Maximally Edge-Connected Realizations and Kundu's k-factor Theorem

James M. Shook[3,4]

Tuesday 15[th] August, 2023

Abstract

A simple graph G with edge-connectivity $\lambda(G)$ and minimum degree $\delta(G)$ is maximally edge connected if $\lambda(G) = \delta(G)$. In 1964, given a non-increasing degree sequence $\pi = (d_1, \ldots, d_n)$, Jack Edmonds showed that there is a realization G of π that is k-edge-connected if and only if $d_n \geq k$ with $\sum_{i=1}^{n} d_i \geq 2(n-1)$ when $d_n = 1$. We strengthen Edmonds's result by showing that given a realization G_0 of π if Z_0 is a spanning subgraph of G_0 with $\delta(Z_0) \geq 1$ such that $|E(Z_0)| \geq n-1$ when $\delta(G_0) = 1$, then there is a maximally edge-connected realization of π with $G_0 - E(Z_0)$ as a subgraph. Our theorem tells us that there is a maximally edge-connected realization of π that differs from G_0 by at most $n-1$ edges. For $\delta(G_0) \geq 2$, if G_0 has a spanning forest with c components, then our theorem says there is a maximally edge-connected realization that differs from G_0 by at most $n-c$ edges. As an application we combine our work with Kundu's k-factor Theorem to show there is a maximally edge-connected realization with a $(k_1, \ldots, k_n)$-factor for $k \leq k_i \leq k+1$ and present a partial result to a conjecture that strengthens the regular case of Kundu's k-factor theorem.

Keywords— edge-connectivity, degree sequence, k-factor, regular graph, perfect matching

1 Introduction

We only consider simple graphs, and see Diestel [9] for terminology not defined here. For a graph $G = (V, E)$ and $v \in V$, we let $deg_G(v)$ denote the number of neighbors of v in G, and let $\Delta(G)$ and $\delta(G)$ denote the maximal and minimal degrees of G, respectively. We let $\lambda(G)$ denote edge-connectivity, and say G is maximally edge-connected if $\lambda(G) = \delta(G)$.

A sequence of non-negative integers $\pi = (d_1, \ldots, d_n)$ is a degree sequence if there exists a graph G with vertex set $V = \{v_1, \ldots, v_n\}$ such that $deg_G(v_i) = d_i$. Such a graph G is said to realize or is a realization of π. We call the sequence π graphic if it is a degree sequence. If instead a

[3]National Institute of Standards and Technology, Computer Security Division, Gaithersburg, MD; james.shook@nist.gov.

graph G with vertex set $V = \{v_1, \ldots, v_n\}$ is given, then we let $\pi(G) = (deg_G(v_1), \ldots, deg_G(v_n))$. If every entry of a degree sequence is non-zero, then we say the degree sequence is positive. We let $\mathcal{R}(\pi)$ denote the set of realizations of the graphic sequence π. For a graph F, with vertex set V, we let $\mathcal{R}(\pi, F) \subseteq \mathcal{R}(\pi)$ be the set of all realizations whose set of edges include $E(F)$. We have $\mathcal{R}(\pi) = \mathcal{R}(\pi, \emptyset)$, and we write $\mathcal{R}(G, F)$ for $\mathcal{R}(\pi(G), F)$.

The conditions for when a graph is maximally edge-connected are well studied [14]. In particular, given a graph G, Bollabás [2] and then extended upon in [8, 13] found degree conditions for $\pi(G)$ that imply G is maximally edge-connected. On the other hand, Jack Edmonds [10] gave necessary and sufficient conditions for when a degree sequence has a realization that is maximally edge-connected.

Theorem 1 ([10])**.** *For a non-increasing degree sequence $\pi = (d_1, \ldots, d_n)$, there is a $G \in \mathcal{R}(\pi)$ that is k-edge-connected if and only if $d_n \geq k$ with $\sum_{i=1}^{n} d_i \geq 2(n-1)$ when $d_n = 1$.*

Edmonds did this work in 1964 while at the National Institute of Standards and Technology (NIST was known as the National Bureau of Standards (NBS) when Edmonds did his work.), and besides a short constructive proof of Theorem 1 provided by Kleitman and Wang [23] in 1974, only within the last ten years has Theorem 1 been extended. Gu and Lai [12] generalized Theorem 1 to k-edge-connected uniform hypergraphs, and around the same time Tian, Meng, Lai, and Zhang [22] gave necessary and sufficient conditions for when a degree sequence has a realization that is super edge-connected. A super edge-connected graph is one where every minimum edge cut isolates a vertex with minimum degree. While studying edge-disjoint perfect matchings we needed to strengthen Theorem 1 so that the realization preserved a subgraph found in the original graph. However, we are able to prove more and so we present that here.

Theorem 2. *If there is a graph $G_0 = (V, E)$ with edge-disjoint spanning subgraphs F and Z_0 with $\delta(Z_0) > \Delta(F)$ such that $|E(Z_0)| \geq |V| - 1$ when $\delta(G_0) = 1$, then there is a $G \in \mathcal{R}(G_0, G_0 - E(Z_0))$ such that $G - E(F)$ is maximally edge-connected.*

It is not hard to derive Theorem 1 from Theorem 2. The first part of Theorem 1 is trivial and the second part follows from Theorem 2 by letting F be empty and $Z_0 = G_0$.

When F is empty Theorem 2 says any graph G_0 differs from a maximally edge-connected realization of $\pi(G_0)$ by at most $|V(G_0)| - 1$ edges. For $\delta(G_0) \geq 2$, if we let Z_0 be a spanning forest with c components, then we can show G_0 differs from some maximally edge-connected realization of $\pi(G_0)$ by at most $|V(G_0)| - c$ edges. Interestingly, this means if G_0 has a perfect matching, then there is some maximally edge-connected realization that differs form G_0 by at most $|V(G_0)|/2$ edges.

Observe in Theorem 2 that if F is maximally edge-connected or $F = \emptyset$, then G is maximally edge connected. Moreover, if $F = \emptyset$ and $Z_0 = G_0 - E(H)$ where H is a subgraph of G_0 such that $\Delta(H) \leq d_n - 1$ and $|E(Z_0)| \geq n - 1$, then Theorem 2 says H is a subgraph of some maximally edge-connected realization of π. With a simpler proof than Theorem 2 we may allow $\delta(Z_0) \geq \Delta(F)$ when $Z_0 = G_0 - E(F)$ at the expense of lowering the edge connectivity of $G - E(F)$ by one when $\delta(G - E(F))$ is odd.

