we considered the Lamé–Wangerin function $W_\nu^n(t,k)$ for $t\in(-K,K)$. In the following we will also need this function for purely imaginary t. More generally, by analytic continuation, we define $W_\nu^n(t,k)$ on the strip $|\Re t|<K$.

LEMMA 2.7. *Let $0<c<b<2K'$. Then there is a constant $p\in(0,1)$ independent of ν, n and s such that*

$$0\le\frac{W_\nu^n(is,k)}{W_\nu^n(ib,k)}\le 2\,p^{n+\nu+1},$$

for $\nu\ge-\frac12$, $n\in\mathbb{N}_0$, $s\in[0,c]$.

PROOF. Let $E(s)$ be the solution of the differential equation

$$\frac{\mathrm{d}^2E}{\mathrm{d}s^2}=q(s)E,\quad q(s)=\Lambda_\nu^n(k)-\nu(\nu+1)+\nu(\nu+1)k'^2\operatorname{sn}^2(s,k'),$$

satisfying the initial conditions $E(0)=1$, $E'(0)=0$ if n is even and $E(0)=0$, $E'(0)=1$ if n is odd. Then $E(s)$ is a constant multiple of $W_\nu^n(is,k)$. If $\nu\ge 0$ then (2.12) gives

$$q(s)\ge\omega^2(n+1)^2+\nu(\nu+1)k'^2\operatorname{sn}^2(s,k')>0.$$

If $-\frac12\le\nu<0$ then Lemma 2.3 and $\omega>k'$ imply

$$q(s)\ge\omega^2(n+\nu+1)^2+\nu(\nu+1)k'^2\operatorname{sn}^2(s,k')\ge\tfrac14\omega^2-\tfrac14k'^2>0.$$

In both cases we apply Lemma 2.6 to obtain the desired result. $\square$

3. Peanut harmonics in flat-ring coordinates

In a previous paper [1], we studied the internal and external harmonics in flat-ring coordinates associated with coordinate surfaces which are flat-rings. In the present work we now investigate the harmonics associated with the coordinate surfaces which are orthogonal to the flat-rings, namely what we refer to as a peanut cyclidic surface.

3.1. Flat-ring coordinates. Flat-ring cyclide coordinates is an orthogonal curvilinear coordinate system in $\mathbb{R}^3$. These coordinates are connected to Cartesian coordinates $\mathbf{r} = (x, y, z)$ by the transformation

$$x = R\cos\phi, \quad y = R\sin\phi, \quad z = -ikR\,\mathrm{sn}(s,k)\,\mathrm{sn}(it,k), \tag{3.1}$$

where $s \in (0, 2K)$, $t \in (-K', K')$, $\phi \in [-\pi, \pi)$, and

$$\frac{1}{R} = \frac{1}{k'}\,\mathrm{dn}(s,k)\,\mathrm{dn}(it,k) + \frac{k}{k'}\,\mathrm{cn}(s,k)\,\mathrm{cn}(it,k). \tag{3.2}$$

If one uses Jacobi's imaginary transformation [6, Table 22.6.1], namely

$$\mathrm{sn}(iz,k) = i\,\mathrm{sc}(z,k'), \quad \mathrm{cn}(iz,k) = \mathrm{nc}(z,k'), \quad \mathrm{dn}(iz,k) = \mathrm{dc}(z,k'), \tag{3.3}$$

then we can rewrite R, z as follows

$$R = \frac{k'\,\mathrm{cn}(t,k')}{k\,\mathrm{cn}(s,k) + \mathrm{dn}(s,k)\,\mathrm{dn}(t,k')}, \quad z = \frac{kk'\,\mathrm{sn}(s,k)\,\mathrm{sn}(t,k')}{k\,\mathrm{cn}(s,k) + \mathrm{dn}(s,k)\,\mathrm{dn}(t,k')}. \tag{3.4}$$

Since the Jacobian elliptic functions $\mathrm{cn}(s,k)$, $\mathrm{sn}(s,k)$, $\mathrm{dn}(s,k)$ depend on the modulus $k \in (0,1)$, this is actually a family of coordinate systems depending on the parameter $k \in (0,1)$. If we set $\phi = 0$, we obtain a coordinate system s, t in the half-plane $x > 0$, $z \in \mathbb{R}$. The rectangle $s \in (0, 2K)$, $t \in (-K', K')$ is mapped bijectively onto the set

$$Q_2 = \big\{(x,z) : x > 0\big\} \setminus \big\{(x,0) : x \in [0,b] \cup [b^{-1}, \infty)\big\}, \tag{3.5}$$

where

$$b = \frac{1-k}{k'} = \sqrt{\frac{1-k}{1+k}} \in (0,1), \tag{3.6}$$

for $k \in (0,1)$. The region Q_2 is the right-hand half-plane with cuts along the x-axis from $x = 0$ to $x = b$ and from b^{-1} to ∞. This was shown in [1, Section 2]. Some coordinate lines of this coordinate system are depicted in Figure 1.

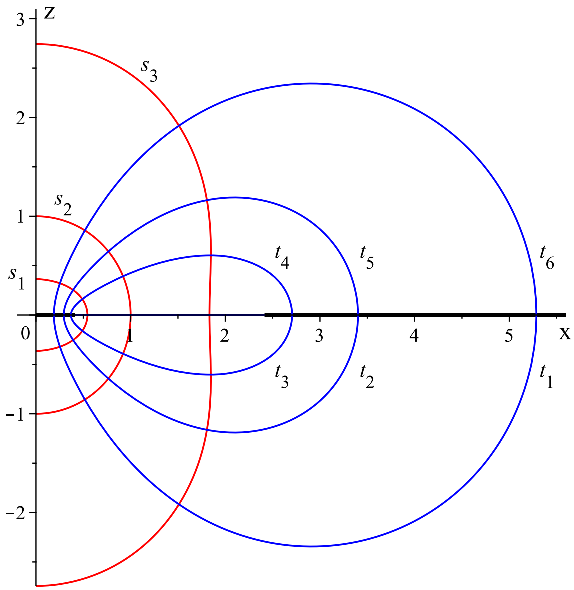

Figure 1: Coordinate lines $s = s_i$, $s_1 = 0.5K$, $s_2 = K$, $s_3 = 1.5K$ and $t = t_i$, $t_1 = -0.7K'$, $t_2 = -0.5K'$, $t_3 = -0.3K'$, $t_4 = 0.3K'$, $t_5 = 0.5K'$, $t_6 = 0.7K'$ of planar flat-ring coordinates for $k = 2^{-1/2}$.

Let $\mathbf{r} \neq \mathbf{0}$ be a point in $\mathbb{R}^3$. The inversion $\sigma(\mathbf{r})$ of $\mathbf{r}$ at the unit sphere $x^2 + y^2 + z^2 = 1$ is given by

$$\sigma(\mathbf{r}) = \|\mathbf{r}\|^{-2}\mathbf{r}.$$

Let s, t, ϕ be flat-ring coordinates of $\mathbf{r}$ and set $\tilde{s} = 2K - s$. Let $\tilde{R}$ be given by (3.2) with s replaced by $\tilde{s}$. Since $\operatorname{sn}(2K - s, k) = \operatorname{sn}(s, k)$, $\operatorname{cn}(2K - s, k) = -\operatorname{cn}(s, k)$, $\operatorname{dn}(2K - s, k) = \operatorname{dn}(s, k)$, a computation shows that $R = \tilde{R}\,\|\mathbf{r}\|^2$ and this implies that the point $\tilde{\mathbf{r}}$ with flat-ring coordinates $\tilde{s}, t, \phi$ agrees with $\sigma(\mathbf{r})$. Therefore, the inversion at the unit sphere is given in flat-ring coordinates by $s \mapsto 2K - s$.

