The Crane Operator's Trick and other Shenanigans with a Pendulum

Stephan Schlamminger,* Leon Chao, Vincent Lee, and David B. Newell

National Institute of Standards and Technology,

100 Bureau Drive, Gaithersburg, MD 20899

Clive C. Speake

University of Birmingham

(Dated: March 9, 2022)

Abstract

The dynamics of a swinging payload suspended from a stationary crane, an unwanted phenomenon on a construction site, can be described as a simple pendulum. However, an experienced crane operator can deliver a swinging payload and have it stop dead on target in a finite amount of time by carefully modulating the speed of the trolley. Generally, a series of precisely timed stop and go movements of the trolley are implemented to damp out the kinetic energy of the simple harmonic oscillator. Here, this mysterious crane operator's trick will be revealed and ultimately generalized to capture the case where the load is initially swinging. Finally, this modus operandi is applied to a torsion balance used to measure G, the universal gravitational constant responsible for the swinging of the crane's payload in the first place.

I. INTRODUCTION

Many engineers have developed robust control algorithms for the automation of industrial cranes to solve the problem of damping the swinging motion of a suspended load.[1–3] Still, few have analyzed the capabilities of the most classic controller – the human crane operator. While the pendulum provides a simple yet fascinating system to study physics,[4] most articles on damping or excitation achieve their goal by changing the effective lengths of the pendulum.[5–7] Moreover, the papers that describe damping rarely take advantage of the Laplace transformation, with 12 being an exception.[8–11] Here, we study a pendulum with a suspension point that is allowed to move horizontally via a trolley. Using the Laplace transform to obtain the pendulum's motion, we show that damping can be achieved simply by timing the suspension point's motion. In short, the authors combine their experience of driving overhead cranes (SS, LC) with their love for physics (all authors) to unveil the principles behind one of the tricks used by crane operators. This trick consists of two stop-and-go motions of the trolley. These can damp out the undesirable swinging motion of the pendulum so that it stops dead on target in a finite amount of time. These movements will be analyzed and explained in the later sections. Finally, this technique is applied to a torsion pendulum physics experiment at the National Institute of Standards and Technology (NIST) for measuring the Newtonian constant of gravitation G, showing the ubiquity of simple harmonic motion, ranging from construction sites to the modern laboratory.

Figure 1 shows an idealized diagram of a typical tower crane with two degrees of freedom: the horizontal motion provided by the trolley and the vertical motion by the hoist. For this analysis, the load is assumed to have already been lifted to the proper height, and vertical motion during liftoff and touchdown are ignored. A point load with mass m is suspended by a rope with fixed length l and negligible mass, and can swing at small angles with a period T_o. The horizontal position of the load (mass) is given by x_m and that of the trolley (on the crane) by x_c. The corresponding velocities are abbreviated by $v_\mathrm{m} = \dot{x}_\mathrm{m}$ and $v_\mathrm{c} = \dot{x}_\mathrm{c}$, respectively. The angle θ of the rope with respect to vertical can be obtained from $\sin\theta = (x_\mathrm{m} - x_\mathrm{c})/l$. With the trolley at rest, the velocity and acceleration of the load tangential to its arcuate motion are $l\dot{\theta}$ and $l\ddot{\theta}$. For small excursions ($\theta \ll 1$), the velocity and acceleration can be approximated as $\dot{x}_\mathrm{m}$ and $\ddot{x}_\mathrm{m}$, respectively. Hence, the linearized