Theorem 3. *If there is a subgraph F of a graph $G_0 = (V, E)$ with $\delta(G_0 - E(F)) \geq \Delta(F)$ such that $|E(G_0 - E(F))| \geq |V| - 1$ when $\delta(G_0) = 1$, then there is a $G \in \mathcal{R}(G_0, F)$ such that $G - E(F)$ is maximally edge-connected when $\delta(G - E(F))$ is even and $(\delta(G - E(F)) - 1)$-edge-connected when $\delta(G - E(F))$ is odd.*

1.1 Kundu's k-factor Theorem

For an application of Theorem 2 and Theorem 3, we look to Kundu's k-factor Theorem for inspiration. Recall a graph G is said to have a $(k_1, \ldots, k_n)$-factor if G has a spanning subgraph with degree sequence $(k_1, \ldots, k_n)$. If $k_i = k$ for all i, then we simply call the spanning subgraph a k-factor.

Theorem 4 (Kundu's k-factor Theorem [16]). *For $k \geq 0$, if $\pi = (d_1, \ldots, d_n)$ and $(d_1 - k_1, \ldots, d_n - k_n)$ are both graphic such that $k \leq k_i \leq k+1$ for $1 \leq i \leq n$, then there exists a realization of π that has a $(k_1, \ldots, k_n)$-factor.*

By requiring each term of the degree sequence to be at least two and $k \geq 1$ we can use Theorem 2 to strengthen Kundu's Theorem.

Theorem 5. *For $k \geq 1$, if $\pi = (d_1, \ldots, d_n)$ and $(d_1 - k_1, \ldots, d_1 - k_n)$ are both graphic such that $d_i \geq 2$ and $k \leq k_i \leq k+1$ for all $i \in \{1, \ldots, n\}$, then there exists a maximally-edge connected realization of π that has a $(k_1, \ldots, k_n)$-factor.*

Proof. By Kundu's k-factor Theorem, there exist a $G \in \mathcal{R}(\pi)$ that has a $(k_1, \ldots, k_n)$-factor H. If we let $F = \emptyset$ and $Z_0 = H$, then Theorem 2 says there exists a maximally edge-connected realization $G' \in R(\pi)$ that contains the subgraph $G - E(H)$. Moreover, the edges of G' not in the subgraph $G - E(H)$ form a $(k_1, \ldots, k_n)$-factor. □

For a sequence $\pi = (d_1, \ldots, d_n)$, we let $\mathcal{D}_k(\pi)$ denote the sequence $(d_1 - k, \ldots, d_n - k)$. Busch, Ferrara, Hartke, Jacobson, Kaul, and West [5] showed that if n is even and both π and $\mathcal{D}_k(\pi)$ are graphic, then for $r \leq \min\{2, k\}$ there is a realization of π with a k-factor that has r edge-disjoint 1-factors. Seacrest [19] improved this for $r \leq \min\{4, k\}$. The work of [5] and [19] on edge-disjoint 1-factors suggests a further strengthening of Kundu's theorem.

Conjecture 1 ([4] and later in [5]). *Some realization of a degree sequence $(d_1, \ldots, d_n)$ with even n has k edge-disjoint 1-factors if and only if $(d_1 - k, \ldots, d_n - k)$ is graphic.*

Conjecture 1 was first posed by Brualdi [4] and then independently by Busch et al. in [5]. The work of Seacrest [19] shows the conjecture is true for $k \leq 5$. Busch et al. showed the conjecture holds for $d_n \geq \frac{n}{2} + k - 2$ and $d_1 \leq \frac{n}{2} + 1$. In this paper we focus on large k.

A 1-factorization of a graph is the partition of its edges into 1-factors. Chetwyn and Hilton [6] described a 1-factorization conjecture that says every k-regular graph with $k \geq 2\lceil \frac{n}{4} \rceil - 1$ has a 1-factorization. Impressively, Csaba, Kühn, Lo, Osthos, and Treglown affirmed this conjecture for n sufficiently large.

Theorem 6 ([7]). *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Let $n, k \in \mathbb{N}$ be such that $n \geq n_0$ is even and $k \geq 2\lceil \frac{n}{4} \rceil - 1$. Then every k-regular graph G on n vertices can be decomposed into k edge-disjoint 1-factors.*

Thus, Theorem 6 says Conjecture 1 is true for large k and n sufficiently large. However, we can say more now that we have Theorem 3.

The following classic result of Berge [1] was expanded upon in [3, 15, 18, 20].

Theorem 7 ([1]). *All even ordered $(k-1)$-edge-connected k-regular graphs have a 1-factor.*

We use the result of Berge to show that for large k we can find a k-factor with many edge-disjoint 1-factors.

Theorem 8. *Let $\pi = (d_1, \ldots, d_n)$ be a non-increasing degree sequence with even n such that $\mathcal{D}_k(\pi)$ is graphic. If $k \geq \frac{d_1}{2} + r$ or $k \geq n - 1 - d_n + 2r$, then π has a realization with a k-factor that has at least $r + 1$ edge-disjoint 1-factors.*

Proof. We will first prove the case $k \geq d_1/2 + r$. By Kundu's k-factor theorem there is some realization of π with a k-factor. Let $i \leq k$ be the largest non-negative integer such that there is a $G_i \in \mathcal{R}(\pi)$ with a $(k-i)$-factor H_i such that $F_i = G_i - E(H_i)$ has i edge-disjoint 1-factors. Since $\delta(H_i) = k-i \geq d_1/2+r-i \geq (k-i+\Delta(F_i))/2+r-i$, we see that $\delta(H_i) \geq \Delta(F_i)+2(r-i)$. Therefore, if $r \leq i$, then we may apply Theorem 3 to find a $G_{i+1} \in \mathcal{R}(\pi, F_i)$ such that $H_{i+1} = G_{i+1} - E(F_i)$ is a $(k-i-1)$-edge-connected $(k-i)$-factor. However, we deduce a contradiction since Theorem 7 implies H_{i+1} has a 1-factor, and therefore, G_{i+1} has a k-factor with $i+1$ edge-disjoint 1-factors. Thus, $i \geq r+1$.