For a fixed value $s_0 \in (0, 2K)$ the coordinate surface $s = s_0$ describes a closed surface (adding two intersection points with the z-axis where the coordinate system is not valid). We call this closed surface a peanut (see Figure 2). The surface $s = 2K - s_0$ is the inversion of the surface $s = s_0$. Note that the surface $s = K$ is the unit sphere. Let D_2 denote the interior of the peanut surface $s = s_0$ which is given by $s < s_0$ (adding parts of the z-axis and the disk $R \leq b$ in the plane $z = 0$).

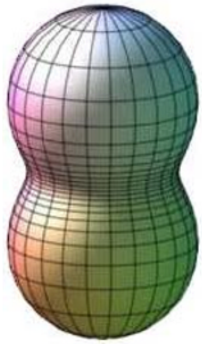

Figure 2: Peanut, $k = 0.5$, $s = 1.7K$.

The surface $s = s_0$ as well as the domain D_2 can be expressed in Cartesian coordinates as follows. If $0 < s_0 < 2K$, $s_0 \neq K$, we define the function

$$\Omega(\mathbf{r}) := \frac{k^2(\|\mathbf{r}\|^2 + 1)^2}{\mathrm{dn}^2(s_0, k)} - \frac{(\|\mathbf{r}\|^2 - 1)^2}{\mathrm{cn}^2(s_0, k)} + \frac{4z^2}{\mathrm{sn}^2(s_0, k)}. \tag{3.7}$$

If $0 < s_0 < K$ then the coordinate surface $s = s_0$ is given by the part of the surface $\Omega(\mathbf{r}) = 0$ [1, (2.11)] which lies inside the unit sphere. If $K < s_0 < 2K$ then the coordinate surface $s = s_0$ is given by the part of the surface $\Omega(\mathbf{r}) = 0$ which lies outside of the unit sphere. All coordinate surfaces $s = s_0$ intersect the plane $z = 0$ in the annulus $b < R < b^{-1}$, where $R = \sqrt{x^2 + y^2}$. Between the surfaces $s = s_0$ and $s = 2K - s_0$ we have $\Omega(\mathbf{r}) > 0$ and interior to the smaller surface and exterior to the larger surface we have $\Omega(\mathbf{r}) < 0$. We can check this by considering $\mathbf{r} = \mathbf{0}$ and $\|\mathbf{r}\| = 1$.

We distinguish three cases:

• If $s_0 = K$ then the surface $s = s_0$ is the unit sphere and D_2 is the open unit ball

$$B = \{\mathbf{r} : \|\mathbf{r}\| < 1\}.$$

Notice that $\Omega(\mathbf{r})$ is not well-defined in this case since [6, Table 22.5.1] $\mathrm{cn}(K, k) = 0$.

• If $0 < s_0 < K$ then the surface $s = s_0$ lies in B and

$$D_2 = B \cap \{\mathbf{r} : \Omega(\mathbf{r}) < 0\}.$$

• If $K < s_0 < 2K$ then the surface $s = s_0$ is exterior to the unit sphere and

$$D_2 = B \cup \{\mathbf{r} : \Omega(\mathbf{r}) > 0\}.$$

3.2. Internal peanut harmonics. Let $m \in \mathbb{Z}$, $\nu = |m| - \frac{1}{2}$, $k \in (0, 1)$, $h \in \mathbb{C}$ and $\lambda = \nu(\nu + 1) - h$. Let $u_1(s)$, $0 < s < 2K$, be a solution of the Lamé equation (2.1), and let $u_2(t)$, $-K' < t < K'$, be a solution of the modified Lamé equation (2.2). Let $\mathbf{r} = (x, y, z)$, $R = \sqrt{x^2 + y^2}$ (3.4). Then, by [1, Theorem 3.1], the function

$$u(\mathbf{r}) = \frac{1}{\sqrt{R}} u_1(s) u_2(t) \mathrm{e}^{im\phi}, \tag{3.8}$$

is harmonic in $\mathbb{R}^3$ except for the z-axis and the set $\{(x, y, 0) : R \le b$ or $R \ge b^{-1}\}$, where

$$b := b(k) = \sqrt{\frac{1-k}{1+k}}, \tag{3.9}$$

is defined in (3.6). Such a function will be called an internal peanut harmonic if it is harmonic on each domain D_2 considered in §3.1. Therefore, an internal peanut harmonic is harmonic on the union of all domains D_2, that is, on all of $\mathbb{R}^3$ except for the set $\{(x, y, 0) : R \ge b^{-1}\}$. In this case $u(\mathbf{r})$ has to stay bounded when we approach the positive and negative z-axis, so the function $|t \mp K'|^{-1/2} u_2(t)$ has to stay bounded as $t \to \pm K'$. Therefore, we arrive at the eigenvalue problem treated in Section 2. Correspondingly, we take $\lambda = \Lambda^n_{|m|-\frac{1}{2}}(k')$ and $u_2(t) = W^n_{|m|-\frac{1}{2}}(t, k')$ Then we require that the function $u_1(s) u_2(t)$ is analytic in the right-hand half plane $x > 0, z \in \mathbb{R}$ except the segment between b^{-1} and $+\infty$ on the x-axis. As in [1, Section 5.1] we see that this implies that u_1 and u_2 are both even or both odd functions. We take $u_1(s) = u_2(is)$. Thus we define internal peanut harmonics by

$$G^n_m(\mathbf{r}) = \frac{1}{\sqrt{R}} W^n_{|m|-\frac{1}{2}}(is, k') W^n_{|m|-\frac{1}{2}}(t, k') \mathrm{e}^{im\phi}, \tag{3.10}$$

for $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$. We now collect some properties of internal peanut harmonics in the following theorem.

THEOREM 3.1. *The internal peanut harmonics G^n_m are harmonic on $\mathbb{R}^3$ except for the set $\{(x, y, 0) : R \ge b^{-1}\}$. Moreover, we have*

$$G^n_m(x, y, -z) = (-1)^n G^n_m(\mathbf{r}). \tag{3.11}$$

PROOF. From our discussion at the beginning of this subsection we know that G^n_m is a harmonic function on $\mathbb{R}^3$ except in the annulus $\{(x, y, 0) : x^2 + y^2 \ge b^{-2}\}$, the z-axis, and the circle centered at the origin with radius b in the xy-plane G^n_m is bounded in a neighborhood of the circle. Therefore, the circle is a removable singularity of G^n_m [9, Theorem XIII, p. 271]. Since G^n_m stays bounded when we approach the z-axis, the z-axis is also a removable singularity. Hence, G^n_m is harmonic on the desired domain. The reflection $z \mapsto -z$ is expressed by $t \mapsto -t$ which implies (3.11). $\square$

3.3. The Dirichlet problem for internal peanut harmonics. Theorem 2.1 implies the following theorem.

THEOREM 3.2. *The system of functions*

$$J^k_{m,n}(t,\phi) := \frac{1}{\sqrt{2\pi}} W^{\,n}_{|m|-\frac{1}{2}}(t,k')\mathrm{e}^{im\phi},$$

$m \in \mathbb{Z}$, $n \in \mathbb{N}_0$, *is an orthonormal basis in the Hilbert space*

$$H_2 = L^2((-K',K') \times (-\pi,\pi)).$$

We now solve the Dirichlet problem for the peanut region D_2 given by $s < s_0$. We say that a harmonic function u defined in D_2 attains the boundary values f on ∂D_2 in the weak sense if $\sqrt{R}u$ (expressed in terms of flat-ring coordinates s,t,ϕ) evaluated at $s_1 \in (0,s_0)$ converges to $\sqrt{R}f$ in the Hilbert space H_2 as $s_1 \to s_0$. Notice that the peanut region D_2 (in contrast to the flat-ring region D_1) meets the z-axis so that the factor $\sqrt{R}$ cannot be omitted in this definition. As in [1, Section 5.2], the solution of the Dirichlet problem is unique.