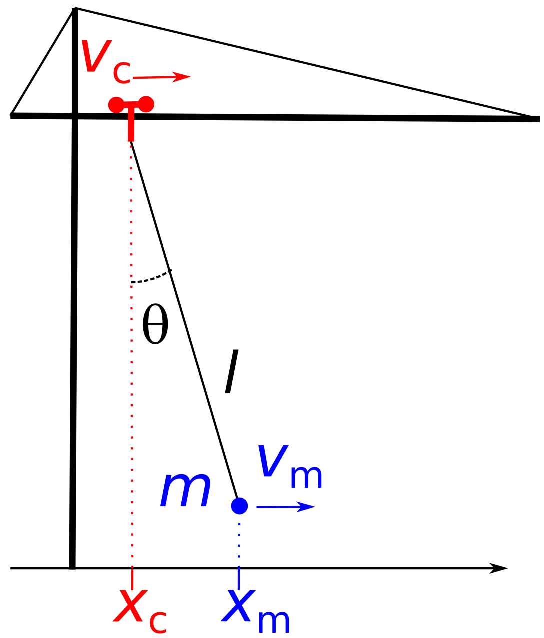

FIG. 1. A simplified diagram of a tower crane. The crane trolley is free to move along the horizontal direction. The point load with mass m is suspended from the trolley via a rope of fixed length l. We denote the horizontal position of the mass with x_{m} and that of the trolley on the crane with x_{c}. As indicated by the arrows, the corresponding velocities are abbreviated by $v_{\mathrm{m}} = \dot{x}_{\mathrm{m}}$ and $v_{\mathrm{c}} = \dot{x}_{\mathrm{c}}$, respectively.

differential equation of motion is

$$m\ddot{x}_{\mathrm{m}} + c\dot{x}_{\mathrm{m}} + (x_{\mathrm{m}} - x_{\mathrm{c}})\frac{mg}{l} = 0, \tag{1}$$

where g is the local gravitational acceleration, and c is a damping coefficient, that accounts for the fluid friction force $c\dot{x}_{\mathrm{m}}$, which is proportional to the velocity of the load. The damping term $c\dot{x}_{\mathrm{m}}$ is included in eq. (1) for completeness, but, from here on, however, we neglect it to avoid complicating the equations and overburdening the physical insight with mathematics. In addition, the experimental oscillator discussed in section V exhibits almost no damping so that this approximation is valid in that case also. Without damping, eq. (1) can be rewritten as

$$\ddot{x}_{\mathrm{m}} + \omega_o^2 x_{\mathrm{m}} = \omega_o^2 x_{\mathrm{c}}, \quad \text{where} \quad \omega_o = \frac{2\pi}{T_o} = \sqrt{\frac{g}{l}}. \tag{2}$$

Appendix A shows the derivation of eq. (2) with the Lagrange formalism. Using the Laplace transform,[13] a popular technique used in control theory for solving ordinary differential equations, eq. (2) can be transformed to the s-domain,

$$s^2 X_{\mathrm{m}}(s) + \omega_o^2 X_{\mathrm{m}}(s) = \omega_o^2 X_{\mathrm{c}}(s). \tag{3}$$

The Laplace transform is similar to the Fourier transform, but uses the complex variable s instead of $i\omega$ for converting between the time and frequency domains. We denote, as is usual, the variables in the Laplace domain with upper case letters and the ones in the time domain with lower case letters. A table in Appendix B gives the transformations that are used in this article.

Finally, the response function, the ratio of output to input, in this case the load position to the trolley position, is given by

$$R(s) := \frac{X_{\mathrm{m}}(s)}{X_{\mathrm{c}}(s)} = \frac{\omega_o^2}{s^2 + \omega_o^2}. \tag{4}$$

II. STARTING WITH THE LOAD AT REST

Let us start with a simple case that does not require any shenanigans. For $t < 0$, both the load and the trolley are initially at rest with $x_{\mathrm{m}} = x_{\mathrm{c}} = 0$. A series of two movements are examined: at $t = 0$, the trolley begins to move at constant velocity (Sec. II A), and at $t = \tau$ it comes to a stop (Sec. II B). Following these two motions, the crane operator's trick to stop the load from swinging is revealed (Sec. II C).