The case $k \geq n-1-d_n+2r$ can be proved with an application of the first part of this theorem to the non-increasing degree sequences $(n-1-d_n+k, \ldots, n-1-d_1+k)$ and $(n-1-d_n, \ldots, n-1-d_1)$, the reverse order of the complements of $\mathcal{D}_k(\pi)$ and π, to find a realization with a k-factor that has $r+1$ edge-disjoint 1-factors. The k-factor can then be mapped to a realization of π □

It would be interesting to see if Theorem 7 or results like it can be used to find more edge-disjoint 1-factors. Considering the 1-factorization conjecture and the work of Csaba et al., it seems likely that as k increases the number of edge-disjoint 1-factors in a maximally edge-connected regular graph increases. Thomassen in [21] showed that the edges of a k-regular k-edge-connected graphs with some restrictions can be partitioned in various ways. Although, some of those restrictions may not be necessary. Thomassen, in the same paper, posed some nice conjectures and problems that avoids them. However, Mattiolo [17], answering Problem 1 in [21], presented k-regular k-edge-connected graphs that cannot be partitioned into a 2-factor and $k-2$ 1-factors. Thus, our strategy maybe limited to finding many edge-disjoint 1-factors, but not k of them.

2 Proofs

Let $G = (V, E)$ be a graph with $X, Y \subseteq V$. We denote $G[X]$ to be the induced graph on X, and let $\overline{X} = V - X$. We denote $E_G(X, Y)$ to be the set of all edges of G that have one end in X and the other in Y, and let $e_G(X, Y) = |E_G(X, Y)|$ and $e_G(x, Y) = e_G(\{x\}, Y)$. We denote $\Gamma_G(X)$ to be the set of all vertices in X that are adjacent in G to vertices in $\overline{X}$.

Let $G = (V, E)$ be a graph with $\lambda(G) < \delta(G)$, and let $A \subset V$. If $e_G(A, \overline{A}) < \delta(G)$, then we say A is weak. If $e_G(A, \overline{A}) = \lambda(G)$, then we say A is minimally weak. If A is weak and $e_G(S, \overline{S}) \geq \delta(G)$ for every $S \subset A$, then we say A is critically weak.

The next two lemmas play an important role in the proofs of Theorem 2 and Theorem 3.

Lemma 1. *For a graph $G = (V, E)$ with $\lambda(G) < \delta(G)$, if $S \subseteq A \subset V$ such that A is critically weak and $e_G(S, \overline{S}) \leq \delta(G)$, then for any $X \subset V$ such that $X \cap S \neq \emptyset$ and $\overline{X} \cap S \neq \emptyset$ we see that*

$$e_G(X \cap S, \overline{X} \cap S) \geq \left\lceil \frac{\delta(G)}{2} \right\rceil \tag{1}$$

when $S \subset A$ and for $A = S$, we see that

$$e_G(X \cap A, \overline{X} \cap A) \geq \left\lceil \frac{\delta(G)+1}{2} \right\rceil \tag{2}$$

where equality in (2) implies

$$\min\{e_G(X \cap A, \overline{A}), e_G(\overline{X} \cap A, \overline{A})\} \geq \left\lfloor \frac{\delta(G)-1}{2} \right\rfloor. \tag{3}$$

Proof. By the definition of a critically weak set we know that $e_G(S \cap X, \overline{S \cap X}) \geq \delta(G)$ and $e_G(S \cap \overline{X}, \overline{S \cap \overline{X}}) \geq \delta(G)$. Thus,

$$\begin{aligned} e_G(S, \overline{S}) &= e_G(X \cap S, \overline{X \cap S}) + e_G(\overline{X} \cap S, \overline{\overline{X} \cap S}) - 2e_G(X \cap S, \overline{X} \cap S) \\ &\geq 2(\delta(G) - e_G(X \cap S, \overline{X} \cap S)). \end{aligned} \tag{4}$$

When $S \subset A$ we have by our condition on S that $e_G(S, \overline{S}) = \delta(G)$. Thus, solving (4) for $e_G(X \cap S, \overline{X} \cap S)$ we establish (1). Using the fact that $e_G(A, \overline{A}) \leq \delta(G) - 1$ when $S = A$ we can establish (2) by solving (4) for $e_G(X \cap A, \overline{X} \cap A)$. Since A is critically weak, when equality holds in (2) we see that

$$\begin{aligned} \min\{e_G(X \cap A, \overline{A}), e_G(\overline{X} \cap A, \overline{A})\} &\geq \delta(G) - e_G(X \cap A, \overline{X} \cap A) \\ &= \delta(G) - \left\lceil \frac{\delta(G)+1}{2} \right\rceil = \left\lfloor \frac{\delta(G)-1}{2} \right\rfloor. \end{aligned}$$ □

The following lemma was proved in different forms in [8, 11]. We present our proof of the form we need.

Lemma 2. *For a graph $G = (V, E)$ with $\lambda(G) < \delta(G)$, if $A \subset V$ is weak, then $A - \Gamma_G(A) \neq \emptyset$ and therefore, every $x \in A - \Gamma_G(A)$ has a neighbor in $A - \Gamma_G(A)$.*

Proof. Suppose $\Gamma_G(A) = A$. In this case every vertex in A must be adjacent in G to at least $\delta(G) - (|A| - 1)$ vertices in $\overline{A}$. Thus,

$$\delta(G) - 1 \geq e_G(A, \overline{A}) \geq |A|(\delta(G) - (|A| - 1)).$$

If we combine all terms of the last inequality onto the right hand side and simplify, then we see the contradiction

$$0 \geq (\delta(G) - |A|)(|A| - 1) + 1$$

since $\delta(G) - 1 \geq |A| \geq 1$. Thus, $A - \Gamma_G(A)$ is not empty, and consequently, $N_G(x) \subseteq A$ for every $x \in A - \Gamma_G(A)$. Since $|\Gamma_G(A)| \leq \delta(G) - 1$, every $x \in A - \Gamma_G(A)$ has a neighbor in $A - \Gamma_G(A)$. □

We will often make use of an edge-exchange. This operation, see Figure 1, consists of exchanging two edges vx_0 and x_1u from a graph G with two edges x_0u and vx_1 from $\overline{G}$ to create another realization of $\pi(G)$ while leaving all other edges the same. We will often refer to edges of $\overline{G}$ as non-edges of G.

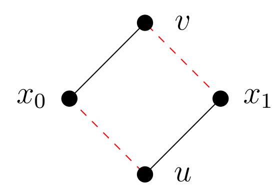

Figure 1: An edge-exchange between edges vx_0 and x_1u of G and non-edges x_0u and vx_1.