THEOREM 3.3. *Let f be a function defined on the boundary ∂D_2 of the region D_2 for some $s_0 \in (0,2K)$. Suppose that f is represented in flat-ring coordinates as*

$$\sqrt{R}f(\mathbf{r}) = g(t,\phi), \quad t \in (-K',K'), \quad \phi \in (-\pi,\pi)$$

such that $g \in H_2$. For all $m \in \mathbb{Z}$ and $n \in \mathbb{N}_0$. Define

$$\begin{aligned} c^n_m &:= \frac{1}{2\pi W^n_{|m|-\frac{1}{2}}(is_0,k')} \int_{-\pi}^{\pi} \mathrm{e}^{-im\phi} \int_{-K'}^{K'} g(t,\phi) W^n_{|m|-\frac{1}{2}}(t,k')\,\mathrm{d}t\,\mathrm{d}\phi \\ &= \frac{1}{2\pi \big\{W^n_{|m|-\frac{1}{2}}(is_0,k')\big\}^2} \int_{\partial D_2} \frac{1}{h(\mathbf{r})} f(\mathbf{r}) G^n_{-m}(\mathbf{r})\,\mathrm{d}S(\mathbf{r}), \end{aligned}$$

where $h(\mathbf{r}) = kR(\mathrm{sn}^2(s,k) - \mathrm{sn}^2(it,k))^{1/2}$. Then the function

$$u(\mathbf{r}) = \sum_{m\in\mathbb{Z}} \sum_{n=0}^{\infty} c^n_m G^n_m(\mathbf{r}) \tag{3.12}$$

is harmonic in D_2 and it attains the boundary values f on ∂D_2 in the weak sense. The infinite series in (3.12) converges absolutely and uniformly in compact subsets of D_2.

PROOF. As in the proof of [1, Theorem 5.3], we see that the two formulas for c_m^n agree. Using (3.2) we find

$$\frac{1}{R}=\frac{1}{k'}\operatorname{dn}(s,k)\operatorname{dc}(t,k')+\frac{k}{k'}\operatorname{cn}(s,k)\operatorname{nc}(t,k')\le\frac{2}{k'}\operatorname{dc}(t,k').$$

Let $t\in(-K',K')$, $\phi\in(-\pi,\pi)$ and $0<s\le s_1<s_0$. Using Lemmas 2.4, 2.5, 2.7 we estimate

$$\frac{1}{\sqrt{R}}\left|\frac{W^n_{|m|-\frac12}(is,k')}{W^n_{|m|-\frac12}(is_0,k')}W^n_{|m|-\frac12}(t,k')\mathrm{e}^{im\phi}\right|\le Cp^{|m|+n}(1+|m|+n),$$

where the constants C and $p\in(0,1)$ are independent of m,n,s,t,ϕ. Since $\{c_m^n\}$ is a bounded double sequences, this proves that the series in (3.12) is absolutely and uniformly convergent on compact subsets of D_2. Consequently, by Theorem 3.1, u defined by (3.12) is harmonic in D_2. We show that u attains the boundary values f on ∂D_2 in the weak sense by the same method as used in the proof of [1, Theorem 5.3]. In this argument we use Theorem 3.2. $\square$

3.4. External peanut harmonics. External peanut harmonics are harmonic functions u of the form (3.8) which are harmonic outside any peanut region D_2, that is, on all of $\mathbb{R}^3$ except the disk centered at the origin with radius b in the xy-plane. External peanut harmonics can simply be defined by the Kelvin transformation [9, Ch. IX, §2] of internal peanut harmonics. Note that this method was not available for flat-ring harmonics because the Kelvin transformation of an internal flat-ring harmonic is again an internal flat-ring harmonic.

More explicitly, we define external peanut harmonics by

$$H_m^n(\mathbf{r})=\frac{1}{\sqrt{R}}W^n_{|m|-\frac12}(2iK-is,k')W^n_{|m|-\frac12}(t,k')\mathrm{e}^{im\phi}, \tag{3.13}$$

for $m\in\mathbb{Z}$, $n\in\mathbb{N}_0$.

THEOREM 3.4. *External peanut harmonics H_m^n are harmonic functions on $\mathbb{R}^3$ except for the disk centered at the origin with radius b in the xy-plane. Moreover,*

$$G_m^n(\sigma(\mathbf{r}))=\|\mathbf{r}\|H_m^n(\mathbf{r}), \tag{3.14}$$

$$H_m^n(x,y,-z)=(-1)^nH_m^n(\mathbf{r}), \tag{3.15}$$

$$\lim_{\|\mathbf{r}\|\to\infty}H_m^n(\mathbf{r})=0. \tag{3.16}$$

Proof. As mentioned in Section 3.1 inversion at the unit sphere is expressed in flat-ring coordinates by $s \mapsto 2K - s$. This implies (3.14). Theorem 3.1 and (3.14) shows that H_m^n is harmonic on the desired domain. Equations (3.15) and (3.16) also follow from Theorem 3.1 and (3.14). $\square$

External harmonics admit an integral representation in terms of internal harmonics. Define the Wronskian w_m^n by

$$w_m^n := V(s)U'(s) - V'(s)U(s), \tag{3.17}$$

where $U(s) := W^n_{|m|-\frac{1}{2}}(is, k')$, $V(s) := U(2K - s)$. The function $U(s)$ is a constant multiple of the solution $E(s)$ of (2.1) determined by initial conditions $E(0) = 1$, $E'(0) = 0$ if n is even and $E(0) = 0$, $E'(0) = 1$ if n is odd. This equation has the form $E'' = q(s)E$ with $q(s) > 0$. Therefore, $E(s) > 0$ and $E'(s) > 0$ for $s > 0$ which implies that $w_m^n \neq 0$.

Theorem 3.5. *Let $s_0 \in (0, 2K)$, $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$, and let $\mathbf{r}^*$ be a point outside $\overline{D}_2$, where D_2 is the region given by $s < s_0$. Then*

$$H_m^n(\mathbf{r}^*) = \frac{w_m^n}{4\pi \big\{ W^n_{|m|-\frac{1}{2}}(is_0, k') \big\}^2} \int_{\partial D_2} \frac{G_m^n(\mathbf{r})}{h(\mathbf{r}) \|\mathbf{r} - \mathbf{r}^*\|} \, \mathrm{d}S(\mathbf{r}). \tag{3.18}$$

We omit the proof of this theorem which is very similar to the proof of [1, Theorem 5.5].

3.5. Expansion of the fundamental solution. We obtain the desired expansion of $\|\mathbf{r} - \mathbf{r}^*\|^{-1}$ in internal and external peanut harmonics by combining Theorems 3.3 and 3.5.

Theorem 3.6. *Let $\mathbf{r}, \mathbf{r}^* \in \mathbb{R}^3$ with flat-ring coordinates $s, s^* \in (0, 2K)$, $t, t^* \in (-K', K')$, respectively. If $s < s^*$ then*

$$\frac{1}{\|\mathbf{r} - \mathbf{r}^*\|} = 2 \sum_{m \in \mathbb{Z}} \sum_{n=0}^{\infty} \frac{1}{w_m^n} G_m^n(\mathbf{r}) H_{-m}^n(\mathbf{r}^*). \tag{3.19}$$

Since we now have (3.19), we can follow [3,5] in order to obtain an addition theorem for the associated Legendre function of the second kind with odd-half-integer degree in terms of Lamé–Wangerin functions. This proceeds through comparison of the azimuthal Fourier component of the $1/r$ potential in rotationally-invariant coordinate systems which separates Laplace's equation, such as flat-ring cyclide coordinates.