A. Trolley starts moving

We assume that for $t < 0$ the trolley is at rest, and at $t = 0$ it instantaneously starts to move with constant velocity v_{c}. Hence, x_{c}, determined by the Heaviside function $u(t)$ multiplied with $v_{\mathrm{c}}t$, is given by

$$x_{\mathrm{c}}(t) = u(t)v_{\mathrm{c}}t \quad \text{and} \quad X_{\mathrm{c}}(s) = \frac{v_{\mathrm{c}}}{s^2} \quad \text{with} \quad u(t) = \begin{cases} 0 & t < 0 \\ 0.5 & t = 0 \\ 1 & t > 0 \end{cases} \tag{5}$$

Note that in the formal definition, $u(0) = 0.5$, but this point is inconsequential here because $u(t)t = 0$ for $t = 0$. The corresponding Laplace transformation $X_{\mathrm{c}}(s)$ is obtained using

Appendix B so that the position of the load in the Laplace domain, $X_m(s)$ can be inferred from eq. (4):

$$X_{\mathrm{m}}(s) = R(s)X_{\mathrm{c}}(s) = \frac{\omega_o^2 v_{\mathrm{c}}}{s^4 + \omega_o^2 s^2} = \frac{v_{\mathrm{c}}}{s^2} - \frac{v_{\mathrm{c}}}{s^2 + \omega_o^2}. \tag{6}$$

Equation (6), when converted back into the time domain for $t > 0$, yields

$$x_{\mathrm{m}}(t) = v_{\mathrm{c}} t - \frac{v_{\mathrm{c}}}{\omega_o} \sin(\omega_o t). \tag{7}$$

The horizontal velocity of the load, v_{m}, is the time derivative of x_{m}:

$$v_{\mathrm{m}} = v_{\mathrm{c}} - v_{\mathrm{c}} \cos(\omega_o t). \tag{8}$$

Equation (8) shows that the average speed of the load, v_{m}, is equal to the steady speed of the trolley, v_{c}. However, its speed oscillates with an amplitude of v_{c}, so, with respect to the construction site, the load's speed varies from 0 to $2v_{\mathrm{c}}$. At $t = kT_o$, the speed of the load is zero. Here, as throughout the text, k denotes a positive integer. If the operator stops the trolley at those instants, both the velocity of the trolley, v_{c}, and the of load, v_{m}, will be zero. Furthermore, as is apparent from eq. (7), the position of both the load and the trolley will then be $v_{\mathrm{c}} k T_o$. The mass will hang straight down, with $\theta = 0$, from the resting trolley. So the pendulum will be at rest and will remain at rest until the next stimulus arises.

B. Trolley starts moving and then stops

In this section, we use the Laplace transformation to find the motion of the load after the trolley is brought abruptly to a stop, and we check the claim that stopping the trolley at times $t = kT_o$ will result in the load hanging at rest. As before, the trolley starts moving at $t = 0$ with a velocity v_{c}, but now comes to a stop at $t = \tau$. In the Laplace domain, this motion is given by

$$X_{\mathrm{c}}(s) = \frac{v_{\mathrm{c}}}{s^2} - \frac{v_{\mathrm{c}}}{s^2} \exp(-\tau s), \tag{9}$$

where the first term represents the motion starting at $t = 0$, and the second term represents the stop at a later time τ. In the Laplace domain, a time shift by τ requires multiplication by $\exp(-\tau s)$. The transformation back to the time domain gives eq. (7) added to a similar term, but with a negative sign and shifted in time by τ. Hence, the distance travelled by the load is given, for $t > \tau$.[14], by

$$x_{\mathrm{m}}(t) = v_{\mathrm{c}} t - \frac{v_{\mathrm{c}}}{\omega_o} \sin(\omega_o t) - v_{\mathrm{c}}(t - \tau) + \frac{v_{\mathrm{c}}}{\omega_o} \sin(\omega_o (t - \tau)). \tag{10}$$

Furthermore, the velocity of the load at the pendulum frequency is determined by the temporal derivative of eq. (10):

$$v_{\mathrm{m}}(t) = -v_{\mathrm{c}} \cos(\omega_o t) + v_{\mathrm{c}} \cos(\omega_o (t - \tau)). \tag{11}$$

Expanding the second term, and subsequent grouping of $\cos(\omega_o t)$ and $\sin(\omega_o t)$ terms yields

$$v_{\mathrm{m}}(t) = v_{\mathrm{c}}\Big(-1 + \cos(\omega_o \tau)\Big)\cos(\omega_o t) + v_{\mathrm{c}} \sin(\omega_o \tau) \sin(\omega_o t). \tag{12}$$