2.1 Proof of Theorem 3

Since the proof of Theorem 3 follows a simplified proof of Theorem 2, we present it here first.

Proof. We choose a $G \in \mathcal{R}(G_0, F)$ such that

(C1) $\lambda(G - E(F))$ is maximized, and

(C2) subject to (C1), we minimize the number of minimally weak sets in $G - E(F)$.

Let $Z = G - E(F)$. For a contradiction, we assume $\lambda(Z) < \delta(Z)$ and $\lambda(Z) < \delta(Z) - 1$ when $\delta(Z)$ is odd. As a consequence, we may choose an arbitrary minimally weak set $A_0 \subseteq V(G)$ and critically weak sets $A \subseteq A_0$ and $B \subseteq \overline{A_0}$.

We choose an $a \in A - \Gamma_Z(A)$ and a $b \in B - \Gamma_Z(b)$ such that we give priority to an adjacent pair in F. If a and b are adjacent, then $N_Z(a) - N_F(b) \neq \emptyset$ and $N_Z(b) - N_F(a) \neq \emptyset$ since $\delta(Z) \geq \Delta(F)$. If a and b are not adjacent, then by our choice of a and b, we deduce that $N_F(a) \cap (B - \Gamma_Z(b)) = \emptyset$ and $N_F(b) \cap (B - \Gamma_Z(a)) = \emptyset$. In either case we may choose an $a' \in N_Z(a) - N_F(b)$ and a $b' \in N_Z(b) - N_F(a)$.

Let W be the realization of $\mathcal{R}(G_0, F)$ created by exchanging the edges aa' and bb' of Z with the non-edges ab' and ba' of G. By (C1) we know that $\lambda(Z) \geq \lambda(W - E(F))$. Let us first examine A_0 in $W - E(F)$. Since the edges aa' and bb' are not in $E_Z(A_0, \overline{A_0})$ and the edges ab' and ba' are in $E_{W-E(F)}(A_0, \overline{A_0})$, we see that $e_{W-E(F)}(A_0, \overline{A_0}) = e_Z(A_0, \overline{A_0}) + 2$. Since A_0 is minimally weak in Z, we see that

$$e_{W-E(F)}(A_0, \overline{A_0}) = e_Z(A_0, \overline{A_0}) + 2 = \lambda(Z) + 2 \geq \lambda(W - E(F)) + 2.$$

Therefore, A_0 is not minimally weak in $W - E(F)$.

We choose an arbitrary $X \subseteq V$ that is minimally weak in $W - E(F)$. Thus,

$$\lambda(Z) \geq \lambda(W - E(F)) = e_{W-E(F)}(X, \overline{X}).$$

Suppose at most one of aa' or bb' is in $E_Z(X, \overline{X})$. We have three cases to consider. If both aa' and bb' are in $E_Z(X, X)$ or in $E_Z(\overline{X}, \overline{X})$, then both ab' and ba' are in $E_{W-E(F)}(X, X)$ or in $E_{W-E(F)}(\overline{X}, \overline{X})$. If one of aa' or bb' is in $E_Z(X, \overline{X})$, then one of ab' or ba' is in $E_{W-E(F)}(X, \overline{X})$. Thus, $e_{W-E(F)}(X, \overline{X}) = e_Z(X, \overline{X})$ in the first two cases. Finally, if one of aa' or bb' is in $E_Z(X, X)$ and the other is in $E_Z(\overline{X}, \overline{X})$, then both ab' and ba' are in $E_{W-E(F)}(X, \overline{X})$. Thus, $e_{W-E(F)}(X, \overline{X}) \geq e_Z(X, \overline{X})$ in all three cases. From this we may deduce that

$$\lambda(Z) \geq \lambda(W - E(F)) = e_{W-E(F)}(X, \overline{X}) \geq e_Z(X, \overline{X}) \geq \lambda(Z).$$

This implies $\lambda(Z) = \lambda(W - E(X))$ and X is minimally weak in Z. Thus, $W - E(F)$ satisfies (C1), and since X is an arbitrarily chosen minimally weak set of $W - E(F)$, we may conclude that every minimally weak set in $W - E(F)$ is minimally weak in Z. However, since A_0 is not minimally weak in $W - E(F)$, we may, in contradiction with (C2), conclude that $W - E(F)$ has fewer minimally weak sets than Z. Thus, W contradicts our choice of Z, and therefore, we may assume $\{aa', bb'\} \subseteq E_Z(X, \overline{X})$. Since the edge-exchange between aa' and bb' only affects the two edges, we may conclude that $e_{W-E(F)}(X, \overline{X}) \geq e_Z(X, \overline{X}) - 2$. By Lemma 1 $e_Z(X \cap A, \overline{X} \cap A)$ and $e_Z(X \cap B, \overline{X} \cap B)$ are at least $\left\lceil \frac{\delta(Z)+1}{2} \right\rceil$. Thus,

$$\lambda(Z) \geq e_{W-E(F)}(X, \overline{X}) \geq e_Z(X, \overline{X}) - 2 \geq 2\left\lceil \frac{\delta(Z)+1}{2} \right\rceil - 2.$$

However, this presents a contradiction since the right hand side of the last inequality is at least $\delta(Z)$ when $\delta(Z)$ is even and at least $\delta(Z) - 1$ when $\delta(Z)$ is odd. Thus, G satisfies Theorem 3. □

2.2 Proof of Theorem 2

Edmonds established Theorem 1 by directly proving the $\delta(G - E(F)) = 1$ case and then used a strategy of reducing the number of weak sets when $\delta(G - E(F)) \geq 2$. We follow the same strategy, but our job is more difficult since we have fewer edges with which we may exchange. We tackle this difficulty by carefully selecting critically weak sets and vertices so we may find edges with useful properties.

Proof. We choose a $G \in \mathcal{R}(G_0, G_0 - E(Z_0))$ such that

(C1) $\lambda(G - E(F))$ is maximized, and

(C2) subject to (C1), we minimize the number of minimally weak sets in $G - E(F)$.

Let $H = G - E(F)$, and by contradiction we assume $\lambda(H) < \delta(H)$. We let $Z = G - E(G_0 - E(Z_0))$, and observe that $Z \in \mathcal{R}(Z_0)$.

Claim 8.1. H is connected, and $\delta(H) \geq 2$.