THEOREM 3.7. *Let* $m \in \mathbb{N}_0$, $0 < s < s^* < 2K$, $t, t^* \in (-K', K')$. *Then*

$$Q_{m-\frac{1}{2}}(\chi) = 2\pi \sum_{n=0}^{\infty} \frac{1}{w_m^n} \times W_{m-\frac{1}{2}}^n(is, k') W_{m-\frac{1}{2}}^n(t, k') W_{m-\frac{1}{2}}^n(2iK - is^*, k') W_{m-\frac{1}{2}}^n(t^*, k'), \tag{3.20}$$

where $\chi\colon ((0, 2K) \times (-K', K'))^2 \times (0, 1) \to \mathbb{R}$ *is given by*

$$\begin{aligned} \chi := \chi(s, t, s^*, t^*; k) := {} & k^2 \operatorname{sn}(s, k) \operatorname{sn}(it, k) \operatorname{sn}(s^*, k) \operatorname{sn}(it^*, k) \\ & - \frac{k^2}{k'^2} \operatorname{cn}(s, k) \operatorname{cn}(it, k) \operatorname{cn}(s^*, k) \operatorname{cn}(it^*, k) \\ & + \frac{1}{k'^2} \operatorname{dn}(s, k) \operatorname{dn}(it, k) \operatorname{dn}(s^*, k) \operatorname{dn}(it^*, k). \end{aligned} \tag{3.21}$$

PROOF. This follows from comparison of (3.19) with the azimuthal Fourier expansion [4, (15)]

$$\frac{1}{\|\mathbf{r} - \mathbf{r}^*\|} = \frac{1}{\pi\sqrt{RR^*}} \sum_{m=0}^{\infty} Q_{m-\frac{1}{2}}(\chi) \mathrm{e}^{im(\phi-\phi^*)},$$

where

$$\chi = \frac{R^2 + R^{*2} + (z - z^*)^2}{2RR^*},$$

with R, R^*, z, z^* given in terms of s, t and s^*, t^* respectively (3.4), and (3.7) is given in [1, Lemma 5.7]. $\square$

The addition Theorem 3.7 leads to an integral relation for Lamé–Wangerin functions.

THEOREM 3.8. *Let* $m, n \in \mathbb{N}_0$, $0 < s < s^* < 2K$, $-K' < t^* < K'$. *Then*

$$\int_{-K'}^{K'} Q_{m-\frac{1}{2}}(\chi) W_{m-\frac{1}{2}}^n(t, k') \, \mathrm{d}t = \frac{2\pi}{w_m^n} W_{m-\frac{1}{2}}^n(is, k') W_{m-\frac{1}{2}}^n(2iK - is^*, k') W_{m-\frac{1}{2}}^n(t^*, k'). \tag{3.22}$$

Theorem 3.8 is a new result. However, we are able to improve upon it by using the method of fundamental solutions employed in [13].

In order to simplify notation we set

$$V_\nu(s) := W_\nu^n(is, k), \tag{3.23}$$

for some $|\Im s| < K'$, $n \in \mathbb{N}_0$, $0 < k < 1$, $\nu \geq -\frac{1}{2}$. Then V_ν is a solution of Lamé's equation (2.1) with $h = \nu(\nu+1) - \Lambda_\nu^n(k)$, and it is a Fuchs-Frobenius solution at both regular singular points $s = \pm iK'$ belonging to the exponent $\nu + 1$.

THEOREM 3.9. *Let $V_\nu(z)$ be as in (3.23). If $0 < s_0 < 2K$, $-s_0 < s_1 < s_0$, $-K' < t_0 < K'$ then*

(3.24)
$$2\pi\, V_\nu(2K - s_0)V_\nu(it_0)V_\nu(s_1) = [\tilde{V}_\nu, V_\nu] \int_{-K'}^{K'} Q_\nu(\chi(s_1, t, s_0, t_0))V_\nu(it)\, \mathrm{d}t,$$

where $[V_\nu, \tilde{V}_\nu]$ denotes the Wronskian of V_ν and $\tilde{V}_\nu(z) = V_\nu(2K - z)$. Since V_ν and $\tilde{V}_\nu$ are both solutions of (2.1), the Wronskian is a constant.

PROOF. We define

$$u(s,t) = V_\nu(2K - s)V_\nu(it), \quad v(s,t) = Q_\nu(\chi(s, t, s_0, t_0)).$$

The function $v(s,t)$ is well-defined for $(s,t) \in \mathbb{R} \times (-K', K')$ except for logarithmic singularities at the points $(s_0 + 4jK, t_0)$ and $(-s_0 + 4jK, -t_0)$ with $j \in \mathbb{Z}$ [13, Lemma 1.3]. Let $s_2 = 2K$ and $-K' < t_1 < t_0 < t_2 < K'$. Let $C_1 + C_2 + C_3 + C_4$ be the rectangular path as shown in Figure 3. This path forms the boundary of the rectangle $[s_1, s_2] \times [t_1, t_2]$. This rectangle contains the point (s_0, t_0) but none of the other singularities of v. According to [13, Theorem 1.11] we have

$$2\pi\, u(s_0, t_0) = \sum_{j=1}^{4} \int_{C_j} \big((u\partial_2 v - v\partial_2 u)\, \mathrm{d}s + (v\partial_1 u - u\partial_1 v)\, \mathrm{d}t\big). \tag{3.25}$$

By our assumption on V_ν and [13, Lemma 2.6], the integral $\int_{C_1}(u\partial_2 v - v\partial_2 u)\, \mathrm{d}s$ converges to 0 as $t_1 \to -K'$, and the integral $\int_{C_3}(u\partial_2 v - v\partial_2 u)\, \mathrm{d}s$ converges to 0 as $t_2 \to K'$. Therefore, one obtains

$$2\pi V_\nu(2K - s_0)V_\nu(it_0) = I_1 + I_2 + I_3 + I_4, \tag{3.26}$$

where

$$I_1 = V_\nu'(2K - s_1) \int_{-K'}^{K'} v(s_1, t)V_\nu(it)\, \mathrm{d}t,$$

$$I_2 = V_\nu(2K - s_1) \int_{-K'}^{K'} \partial_1 v(s_1, t)V_\nu(it)\, \mathrm{d}t,$$

$$I_3 = -V_\nu'(0) \int_{-K'}^{K'} v(2K, t)V_\nu(it)\, \mathrm{d}t, \quad I_4 = -V_\nu(0) \int_{-K'}^{K'} \partial_1 v(2K, t)V_\nu(it)\, \mathrm{d}t.$$

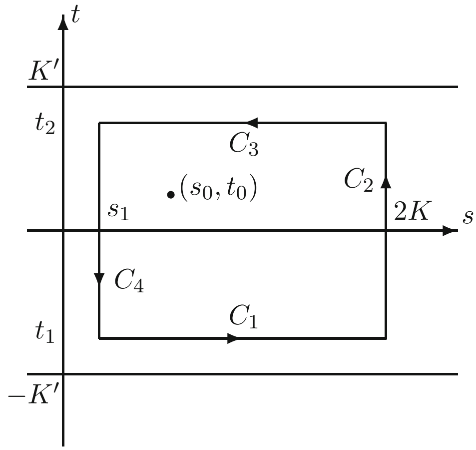

Figure 3: Path of integration used in (3.25)

The function V_ν is even or odd. If V_ν is even then $V_\nu'(0) = 0$ so $I_3 = 0$. If V_ν is odd then $v(2K, t)V_\nu(it)$ is an odd function of t, so again $I_3 = 0$. In a similar way, we see that $I_4 = 0$.