Using half-angle identities for $\omega_o \tau$, one obtains

$$v_{\mathrm{m}}(t) = 2v_{\mathrm{c}} \sin(\omega_o \tau/2)\Big[-\sin(\omega_o \tau/2)\cos(\omega_o t) + \cos(\omega_o \tau/2)\sin(\omega_o t)\Big]. \tag{13}$$

The expression in the square bracket is $\sin(\omega_o t - \omega\tau/2)$. Hence,

$$v_{\mathrm{m}}(t) = 2v_{\mathrm{c}} \sin\left(\frac{\omega_o \tau}{2}\right) \sin\left(\omega_o t - \frac{\omega_o \tau}{2}\right). \tag{14}$$

The equation is different from eq. (8) as it is the outcome of two changes in the trolley velocity instead of one. The second sine in eq. 14 describes the free pendulum motion with a phase shift. Its amplitude is modulated by the first sine that is dependent on τ. Thus, the load velocity becomes zero when $\omega_o \tau = k2\pi$, or rather $\tau = kT_o$, exactly as conjectured above.

C. The crane operator's trick

The preceding section describes a trajectory where the load is at rest both initially and finally. However, during the translation, the horizontal displacement of the load with respect to the trolley can be quite large, up to v_{c}/ω_o. To avoid pendulum motion during transport, the trolley starts at $t = 0$ with $v_{\mathrm{c}}/2$ and then, at $t = \tau$, the velocity is increased to v_{c}. The crane operator's secret is in calculating the value for τ. In the s domain that stimulus is

$$X_{\mathrm{c}}(s) = \frac{v_{\mathrm{c}}/2}{s^2} + \frac{v_{\mathrm{c}}/2}{s^2} \exp(-\tau s). \tag{15}$$

The solution for x_{m} in the time domain for $t > \tau$ is

$$x_{\mathrm{m}}(t) = v_{\mathrm{c}} t - \frac{v_{\mathrm{c}}}{2}\tau - \frac{v_{\mathrm{c}}}{2\omega_o} \sin(\omega_o t) - \frac{v_{\mathrm{c}}}{2\omega_o} \sin(\omega_o t - \omega_o \tau). \tag{16}$$

Thus, if $\omega_o \tau = (2k+1)\pi$, or $\tau = T_o/2 + kT_o$, the pendulum motion is cancelled. The crane operator's trick is to start the trolley with half the desired velocity. Half a period later, the load passes directly below the trolley with a velocity that is twice as fast as the trolley. At that point, the crane operator doubles the trolley speed to match the speed of the load. After this maneuver, or set of movements, the load glides along with constant velocity.

Since all good things must come to an end, the trolley must be stopped before the end of the track. Stopping with zero swing can be accomplished by performing the same trick in reverse: the trolley's speed is reduced to $v_\mathrm{c}/2$ and then, an odd multiple of $T_o/2$ later, reduced to zero.

D. Finite acceleration of the trolley

The analysis above has one shortcoming. It assumes that the trolley speed can go from 0 to v_c or $v_\mathrm{c}/2$ instantaneously, which, unfortunately, is impossible. Instead, assume that, at $t=0$, the trolley starts accelerating with a_c over a duration of δt. A second acceleration with the same duration begins at $t=\tau$. In the s domain, the stimulus is

$$X_\mathrm{c}(s) = \frac{a_\mathrm{c}}{s^3} - \frac{a_\mathrm{c}}{s^3}\exp(-\delta t s) + \frac{a_\mathrm{c}}{s^3}\exp(-\tau s) - \frac{a_\mathrm{c}}{s^3}\exp(-(\tau+\delta t)s). \tag{17}$$

For $t > \tau + \delta t$, the response in the time domain is

$$\begin{aligned} x_\mathrm{m}(t) = a_\mathrm{c}&\left(2t\delta t - \tau\delta t - \delta t^2\right) \\ &+\frac{a_\mathrm{c}}{\omega_o^2}\cos\left(\omega_o t\right) - \frac{a_\mathrm{c}}{\omega_o^2}\cos\left(\omega_o t - \omega_o \delta t\right) \\ &+\frac{a_\mathrm{c}}{\omega_o^2}\cos\left(\omega_o t - \omega_o \tau\right) - \frac{a_\mathrm{c}}{\omega_o^2}\cos\left(\omega_o t - \omega_o\delta t - \omega_o \tau\right). \end{aligned} \tag{18}$$