Proof. For a contradiction, we assume H can be partitioned into the components $C_1, \ldots, C_t$. Suppose there is a component C_l that has a cycle containing an edge $aa' \in E(Z)$. Since $\delta(Z) > \Delta(F)$, we can choose an edge bb' of Z in some other component C_j such that $b \notin N_F(a')$ and $a \notin N_F(b')$. We exchange the edges aa' and bb' of Z with the non-edges ab' and ba' of G to create new realizations $G' \in \mathcal{R}(G_0, G_0 - E(Z_0))$. Since aa' was in a cycle of C_l the vertices in $V(C_l) \cup V(C_i)$ form a component of G', and therefore, G' contradicts (C2) since it has fewer components than G. Thus, to complete the proof of this claim we need to find an edge of Z in a cycle of H. For each i, we let T_i represent a spanning tree of C_i. For some i, if some $aa' \in E(Z) \cap E(C_i)$ is not in T_i, then that edge forms a cycle with edges of T_i. Consider the situation $E(Z) \cap E(C_i) \subseteq E(T_i)$ for all T_i. If $\delta(G_0) = 1$, then we have the contradiction

$$|V| - 1 \leq |E(Z)| \leq \sum_{i=1}^{t} |E(T_i)| = \sum_{i=1}^{t}(|C_i| - 1) \leq |V| - 2.$$

If $\delta(G_0) \geq 2$, then given a leaf $a \in V(C_i)$ there is an edge $aa' \in E(T_i) \cap E(Z)$ and an edge $ab \notin E(T_i)$. However, ab forms a cycle with edges of T_i that includes aa'. Since H is connected, we see by our choice of G that $\delta(H) \geq 2$. □

We choose a minimally weak set A_0 of H such that there are critically weak sets $A \subseteq A_0$ and $B \subseteq \overline{A_0}$ of H with $|\Gamma_H(A)| \geq |\Gamma_H(B)|$.

Claim 8.2. If $aa' \in E(Z[A])$ and $bb' \in E(Z[B])$ such that ab' and ba' are non-edges of G, then there exists an $X \subset V$ with $\{a, b'\} \subseteq X$ and $\{a', b\} \subseteq \overline{X}$ such that $e_H(X, \overline{X}) \leq \lambda(H)+2 \leq \delta(H)+1$.

Proof. For a contradiction, we suppose $e_H(X, \overline{X}) \geq \lambda(H)+3$ for every $X \subset V$ with $\{a, b'\} \subseteq X$ and $\{a', b\} \subseteq \overline{X}$. Let Z' be the realization of $\mathcal{R}(Z)$ created by exchanging the edges aa' and bb' in Z with the non-edges ab' and ba' of G. Thus, the graph $W = G - E(Z) + E(Z')$ is in $\mathcal{R}(G_0, G_0 - E(Z_0))$, and by (C1) we know that $\lambda(W - E(F)) = \lambda(H)$. We choose an arbitrary $X \subseteq V$. If either aa' or $b'b$ is not in $E_H(X, \overline{X})$, then $e_W(X, \overline{X}) = e_H(X, \overline{X})$. If $\{aa', b'b\} \subseteq E_H(X, \overline{X})$, then $e_{W-E(F)}(X, \overline{X}) \geq e_H(X, \overline{X}) - 2 \geq \lambda(H) + 1$. Thus, if X is minimally weak in $W - E(F)$, then it was minimally weak in H. Since $e_{W-E(F)}(A_0, \overline{A_0}) = e_H(A_0, \overline{A_0}) + 2$, we know that A_0 is minimally weak in H and not in $W - E(F)$. Thus, $W - E(F)$ violates (C2). □

Claim 8.3. Let $aa' \in E(Z[A])$ and $bb' \in E(Z[B])$ such that ab' and ba' are non-edges of G. If $e_H(X, \overline{X}) \leq \lambda(H) + 2$ for some $X \subseteq V$ with $\{a, b'\} \subseteq X$ and $\{a', b\} \subseteq \overline{X}$, then $\delta(H)$ is odd, $\lambda = \delta(H) - 1$, $e_H(X, \overline{X}) = \delta(H) + 1$,

$$e_H(X \cap A, \overline{X} \cap A) = e_H(X \cap B, \overline{X} \cap B) = \frac{\delta(H)+1}{2}, \text{and}$$

$$e_H(X \cap A, \overline{A}) = e_H(\overline{X} \cap A, \overline{A}) = \frac{\delta(H)-1}{2}.$$

Proof. We have $e_H(X, \overline{X}) \leq \lambda(H) + 2 \leq \delta(H) + 1$. Applying Lemma 1 to both A and B we see that both $e_H(X \cap A, \overline{X} \cap A)$ and $e_H(X \cap B, \overline{X} \cap B)$ are at least $\left\lceil \frac{\delta(H)+1}{2} \right\rceil$. Thus,

$$\delta(H) + 1 \geq \lambda(H) + 2 \geq e_H(X, \overline{X}) \geq e_H(X \cap A, \overline{X} \cap A) + e_H(X \cap B, \overline{X} \cap B) \geq 2\left\lceil \frac{\delta(H)+1}{2} \right\rceil.$$

This can only be true if $\delta(H)$ is odd, $\lambda(H) = \delta(H) - 1$, $e_H(X, \overline{X}) = \delta(H) + 1$, and

$$e_H(X \cap A, \overline{X} \cap A) = e_H(X \cap B, \overline{X} \cap B) = \frac{\delta(H)+1}{2}.$$

Furthermore, we see by Lemma 1 and (3) that

$$\delta(H) - 1 = e_H(A, \overline{A}) = e_H(X \cap A, \overline{A}) + e_H(\overline{X} \cap A, \overline{A}) \geq \delta(H) - 1.$$

Thus,

$$e_H(X \cap A, \overline{A}) = e_H(\overline{X} \cap A, \overline{A}) = \frac{\delta(H)-1}{2}.$$ □

Claim 8.4. If there is a path P from a vertex $y \in \Gamma_H(A)$ to a $y' \in \Gamma_H(B)$ with no internal vertices in $A \cup B$, then either y is adjacent in F to every vertex in $N_Z(y') \cap (B - \Gamma_H(B))$ or y' is adjacent in F to every vertex in $N_Z(y) \cap (A - \Gamma_H(A))$.