We now apply [13, Theorem 1.11] to the counter-clockwise rectangular path $\tilde{C}$ defined using the vertices (s_1, t_1), (s_1, t_2), $(0, t_1)$, $(0, t_2)$. This time we replace u by $\tilde{u}(s, t) = V_\nu(s)V_\nu(it)$. The path $\tilde{C}$ does not wind around a singularity of $v(s, t)$ so

$$\int_{\tilde{C}} (\tilde{u}\partial_2 v - v\partial_2 \tilde{u})\, \mathrm{d}s + (v\partial_1 \tilde{u} - \tilde{u}\partial_1 v)\, \mathrm{d}t = 0.$$

As before, we let $t_1 \to -K'$, $t_2 \to K'$, and note that the line integral along the segment from $(0, -K')$ to $(0, K')$ vanishes. Therefore, we obtain

$$V_\nu'(s_1) \int_{-K'}^{K'} v(s_1, t)V_\nu(it)\, \mathrm{d}t = V_\nu(s_1) \int_{-K'}^{K'} \partial_1 v(s_1, t)V_\nu(it)\, \mathrm{d}t. \tag{3.27}$$

If we combine (3.26) with $I_3 = I_4 = 0$ and (3.27) we obtain (3.24). $\square$

Theorem 3.9 implies Theorem 3.8 when we set $s_0 = s^*$, $s_1 = s$, $t_0 = t^*$, $\nu = m - \frac{1}{2}$ and replace k by k'. In [7, pp. 79- 80] integral equations for Lamé–Wangerin functions are mentioned. An improved version of these integral equations can be obtained from (3.24) by a limiting process as shown in the following theorem.

THEOREM 3.10. *Let $\nu \ge -\frac{1}{2}$ and $V_\nu(s)$ be as in (3.23). If $t_0 \in (-K', K')$ then*

(3.28)
$$V_\nu(it_0) = \frac{\mathrm{e}^{\frac{1}{2}(\nu+1)i\pi}\Gamma(\nu+1)}{2^{\nu+2}\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}\frac{[\tilde{V}_\nu, V_\nu]}{L_\nu(k)V_\nu(K-iK')}\int_{-K'}^{K'}(f(t,t_0))^{-\nu-1}V_\nu(it)\,\mathrm{d}t,$$

where

$$L_\nu(k) := \lim_{u\to K'-}\mathrm{cn}(u,k')^{-\nu-1}V_\nu(iu),$$

$$f(t,t_0) := k\,\mathrm{sn}(it,k)\,\mathrm{sn}(it_0,k) + \frac{k}{k'}\,\mathrm{cn}(it,k)\,\mathrm{cn}(it_0,k).$$

PROOF. Let $t, t_0, u \in (-K', K')$. Then the function χ defined in (3.7) satisfies

$$\chi(iu,t,K+iu,t_0;k) = ikf(t,t_0)\,\mathrm{sc}(u,k')\,\mathrm{nd}(u,k') + \frac{1}{k'}\,\mathrm{dn}(it,k)\,\mathrm{dn}(it_0,k).$$

It follows that

$$\Re\chi(iu,t,K+iu,t_0;k) = \frac{1}{k'}\,\mathrm{dn}(it,k)\,\mathrm{dn}(it_0,k) \ge \frac{1}{k'} > 1. \tag{3.29}$$

Therefore, $Q_\nu(\chi(iu,t,K+iu,t_0;k))$ is an analytic function of $(t,t_0,u) \in (-K',K')^3$. By analytic continuation it can be shown that (3.24) implies

$$\begin{aligned} &2\pi\,V_\nu(K-iu)V_\nu(it_0)V_\nu(iu) \\ &\quad= [\tilde{V}_\nu, V_\nu]\int_{-K'}^{K'}Q_\nu(\chi(iu,t,K+iu,t_0;k))V_\nu(it)\,\mathrm{d}t. \end{aligned} \tag{3.30}$$

We multiply (3.30) on both sides by $\mathrm{cn}^{-\nu-1}(u,k')$ and take the limit as $u \to K'$. Then we have on the left-hand side

$$\begin{aligned} &\lim_{u\to K'}\mathrm{cn}^{-\nu-1}(u,k')2\pi\,V_\nu(K-iu)V_\nu(it_0)V_\nu(iu) \\ &\quad= 2\pi\,V_\nu(K-iK')V_\nu(it_0)L_\nu(k). \end{aligned}$$

On the right-hand side we have

$$[\tilde{V},V]\lim_{u\to K'}\int_{-K'}^{K'}\mathrm{cn}^{-\nu-1}(u,k')Q_\nu(\chi(iu,t,K+iu,t_0;k))V(it)\,\mathrm{d}t.$$

Suppose that the limit can be taken inside the integral. So we consider

$$\lim_{u\to K'}\mathrm{cn}^{-\nu-1}(u,k')Q_\nu(\chi(iu,t,K+iu,t_0;k)) \tag{3.31}$$

$$= \mathrm{e}^{-\frac{1}{2}(\nu+1)i\pi} \frac{\sqrt{\pi}\,\Gamma(\nu+1)}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})} \left(f(t,t_0)\right)^{-\nu-1}.$$

This follows from the asymptotic behavior of the Legendre function Q_ν [6, (14.8.15)]

$$Q_\nu(z) = \frac{\sqrt{\pi}\,\Gamma(\nu+1)}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})} z^{-\nu-1}(1+O(z^{-2})),$$

as $|z| \to \infty$.

We now justify the interchange of limit and integral. Note that (3.29) implies that there is a constant $C > 0$ (independent of t, t_0, u) such that

$$|Q_\nu(\chi(iu,t,K+iu,t_0;k))| \le C|\chi(iu,t,K+iu,t_0;k)|^{-\nu-1}. \tag{3.32}$$

Moreover, we have

$$f(t,t_0) = \frac{k(k'^{-1} - \mathrm{sn}(t,k')\,\mathrm{sn}(t_0,k'))}{\mathrm{cn}(t,k')\,\mathrm{cn}(t_0,k')} \ge k\Big(\frac{1}{k'} - 1\Big) > 0. \tag{3.33}$$

Now (3.32) and (3.33) give

$$\begin{aligned} &\mathrm{cn}^{-\nu-1}(u,k')\big|Q_\nu(\chi(iu,t,K+iu,t_0;k))\big| \\ &\le C\Big\{\mathrm{cn}(u,k')|\chi(iu,t,K+iu,t_0;k)|\Big\}^{-\nu-1} \\ &\le C\Big\{f(t,t_0)\,\mathrm{sn}(u,k')\Big\}^{-\nu-1} \le C\Big\{\frac{k}{2}\Big(\frac{1}{k'}-1\Big)\Big\}^{-\nu-1}, \end{aligned}$$

provided that $\mathrm{sn}(u,k') \ge \frac{1}{2}$. This justifies the interchange of limit and integral by Lebesgue's bounded convergence theorem, and therefore the proof of (3.28) is complete. □

Note that we have also verified the above addition theorem and integral formulas numerically.

4. Application of the $k \to 1$ limit of flat-ring coordinates

In this section we show that, as $k \to 1$, flat-ring coordinates becomes spherical coordinates. Then we show how our expansion of the $1/r$ potential in peanut harmonics becomes the multipole expansion of the $1/r$ potential [11, pp. 1273-1274, (10.3.37)] in spherical coordinates.