The term in the first line describes the motion with constant velocity and is identical to that of the trolley. Note that, for $t > \tau + \delta t$, the velocity of the trolley is $v_\mathrm{c} = 2a_\mathrm{c}\delta t$. For $\tau = T_o/2 + kT_o$, the first and third cosines as well as the second and fourth cosines are the same with opposite sign and cancel. Hence, all oscillatory terms vanish for $\tau = T_o/2 + kT_o$, and the load tracks the trolley.

In summary, both crane operator tricks are valid even if the trolley's acceleration is finite. Note that we assume that both accelerations, at $t=0$ and at $t=\tau$, are identical and applied for the same duration, δt. This assumption may not hold in a real-world situation, such as if the motor moving the trolley outputs a constant mechanical power.

III. STARTING WITH A SWINGING LOAD

The scenario discussed in this section may be uncommon in the world of crane operators, but is applicable to the physics experiment discussed later. As previously, we assume the trolley to be initially at rest at $x_{\mathrm{c}} = 0$, but the load is now assumed to be swinging. Without loss of generality, we set $t = 0$ as the time when the load swings directly under the trolley at $x_{\mathrm{m}} = 0$ with positive velocity $v_{\mathrm{m}}(0) = v_o$. One could, for instance, imagine that a gust of wind causes the load to swing at the beginning of the crane operator's shift in which the task is to move the load from $x_{\mathrm{m}} = 0$ to $x_{\mathrm{m}} = x_g$, so that it arrives with the smallest possible pendulum amplitude. To avoid unnecessary complication, we assume that the trolley can move with infinite acceleration but has a maximum speed of v_{c}.

The load's motion will be damped using two identical trolley movements performed at a known speed v_c each of duration δt. The combined distance travelled by the trolley is $x_g = 2v_{\mathrm{c}}\delta t$. To succeed in the task, the crane operator has to quickly solve the equations of motion and determine the times t_1 and $t_1 + \tau$ when these trolley operations are to be executed, as the ground recipients are notoriously impatient.

The two moves can be described in the Laplace domain similar to eq. (15). But how should the initial conditions be taken into account? The easiest way is to assume that the pendulum is at rest, and add a delta-function impulse term to the stimulus which reproduces the initial conditions. This term is virtual because the crane operator does not actually have to physically execute this step. If the trolley experiences an initial impulse given by $x_{\mathrm{c}} = \delta(t)$ at $t = 0$, then the pendulum at rest has, by definition of the response function

$$X_{\mathrm{m}}(s) = \frac{\omega_o^2}{s^2 + \omega_o^2} \quad \longrightarrow \quad x_{\mathrm{m}}(t) = \omega_o \sin(\omega_o t) \quad \longrightarrow \quad v_{\mathrm{m}}(t) = \omega_o^2 \cos(\omega_o t) \ \text{ for } \ t > 0. \tag{19}$$

To obtain the desired initial condition, i.e., a velocity v_o, the virtual stimulus must therefore be $x_c = (v_o/\omega_o^2)\,\delta(t)$. The combination of the virtual and real trolley motions in the Laplace domain is

$$\begin{aligned} X_{\mathrm{c}}(s) = {} & \frac{v_o}{\omega_o^2} + \frac{v_{\mathrm{c}}}{s^2}\exp(-t_1 s) \\ & - \frac{v_{\mathrm{c}}}{s^2}\exp(-t_1 s - \delta t\, s) + \frac{v_{\mathrm{c}}}{s^2}\exp(-t_1 s - \tau s) - \frac{v_{\mathrm{c}}}{s^2}\exp(-t_1 s - \tau s - \delta t\, s). \end{aligned} \tag{20}$$