Proof. Suppose there is an $a \in N_Z(y) \cap (A - \Gamma_H(A))$ and a vertex $b \in N_Z(y') \cap (B - \Gamma_H(B))$ such that $a \notin N_G(y')$ and $b \notin N_G(y)$. Let X be an arbitrary subset of V with $\{a, y'\} \subseteq X$ and $\{b, y\} \subseteq \overline{X}$. By Claim 8.3 we see that $e_H(X \cap A, \overline{X} \cap A)$ and $e_H(X \cap B, \overline{X} \cap B)$ are both equal to $\frac{\delta(H)+1}{2}$. Since P is a path from y to y' that has no internal vertices in $A \cup B$, there must be an edge from $X - B$ to $\overline{X} - A$. Therefore, since X is arbitrary, we see by Claim 8.2 the contradiction

$$\begin{aligned} e_H(X, \overline{X}) &\geq e_H(X \cap A, \overline{X} \cap A) + e_H(X \cap B, \overline{X} \cap B) + e_H(X - B, \overline{X} - A) \\ &\geq 2\left(\frac{\delta(H)+1}{2}\right) + 1 \geq \delta(H) + 2. \end{aligned}$$

□

We choose an $x \in A$ such that $e_H(x, \overline{A})$ is maximized and subject to that we minimize $e_Z(x, \overline{A})$.

Claim 8.5. $\delta(H) \geq 3$ and odd, $\lambda(H) = \delta(H) - 1$, and $e_H(x, \overline{A}) \leq \frac{\delta(H)-1}{2}$.

Proof. By Lemma 2 there is an $a \in A - \Gamma_H(A)$ and $b \in B - \Gamma_H(B)$. Since $\delta(Z) > \Delta(F)$, we know there exists an $a' \in N_Z(a) - N_F(b)$ and a $b' \in N_Z(b) - N_F(a)$. By Claim 8.2 there exists an $X \subset V$ with $\{a, b'\} \subseteq X$ and $\{a', b\} \subseteq \overline{X}$ such that $e_H(X, \overline{X}) \leq \lambda(H) + 2 \leq \delta(H) + 1$. By Claim 8.1 and Claim 8.3, $\delta(H)$ is at least three and odd, $\lambda = \delta(H) - 1$, and

$$e_H(X \cap A, \overline{A}) = e_H(\overline{X} \cap A, \overline{A}) = \frac{\delta(H)-1}{2}.$$

Thus, $e_H(x, \overline{A}) \leq \frac{\delta(H)-1}{2}$. □

Since A is weak, we see by Claim 8.5 that A is both minimally and critically weak. We let A' denote all $u \in \Gamma_H(A)$ with $e_H(u, \overline{A}) = \frac{\delta(H)-1}{2}$. Note that $|A'| \leq 2$, and if A' is not empty, then $x \in A'$ by our choice of x and Claim 8.5.

We say edges xy and uv of a graph W are crossable in W if $E_G(\{x, y\}, \{u, v\}) \neq \emptyset$ and not crossable in W, otherwise.

Claim 8.6. Every $aa' \in E(Z[A])$ and $bb' \in E(Z[B])$ are crossable in G.

Proof. By contradiction suppose aa' and bb' are not crossable in G. By Claim 8.2 there must exist an $X \subset V$ with $\{a, b'\} \subseteq X$ and $\{b, a'\} \subseteq \overline{X}$ and a $Y \subset V$ with $\{a, b\} \subseteq Y$ and $\{b', a'\} \subseteq \overline{Y}$ such that $e_H(X, \overline{X}) \leq \delta(H) + 1$ and $e_H(Y, \overline{Y}) \leq \delta(H) + 1$. By Claim 8.3

$$e_H(X \cap A, \overline{X} \cap A) = e_H(X \cap B, \overline{X} \cap B) = e_H(Y \cap A, \overline{Y} \cap A) = e_H(Y \cap B, \overline{Y} \cap B) = \frac{\delta(H)+1}{2}, \text{ and}$$

$$e_H(X \cap A, \overline{A}) = e_H(X \cap B, \overline{B}) = \frac{\delta(H)-1}{2}.$$

This implies $e_H(X \cap A, \overline{X \cap A}) = e_H(X \cap A, \overline{X} \cap A) + e_H(X \cap A, \overline{A}) = \delta(H)$.

Suppose both Y and $\overline{Y}$ intersect $X \cap A$. Since $\delta(H)$ is odd and $e_H(X \cap A, \overline{X \cap A}) = \delta(H)$, we see by Lemma 1 and (1) that $e_H(Y \cap X \cap A, \overline{Y} \cap X \cap A) \geq \frac{\delta(H)+1}{2}$. However, since $aa' \in E_H(Y \cap X \cap A, \overline{Y} \cap \overline{X} \cap A)$, we deduce the contradiction

$$e_H(Y \cap A, \overline{Y} \cap A) = e_H(Y \cap X \cap A, \overline{Y} \cap X \cap A) + E_H(Y \cap X \cap A, \overline{Y} \cap \overline{X} \cap A) \geq 1 + \frac{\delta(H)+1}{2}.$$

An identical argument can be made if Y and $\overline{Y}$ both intersect $X \cap B$. Thus, we assume $X \cap A \subseteq Y$ and $X \cap B \subseteq \overline{Y}$.

Since H is $(\delta(H)-1)$-edge-connected, $e_H(X \cap A, \overline{A}) = \frac{\delta(H)-1}{2}$, and $\delta(H) \geq 3$, there exists a path P from some $u \in X \cap \Gamma_H(A)$ to some $v \in \Gamma_H(B)$ such that no internal vertex of P is in $A \cup B$. If $v \in \overline{X}$, then $E(X-B, \overline{X}-A)$ is not empty. However, this leads to the contradiction

$$e_H(X,\overline{X}) \geq e_H(X \cap A, \overline{X} \cap A) + e_H(X \cap B, \overline{X} \cap B) + e_H(X-B, \overline{X}-A) \geq \delta(H)+2.$$

If $v \in X$, then $e_H(Y-B, \overline{Y}-A)$ is not empty and we have the contradiction

$$e_H(Y,\overline{Y}) \geq e_H(Y \cap A, \overline{Y} \cap A) + e_H(Y \cap B, \overline{Y} \cap B) + e_H(Y-B, \overline{Y}-A) \geq \delta(H)+2. \qquad \square$$

Claim 8.7. $\Delta(F) \geq 1$.