4.1. Flat-ring coordinates in the limit $k \to 1$ are spherical coordinates. Spherical coordinates in $\mathbb{R}^3$, $r \ge 0$, $\theta \in [0,\pi]$, $\phi \in [-\pi,\pi)$ are connected to Cartesian coordinates x, y, z by the transformation

$$x = r\,\cos\phi\,\sin\theta, \quad y = r\,\sin\phi\,\sin\theta, \quad z = r\,\cos\theta. \tag{4.1}$$

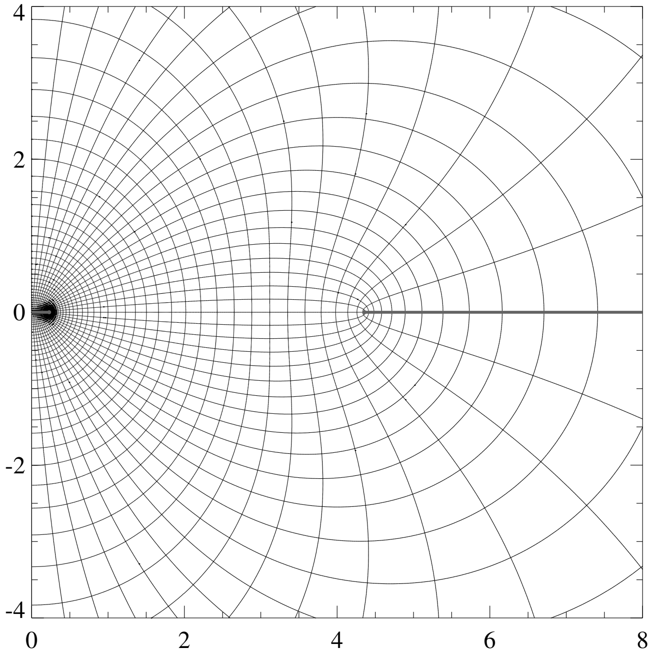

Figure 4: Flat-ring cyclide coordinates with $k = \frac{9}{10}$. Then the asymptotic inner and outer radii of the flat-rings, are respectively given $b = 1/\sqrt{19} \simeq 0.2294$ and $b^{-1} = \sqrt{19} \simeq 4.359$. Small dark grey circles are drawn at the points $(0, b)$ and $(0, b^{-1})$ (see (3.9)) and dark grey thick line segments are drawn to represent the intervals $[0, b^{-1}]$ and $[b, \infty)$. The abscissa represents the radial coordinate $R = (x^2 + y^2)^{1/2}$ and the ordinate represents the z-axis. One can see that as k is approaching 1, the coordinate system close to the origin is starting to resemble spherical coordinates. Similarly, as k approaches 1, the flat-ring coordinate surfaces near the plane $z = 0$ have larger and larger outer-radii.

To demonstrate their connection to flat-ring coordinates, let $\sigma \in \mathbb{R}$ and $\tau \in (0, \pi)$. We set $s = K + \sigma$ and $t = K' - \tau$. Then $t \in (-K', K')$ and, for k sufficiently close to 1, $s \in (0, 2K)$. Now

$$R = \frac{\mathrm{dn}(\sigma, k)\,\mathrm{sn}(\tau, k')}{1 - \mathrm{sn}(\sigma, k)\,\mathrm{dn}(\tau, k')}. \tag{4.2}$$

Therefore,

$$R \to \frac{\operatorname{sech}\sigma \sin\tau}{1 - \tanh\sigma} = \mathrm{e}^{\sigma} \sin\tau, \tag{4.3}$$

as $k \to 1$, and so

$$\lim_{k\to 1} x = \mathrm{e}^{\sigma} \sin\tau \cos\phi, \quad \lim_{k\to 1} y = \mathrm{e}^{\sigma} \sin\tau \sin\phi.$$

Moreover,

$$\lim_{k\to 1} z = \lim_{k\to 1} \frac{\operatorname{cn}(\sigma,k)\operatorname{cn}(\tau,k')}{1-\operatorname{sn}(\sigma,k)\operatorname{dn}(\tau,k')} = \frac{\operatorname{sech}\sigma\cos\tau}{1-\tanh\sigma} = \mathrm{e}^{\sigma}\cos\tau.$$

Therefore, flat-ring coordinates approach spherical coordinates r,θ,ϕ in the limit $k\to 1$ (with $\theta=\tau$, $r=\mathrm{e}^{\sigma}$.)

4.2. The limit of the Lamé–Wangerin functions $W_\nu^n(t,k)$ as $k\to 0$. Note that in our above expansions we have Lamé–Wangerin functions which are a function of $k'=\sqrt{1-k^2}$. So for those Lamé–Wangerin functions, the limit as $k\to 1$ is equivalent to the limit of the Lamé–Wangerin functions with argument k as $k\to 0$.

Let $w(t)=\mathrm{F}_\nu(t,\lambda,k)$ be the solution of equation (2.3) such that (2.4) holds with $a_0=1$. By analytic continuation, this function is well-defined in the strip $-K<\Re t<K$. We note that the Lamé–Wangerin functions W_ν^n can be written as

$$W_\nu^n(t,k)=d_\nu^n(k)\,\mathrm{F}_\nu(t,\Lambda_\nu^n(k),k), \tag{4.4}$$

where the constants $d_\nu(k)>0$ are chosen such that the normalization integral (2.7) is satisfied.

We recall the following lemma.

Lemma 4.1 [1, Lemma 6.3]. *Let D be a simply-connected domain in the complex plane $\mathbb{C}$ containing 0. Let $p_n\colon D\to\mathbb{C}$ be a sequence of analytic functions for each $n\in\mathbb{N}\cup\{\infty\}$ such that $p_n(z)\to p_\infty(z)$ locally uniformly on D, and $p_n(0)=\nu(\nu+1)$ for all $n\in\mathbb{N}\cup\{\infty\}$, where $\nu\ge-\frac12$. For each $n\in\mathbb{N}\cup\{\infty\}$, let $u_n\colon D\to\mathbb{C}$ be the unique analytic function such that $u_n(0)=1$ and $y_n(z):=z^{\nu+1}u_n(z)$ solves*

$$z^2y_n''=p_n(z)y_n. \tag{4.5}$$

Then $u_n(z)\to u_\infty(z)$ locally uniformly on D.

Theorem 4.2. *For every $n\in\mathbb{N}_0$ and $\nu\ge-\frac12$, we have*

$$\tau^{-\nu-1}\,\mathrm{F}_\nu(K-\tau,\Lambda_\nu^n(k),k)\to\left(\frac{\sin\tau}{\tau}\right)^{\nu+1}{}_2F_1\left({-n,n+2\nu+2 \atop \nu+\frac32};\sin^2\tfrac{\tau}{2}\right) \tag{4.6}$$

as $k\to 0$ locally uniformly for $|\Re\tau|<\pi$.

Proof. The function $w(\tau):=\mathrm{F}_\nu(K-\tau,\Lambda_\nu^n(k),k)$ satisfies the differential equation

$$w''+\left(\Lambda_\nu^n(k)-\nu(\nu+1)\operatorname{ns}^2(\tau,k)\right)w=0. \tag{4.7}$$

It follows from Lemma 2.3 that $\Lambda_\nu^n(k) \to (n+\nu+1)^2$ as $k \to 0$. Moreover,

$$\operatorname{sn}(z,k) \to \sin z, \quad \operatorname{cn}(z,k) \to \cos z, \quad \operatorname{dn}(z,k) \to 1 \tag{4.8}$$

locally uniformly on $\mathbb{C}$ [6, (22.5.3)]. Now (4.8) and the maximum principle for analytic functions give

$$\frac{\tau}{\operatorname{sn}(\tau,k)} \to \frac{\tau}{\sin\tau}$$

as $k \to 0$ locally uniformly for $|\Re\tau| < \pi$. Therefore, we can apply Lemma 4.1 to the differential equations (4.7) with the limit differential equation

$$v'' + \big((n+\nu+1)^2 - \nu(\nu+1)\csc^2\tau\big)v = 0. \tag{4.9}$$

The right-hand side of (4.6) tends to 1 as $\tau \to 0$, and when multiplied by $\tau^{\nu+1}$ it is a solution of (4.9) belonging to the exponent $\nu+1$ at $\tau = 0$. Therefore, (4.6) follows from Lemma 4.1. $\square$