Proof. By contradiction suppose F has no edges, and therefore, $H = G$. We choose an $a \in A - \Gamma_H(A)$, $a' \in N_Z(a)$, $b \in B - \Gamma_H(B)$, and $b' \in N_z(b)$. Since $\{b,b'\} \cap N_G(a) = \emptyset$ and $\{a,a'\} \cap N_G(b) = \emptyset$, we see by Claim 8.6 that $a'b' \in E(G)$. By Claim 8.2 there is an $X \subset V$ with $\{a,b'\} \subseteq X$ and $\{b,a'\} \subseteq \overline{X}$ such that $e_H(X,\overline{X}) \leq \delta(H)+1$. By Claim 8.3 both $e_H(X \cap A, \overline{X} \cap A)$ and $e_H(X \cap B, \overline{X} \cap B)$ are equal to $\frac{\delta(H)+1}{2}$. Since $a'b'$ is an edge, we see that $e_H(X-B, \overline{X}-A) \geq 1$. However, this gives us the contradiction

$$e_H(X,\overline{X}) \geq e_H(X \cap A, \overline{X} \cap A) + e_H(X \cap B, \overline{X} \cap B) + e_H(X-B, \overline{X}-A) \geq \delta(H)+2. \qquad \square$$

Since $\Delta(F) \geq 1$, we see that $\delta(Z) \geq 2$.

Claim 8.8. There exists an $S \subseteq A - \{x\}$ with $|S| \geq 2$ such that $e_H(S, A-S) = \frac{\delta(H)+1}{2}$, $e_H(S,\overline{A}) = \frac{\delta(H)-1}{2}$, and $e_H(S,\overline{S}) = \delta(H)$.

Proof. We choose an $a \in A - \Gamma_H(A)$ such that we give priority to those vertices adjacent in F to some vertex in $B - \Gamma_H(B)$. Suppose every vertex in $N_Z(a) - (A' - \{x\})$ is adjacent in F to every vertex in $B - \Gamma_H(B)$. Thus, $\Delta(F) \geq |N_Z(a) - (A' - \{x\})|$. However, since $\delta(Z) \geq \Delta(F)+1$ and $|A'| \leq 2$, we see that $|N_Z(a) - (A' - \{x\})| = \Delta(F)$, and therefore, a must be adjacent in Z to a vertex in $A' - \{x\}$ and a must not be adjacent in F to a vertex in $B - \Gamma_H(B)$. Thus, $A' = \Gamma_H(A)$, and since $\Delta(F) \geq 1$, we know $N_Z(a) - (A' - \{x\})$ is not empty. Since $|B - \Gamma_H(B)| \geq 2$, we see that $\delta(Z) > \Delta(F) \geq 2$ and there must be a vertex in $A - A'$ adjacent to a vertex in $B - \Gamma_H(B)$. However, this contradicts our choice of a. Thus, there must be some $b \in B - \Gamma_H(B)$ not adjacent in G to some $a' \in N_Z(a) - (A' - \{x\})$. Let $b' \in N_Z(b) - N_F(a)$. By Claim 8.2 there is an $X \subset V$ with $\{a,b'\} \subseteq X$ and $\{b,a'\} \subseteq \overline{X}$ such that $e_H(X,\overline{X}) \leq \delta(H)+1$. By Claim 8.3 $e_H(X \cap A, \overline{X} \cap A) = \frac{\delta(H)+1}{2}$ and $e_H(X \cap A, \overline{A}) = e_H(\overline{X} \cap A, \overline{A}) = \frac{\delta(H)-1}{2}$. If $x \in \overline{X}$, then since $a \notin \Gamma_H(A)$, we see that $|X \cap A| \geq 2$. Suppose $x \in X$. If $a' \notin \Gamma_H(A)$, then $|\overline{X} \cap A| \geq 2$, and if $a \in \Gamma_H(A)$, then since $a' \notin A' - \{x\}$ and $e_H(\overline{X} \cap A, \overline{A}) = \frac{\delta(H)-1}{2}$, we see that $|\overline{X} \cap A| \geq 2$. Thus, either $X \cup A$ or $\overline{X} \cup A$ satisfies the conditions of the claim. $\square$

By Claim 8.8 we may choose S to be the smallest such set.

Claim 8.9. If there is an edge $aa' \in E(Z[S])$ and a $b \in B - \Gamma_H(B) - N_F(a')$, then $\{a,a'\} \cap A' \neq \emptyset$.

Proof. Suppose $aa' \in E(Z[S])$ and a vertex $b \in B - \Gamma_H(B) - N_F(a')$. Let $b' \in N_Z(b) - N_F(a)$. By Claim 8.2 there is an $X \subset V$ with $\{a, b'\} \subseteq X$ and $\{b, a'\} \subseteq \overline{X}$ such that $e_H(X, \overline{X}) \leq \delta(H) + 1$. By Claim 8.3 $e_H(X \cap A, \overline{X} \cap A) = \frac{\delta(H)+1}{2}$ and $e_H(X \cap A, \overline{A}) = e_H(\overline{X} \cap A, \overline{A}) = \frac{\delta(H)-1}{2}$. Since $e_H(S, \overline{S}) = \delta(H)$, we see by Lemma 1 and (1) that $e_H(X \cap S, \overline{X} \cap S) \geq \frac{\delta(H)+1}{2}$. This implies $e_H(X \cap (A - S), \overline{X} \cap (A - S)) = 0$ since

$$e_H(X \cap A, \overline{X} \cap A) = e_H(X \cap S, \overline{X} \cap S) + e_H(X \cap (A - S), \overline{X} \cap (A - S)) \geq \frac{\delta(H) + 1}{2}.$$

If $A - S \subset X$, then $\overline{X} \cap A = \overline{X} \cap S$, $|\overline{X} \cap S| < |S|$, $e_H(X \cap S, \overline{X} \cap S) = \frac{\delta(H)+1}{2}$, and $e_H(\overline{X} \cap S, \overline{A}) = \frac{\delta(H)-1}{2}$. By our choice of S, it must be the case $\overline{X} \cap S = \{a'\}$, and therefore, $a' \in A'$. If $A - S \subset \overline{X}$, then $X \cap A = X \cap S$, $|X \cap S| < |S|$, $e_H(X \cap S, \overline{X} \cap S) = \frac{\delta(H)+1}{2}$, and $e_H(X \cap S, \overline{A}) = \frac{\delta(H)-1}{2}$. By our choice of S, it must be the case $X \cap S = \{a\}$, and therefore, $a \in A'$. □

We choose a $q \in S$ that is adjacent in H to the most vertices in $\overline{A}$. If $A' \cap S \neq \emptyset$, then $\{x, q\} = A'$ by our choice of x.