Gegenbauer polynomials $C_n^\lambda(x)$ are given by [6, (15.9.2)]

$$C_n^\lambda(x) = \frac{(2\lambda)_n}{n!}\,{}_2F_1\!\left(\begin{matrix}-n, 2\lambda+n\\ \lambda+\frac12\end{matrix}; \frac{1-x}{2}\right).$$

Therefore, the limit in (4.6) can be expressed in terms of Gegenbauer polynomials

$${}_2F_1\!\left(\begin{matrix}-n, n+2\nu+2\\ \nu+\frac32\end{matrix}; \sin^2\tfrac{\tau}{2}\right) = \frac{n!}{(2\nu+2)_n} C_n^{\nu+1}(\cos\tau).$$

Since the convergence is uniform for $\tau \in [0, \pi-\delta]$ for every $\delta > 0$, we obtain for $w(\tau) := \mathrm{F}_\nu(K-\tau, \Lambda_\nu^n(k), k)$

$$\int_{-K}^{K} w(\tau)^2\,\mathrm{d}\tau = 2\int_0^K w(\tau)^2\,\mathrm{d}\tau$$
$$\to \left(\frac{n!}{(2\nu+2)_n}\right)^2 \int_{-1}^{1} (1-x^2)^{\nu+\frac12}\left(C_n^{\nu+1}(x)\right)^2\,\mathrm{d}x,$$

as $k \to 0$. Define e_ν^n such that

$$e_\nu^n := \int_{-1}^{1} (1-x^2)^{\nu+\frac12}\big(C_n^{\nu+1}(x)\big)^2\,\mathrm{d}x = \frac{\pi}{2^{2\nu+1}n!}\frac{\Gamma(n+2\nu+2)}{(n+\nu+1)\Gamma(\nu+1)^2},$$

whose value follows from [12, (4.7.15)]. Therefore, Theorem 4.2 implies the following result.

THEOREM 4.3. *For every* $n \in \mathbb{N}_0$ *and* $\nu \geq -\frac{1}{2}$, *we have*

$$\tau^{-\nu-1} W_\nu^n(K-\tau,k) \to (e_\nu^n)^{-1/2} \left(\frac{\sin\tau}{\tau}\right)^{\nu+1} C_n^{\nu+1}(\cos\tau)$$

as $k \to 0$ *locally uniformly for* $|\Re\tau| < \pi$.

In the application to peanut harmonics we are interested in the special case $\nu = m - \frac{1}{2}$ with $m \in \mathbb{N}_0$. The Ferrers function of the first kind P_ν^μ satisfies the identity [6, (18.11.1)]

$$\mathsf{P}_{m+n}^m(x) = \left(-\tfrac{1}{2}\right)^m \frac{(2m)!}{m!} (1-x^2)^{m/2} C_n^{m+\frac{1}{2}}(x),$$

where $m, n \in \mathbb{N}_0$. Also using the duplication formula [6, (5.5.5)]

$$\Gamma(m+\tfrac{1}{2})\Gamma(m+1) = 2^{-2m}\sqrt{\pi}\,\Gamma(2m+1)$$

we obtain the following corollary.

COROLLARY 4.4. *For* $m, n \in \mathbb{N}_0$ *we have*

$$\tau^{-m-\frac{1}{2}} W_{m-\frac{1}{2}}^n(K-\tau,k)$$
$$\to \left(\frac{(m+n+\frac{1}{2})n!}{(2m+n)!}\right)^{1/2} (-\tau)^{-m} \left(\frac{\sin\tau}{\tau}\right)^{1/2} \mathsf{P}_{m+n}^m(\cos\tau),$$

as $k \to 0$ *locally uniformly for* $|\Re\tau| < \pi$.

We now determine the limit of Lamé–Wangerin functions W_ν^n on the imaginary axis. Note that W_ν^n takes on real values on this line if n is even and purely imaginary values if n is odd. We recall [1, Lemma 6.1].

LEMMA 4.5. (a) *Let* $a \in \mathbb{R}$ *and let* $\{b_n\}$ *be a sequence of real numbers such that* $a < b_n \to \infty$ *as* $n \to \infty$.

(b) *For* $n \in \mathbb{N}$, *let* $p_n, q_n \colon [a, b_n] \to \mathbb{R}$ *be continuous functions such that* $q_n(x) < 0$ *for all* $x \in [a, b_n]$. *For every* $n \in \mathbb{N}$, *let* $y_n \colon [a, b_n] \to \mathbb{R}$ *be a non-trivial solution of the differential equation*

$$y_n'' + p_n(x) y_n' + q_n(x) y_n = 0 \tag{4.10}$$

such that $y_n(b_n) = 0$.

(c) *Let* $p_\infty, q_\infty \colon [a, \infty) \to \mathbb{R}$ *be continuous functions such that* $p_n(x) \to p_\infty(x)$ *and* $q_n(x) \to q_\infty(x)$ *as* $n \to \infty$ *uniformly on each compact interval* $[a, b]$. *Suppose that the differential equation*

$$y_\infty'' + p_\infty(x) y_\infty' + q_\infty(x) y_\infty = 0 \tag{4.11}$$

admits a bounded nontrivial solution $y_\infty\colon [a,\infty) \to \mathbb{R}$, *and that every solution of* (4.11) *which is linearly independent of* y_∞ *is unbounded as* $x \to \infty$.

Under assumptions (*a*), (*b*), (*c*), *we have*

$$\frac{y_n(x)}{y_n(a)} \to \frac{y_\infty(x)}{y_\infty(a)} \quad \textit{and} \quad \frac{y_n'(x)}{y_n(a)} \to \frac{y_\infty'(x)}{y_\infty(a)} \tag{4.12}$$

as $n \to \infty$ *uniformly on every compact interval* $[a,b]$. *The same result is true if the condition* $y_n(b_n) = 0$ *is replaced by* $y_n'(b_n) = 0$.

We also use the following well-known lemma.

LEMMA 4.6. *Let* D *be a simply-connected domain in* $\mathbb{C}$, *and* $a \in D$. *For* $n \in \mathbb{N}$, *let* $p_n, q_n, p_\infty, q_\infty\colon D \to \mathbb{C}$ *be analytic functions such that* $p_n(z) \to p_\infty(z)$ *and* $q_n(z) \to q_\infty(z)$ *locally uniformly for* $z \in D$. *For each* $n \in \mathbb{N}$ *let* $y_n\colon D \to \mathbb{C}$ *be a solution of the differential equation*

$$y_n'' + p_n(z)y_n' + q_n(z)y_n = 0,$$

and let $y_\infty\colon D \to \mathbb{C}$ *be a solution of*

$$y_\infty'' + p_\infty(z)y_\infty' + q_\infty(z)y_\infty = 0.$$

If

$$y_n(a) \to y_\infty(a), \quad y_n'(a) \to y_\infty'(a),$$

as $n \to \infty$, *then*

$$y_n(z) \to y_\infty(z),$$

as $n \to \infty$ *locally uniformly for* $z \in D$.

THEOREM 4.7. *For* $n \in \mathbb{N}_0$ *and* $\nu \geq -\frac{1}{2}$, *we have*

$$\frac{W_\nu^n(i(K'-\sigma),k)}{W_\nu^n(iK',k)} \to \mathrm{e}^{-(n+\nu+1)\sigma} \tag{4.13}$$

as $k \to 0$ *locally uniformly for* $|\Im\sigma| < \frac{1}{2}\pi$.