Suppose there is an $a \in S$ such that $N_Z(a) \subseteq S$. Since $\delta(Z) \geq 2$, we know that $N_Z(a) - \{q\}$ is not empty. Thus, by Claim 8.9 a and every vertex in $N_Z(a) - \{q\}$ must be adjacent in F to every $b \in B - \Gamma_H(B)$. However, this is a contradiction since $|N_Z[a] - \{q\}| \geq \delta(Z) > \Delta(F)$. Therefore, every vertex in S is adjacent in Z to a vertex not in S.

If $|S| < \delta(H)$, then

$$(\delta(H) - (|S| - 1))|S| \leq e_H(S, \overline{S}) = \delta(H).$$

After rearranging and simplifying, we deduce the contradiction $(|S| - 1)(\delta(H) - |S|) \leq 0$. Thus, $|S| \geq \delta(H)$, and therefore,

$$\delta(H) = e_H(S, \overline{S}) \geq e_Z(S, \overline{S}) \geq |S| \geq \delta(H)$$

implies $|S| = \delta(H)$, $E_H(S, \overline{S}) = E_Z(S, \overline{S})$, and every vertex in S is adjacent in Z to exactly one vertex in $\overline{S}$.

Since $|S| = \delta(H)$ and $\delta(Z) \geq 2$, there must be an edge $aa' \in E(Z[S])$ such that $a \in S - N_F(b)$ for some $b \in B - \Gamma_H(B)$. By Claim 8.9 a or a' must be in A', and therefore, a or a' is q. Thus, $A' = \{x, q\}$, and by our choice of A we see that $|\Gamma_H(B)| \leq |\Gamma_H(A)| = 2$. Furthermore, since q is only adjacent to one vertex not in S, we see that $\delta(H) = 3$.

Since $|S| = 3$ and $q \in A'$, there are two vertices a and a' of $A - \Gamma_H(A)$ such that $S = \{q, a, a'\}$. Moreover, Since every vertex in S is adjacent in Z to exactly one vertex in $\overline{S}$ and $\delta(Z) > \Delta(F) \geq 1$, we see by Claim 8.9 that aq and $a'q$ are edges of Z. Furthermore, aa' is an edge of Z if and only if $\delta(Z) = 3$. To see this, consider that Claim 8.9 says $\Delta(F) = 2$ since both a and a' are adjacent in F to every vertex in $\Gamma_H(B)$ when aa' is an edge of Z. On the other hand, when $\delta(Z) = 3$, aa' must be an edge of Z.

Since H is 2 edge-connected and $A' = \{x, q\}$, we know there is a path P_q from q to some $q' \in \Gamma_H(B)$ and a path P_x from x to some $x' \in \Gamma_H(B)$ such that P_q and P_x are edge-disjoint and each have no interval vertices in $A \cup B$. Moreover, if $x' \neq q'$, then the existence of P_x and P_q lets us reason that xq' and $x'q$ are non-edges of H since $e_H(B, \overline{B}) < \delta(H) = 3$.

Suppose $\Delta(F) = 2$. Since $\delta(Z) > \Delta(F) = 2$, $\delta(Z) = 3$, and therefore, aa, aq, and $a'q$ are edges of Z. Since $|\Gamma_H(B)| \leq |\Gamma_H(A)| \leq 2$, $e_H(B, \overline{B}) = \lambda(H) = 2 < \delta(H) = 3$, $\delta(Z) = 3$, and P_q and P_x are edge-disjoint, we may conclude that q' is adjacent in Z to a vertex $b \in B - \Gamma_H(B)$. Let

$b' \in N_Z(b) \cap (B - \Gamma_H(B))$. By Claim 8.9 b and b' are adjacent in F to both a and a', and therefore, since $\Delta(F) = 2$, $b \notin N_F(q)$ and $a \notin N_F(q')$. However, this contradicts Claim 8.4.

Thus, we are left with the case $\Delta(F) = 1$. We let $u_0 \in N_Z(a) \cap (A - S)$ and $u_1 \in N_Z(a') \cap (A - S)$.

Suppose q' is adjacent in Z to a vertex $b \in B - \Gamma_H(B)$. Let $b' \in N_Z(b) \cap (B - \Gamma_H(B))$. Since both a and a' cannot be adjacent in F to q', Claim 8.4 implies q is adjacent in F to b. By Claim 8.6 both b' and q' are adjacent in G to at least one end of each of the edges au_0 and $a'u_1$. Thus, one of b' and q' are adjacent in G to two vertices in $\{a, a', u_0, u_1\}$. Recall that $\{u_0, u_1\} \subseteq A - S$, $\Gamma_H(A) = \{x, q\}$, and x could be one or both of u_0 and u_1. However, since xq' is a non-edge of H, any edge from b' and q' to $\{a, a', u_0, u_1\}$ must be an edge of F. This contradicts $\Delta(F) = 1$. Thus, q' is not adjacent in Z to a vertex in $B - \Gamma(B)$. Since $\delta(Z) \geq 2$, there must be a cycle C in Z such that $V(C) \subseteq B - \{q'\}$.

Since $x'q$ is a non-edge of H and $V(C) - \{x'\} \subseteq \Gamma_H(B)$, we may deduce that a, a', and q are not adjacent in H to vertices in C. Let $v \in \{a, a', q, u_0, u_1\}$. If there is a $v' \in N_Z(v) \cap A$ that is not adjacent in H to a vertex of C, then we have a contradiction since Claim 8.6 implies that v must be adjacent in F to at least two vertices on $V(C)$. Thus, every vertex in $N_Z(v)$ is adjacent in H to a vertex in $V(C)$. Suppose $|C| \geq 4$, and let v_0v_1, v_1v_2, and v_2v_3 be consecutive edges along C such that q is adjacent in F to v_0. Thus, q is not adjacent in F to $\{v_1, v_2, v_3\}$. By Claim 8.6 either a or a' must be adjacent in F to at least two vertices in $\{v_1, v_2, v_3\}$. This contradiction implies $|C| = 3$. Since $\{a, a', q\}$ are all adjacent in F to distinct vertices in C and u_0 and u_1 are adjacent in H to vertices in C we may conclude that $u_0 = u_1 = x$ and x must be adjacent in H to x'. This implies x' is adjacent along C to two vertices in $B - \Gamma_H(B)$. We have a contradiction since Claim 8.4 says either x' is adjacent in F to both a and a' or x is adjacent in F to two vertices on C. This completes the proof of Theorem 2. □

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