PROOF. The function $w(\sigma) = W_\nu^n(i(K'-\sigma),k)$ satisfies the differential equation

$$\frac{\mathrm{d}^2 w}{d\sigma^2} + q(\sigma,k)w = 0, \tag{4.14}$$

where

$$q(\sigma,k) := -\Lambda_\nu^n(k) + \nu(\nu+1)k^2\,\mathrm{nd}^2(\sigma,k').$$

It follows from Lemma 2.3 and (2.12) that $q(\sigma,k)<0$ for all $\sigma\in\mathbb{R}$. Moreover, $\mathrm{nd}(\sigma,k')\to\cosh\sigma$ as $k\to 0$ locally uniformly for $|\Im\sigma|<\frac{1}{2}\pi$, so

$$q(\sigma,k)\to-(n+\nu+1)^2 \tag{4.15}$$

as $k\to 0$ locally uniformly for $|\Im\sigma|<\frac{1}{2}\pi$. We have $w(K')=0$ if n is even and $w'(K')=0$ if n is odd. Since $K'(k)\to\infty$ as $k\to 0$, we can apply Lemma 4.5 (with $a=0$) and obtain (4.13) and its differentiated form with uniform convergence for $\sigma\in[0,1]$. Local uniform convergence for $|\Im\sigma|<\frac{1}{2}\pi$ follows from Lemma 4.6. □

4.3. The peanut expansion of the $1/r$ potential in the limit $k\to 1$. In spherical coordinates we have internal spherical harmonics

$$G_n^m(\mathbf{r})=r^n\,\mathsf{P}_n^m(\cos\theta)\mathrm{e}^{im\phi}$$

and external spherical harmonics

$$H_n^m(\mathbf{r})=r^{-n-1}\,\mathsf{P}_n^m(\cos\theta)\mathrm{e}^{im\phi},$$

where P_n^m is the Ferrers function of the first kind (associated Legendre function of the first kind on-the-cut) [6, (14.3.1)] with integer degree n and integer order m. Spherical harmonics are harmonic functions (solutions $u(\mathbf{r})$ of Laplace's equation $-\Delta u=0$) of $\mathbf{r}=(x,y,z)$ expressed in spherical coordinates. The functions G_n^m are harmonic on $\mathbb{R}^3$ whereas the functions H_m^n are harmonic on $\mathbb{R}^3\setminus\{\mathbf{0}\}$. Let $\mathbf{r},\mathbf{r}^*\in\mathbb{R}^3$ be two points with $r<r^*$. Then we have the well-known multipole expansion of the fundamental solution of Laplace's equation which is crucial in a great many applications in mathematical physics [11, pp. 1273-1274, (10.3.37)]

$$\frac{1}{\|\mathbf{r}-\mathbf{r}^*\|}=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{(n-m)!}{(n+m)!}\,G_n^m(\mathbf{r})\,\overline{H_n^m(\mathbf{r}^*)} \tag{4.16}$$

$$=\sum_{n=0}^{\infty}\frac{r^n}{(r^*)^{n+1}}\sum_{m=-n}^{n}\frac{(n-m)!}{(n+m)!}\,\mathsf{P}_n^m(\cos\theta)\mathsf{P}_n^m(\cos\theta^*)\mathrm{e}^{im(\phi-\phi^*)} \tag{4.17}$$

$$=\sum_{n=0}^{\infty}\frac{r^n}{(r^*)^{n+1}}P_n(\cos\theta\cos\theta^*+\sin\theta\sin\theta^*\cos(\phi-\phi^*)), \tag{4.18}$$

where $P_n=\mathsf{P}_n^0$ is the Legendre polynomial, and (4.18) is the famous Laplace expansion of the $1/r:=1/\|\mathbf{r}-\mathbf{r}^*\|$ potential. This expansion can be written in the form

$$\frac{1}{\|\mathbf{r}-\mathbf{r}^*\|}=\sum_{m\in\mathbb{Z}}\mathrm{e}^{im(\phi-\phi^*)}\sum_{n=0}^{\infty}B_{m,n}(r,r^*,\theta,\theta^*), \tag{4.19}$$

where

$$B_{m,n} = \frac{r^{|m|+n}}{(r^*)^{|m|+n+1}} \frac{n!}{(2|m|+n)!} \mathsf{P}^{|m|}_{|m|+n}(\cos\theta)\,\mathsf{P}^{|m|}_{|m|+n}(\cos\theta^*), \tag{4.20}$$

and r, θ, ϕ are spherical coordinates of $\mathbf{r}$ and r^*, θ^*, ϕ^* are spherical coordinates of $\mathbf{r}^*$ provided $r < r^*$.

Returning to flat-ring coordinates, if we substitute (3.10), (3.13) and (4.2) in (3.19), we obtain

$$\frac{1}{\|\mathbf{r}-\mathbf{r}^*\|} = \sum_{m\in\mathbb{Z}} \mathrm{e}^{im(\phi-\phi^*)} \sum_{n=0}^{\infty} A_{m,n}(\sigma,\sigma^*,\tau,\tau^*,k),$$

where

$$\begin{aligned} A_{m,n} = 2\Big(&\frac{1-\mathrm{sn}(\sigma,k)\,\mathrm{dn}(\tau,k')}{\mathrm{dn}(\sigma,k)\,\mathrm{sn}(\tau,k')}\Big)^{1/2}\Big(\frac{1-\mathrm{sn}(\sigma^*,k)\,\mathrm{dn}(\tau^*,k')}{\mathrm{dn}(\sigma^*,k)\,\mathrm{sn}(\tau^*,k')}\Big)^{1/2} \\ &\times W^n_{|m|-\frac12}(K'-\tau,k')W^n_{|m|-\frac12}(K'-\tau^*,k') \\ &\times \frac{1}{w^n_m(k)} W^n_{|m|-\frac12}(i(K+\sigma),k')W^n_{|m|-\frac12}(i(K-\sigma^*),k'), \end{aligned}$$

where $\sigma = s-K$, $\sigma^* = s^*-K$, $s<s^*$, $\tau = K'-t$, $\tau^* = K'-t^*$.

We now prove the main result of this section.

THEOREM 4.8. *Let $m\in\mathbb{Z}$, $n\in\mathbb{N}_0$, $\tau,\tau^*\in(0,\pi)$, $\sigma,\sigma^*\in\mathbb{R}$. Then*

$$A_{m,n}(\sigma,\sigma^*,\tau,\tau^*,k) \to B_{m,n}(\mathrm{e}^{\sigma},\mathrm{e}^{\sigma^*},\tau,\tau^*),$$

as $k\to 1$.

PROOF. It is enough to consider $m\in\mathbb{N}_0$. By (4.3) we have as $k\to1$,

$$\Big(\frac{1-\mathrm{sn}(\sigma,k)\,\mathrm{dn}(\tau,k')}{\mathrm{dn}(\sigma,k)\,\mathrm{sn}(\tau,k')}\Big)^{1/2} \to \mathrm{e}^{-\frac12\sigma}(\sin\tau)^{-1/2}.$$

By Corollary 4.4, we have as $k\to1$ that

$$\begin{aligned} &W^n_{m-\frac12}(K'-\tau,k')W^n_{m-\frac12}(K'-\tau^*,k') \\ &\to \frac{(m+n+\frac12)n!}{(2m+n)!}(\sin\tau)^{1/2}\,\mathsf{P}^m_{m+n}(\cos\tau)(\sin\tau^*)^{1/2}\,\mathsf{P}^m_{m+n}(\cos\tau^*). \end{aligned}$$

By Theorem 4.7, we have as $k\to1$ that

$$\frac{2}{w^n_m(k)}W^n_{m-\frac12}(i(K+\sigma),k')W^n_{m-\frac12}(i(K-\sigma^*),k') \to \frac{\mathrm{e}^{(m+n+\frac12)\sigma}\mathrm{e}^{-(m+n+\frac12)\sigma^*}}{m+n+\frac12}.$$

Combining our results we obtain the statement of the theorem. □

This proves our assertion that the peanut expansion of the $1/r$ potential in the limit as $k \to 1$ becomes the famous multipole expansion of the $1/r$ potential in spherical coordinates.

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