Multiplying this by the response function and transforming back into the time domain gives for $t > t_1 + \tau + \delta t$

$$\begin{aligned} x_m(t) = 2v_c\delta t + \frac{v_o}{\omega_o}\sin(\omega_o t) \\ -\frac{v_c}{\omega_o}\sin(\omega_o t - \omega_o t_1) - \frac{v_c}{\omega_o}\sin(\omega_o t - \omega_o t_1 - \omega_o \tau) \\ +\frac{v_c}{\omega_o}\sin(\omega_o t - \omega_o t_1 - \omega_o \delta t) + \frac{v_c}{\omega_o}\sin(\omega_o t - \omega_o t_1 - \omega_o \tau - \omega_o \delta t). \end{aligned} \tag{21}$$

Since we are interested in the pendulum motion of the load after the maneuver, we take the temporal derivative of the previous equation which yields

$$v_m(t) = \cos(\omega_o t)\Big[v_o - v_c\cos(\omega_o t_1) + v_c\cos(\omega_o t_1 + \omega_o \delta t) - v_c\cos(\omega_o t_1 + \omega_o \tau) \tag{22}$$

$$+ v_c\cos(\omega_o t_1 + \omega_o\delta t + \omega_o \tau)\Big] +$$

$$\sin(\omega_o t)\left[4v_c\cos\left(\frac{\omega_o \tau}{2}\right)\cos\left(\frac{2\omega_o t_1 + \omega_o\tau + \omega_o\delta t}{2}\right)\sin\left(\frac{\omega_o\delta t}{2}\right)\right]. \tag{23}$$

For the pendulum to be at rest, the terms within the two square brackets must be equal to zero. Inspecting the terms inside the second set of square brackets yields two possible ways the product can be zero. First, one could choose $\omega_o\tau = \pi$. However, for that case, the result would be

$$v_m(t) = v_o\cos(\omega_o t). \tag{24}$$

As we have seen in the previous section, making two moves an odd multiple of half a period apart will not change the energy stored in the pendulum's motion, and the load's amplitude will not be damped. The second way to make the terms vanish would be

$$2\omega_o t_1 + \omega_o\tau + \omega_o\delta t = \pi + 2k\pi \implies t_1 = \frac{T_o}{4} - \frac{\tau}{2} - \frac{\delta t}{2} + kT_o/2. \tag{25}$$

Then the velocity of the pendulum for $t > t_1 + \tau + \delta t$ is

$$v_m(t) = \cos(\omega_o t)\left(v_o \pm 4v_c\cos\left(\frac{\tau\omega_o}{2}\right)\sin\left(\frac{\delta t\omega_o}{2}\right)\right). \tag{26}$$

The sign before $4v_c$ is positive if k in eq. (25) is even and negative if it is odd. The pendulum motion can be damped to zero as long as $v_o \leq 4v_c$. For the case $v_o = 4v_c$, the choices for τ and δt would be $\tau = T_o/2 + k_1 T_o$ and $\delta t = 2k_2 T_o$, with $\tau > \delta t$ and k_1, k_2 positive integers. Usually δt is determined by the distance $x_g = 2v_c\delta t$ over which the load needs to be transported. For a given δt, the largest possible amplitude that can be damped to zero is

$$v_o \leq 4v_c\sin\left(\frac{\delta t\omega_o}{2}\right). \tag{27}$$

To accomplish this, τ must be chosen as

$$\tau = \frac{T_o}{\pi} \arccos\left(\frac{v_o}{\mp 4 v_\mathrm{c} \sin\left(\frac{\delta t \omega_o}{2}\right)}\right) + 2kT_o. \tag{28}$$

Furthermore, if

$$v_o > 4 v_\mathrm{c} \sin\left(\frac{\delta t \omega_o}{2}\right), \tag{29}$$

then the load will not be at rest after the maneuver. However, the energy of the pendulum can be minimized by setting $\cos\left(\frac{\tau\omega_o}{2}\right) = 1$, or in other words

$$\tau = 2kT_o. \tag{30}$$

IV. EXAMPLES BY SIMULATION

In this section, we examine two examples based on the equations derived previously. We assume $l = 20\,\mathrm{m}$, resulting in a pendulum period of 8.97 s for small angles. The goal is to move the load by 6 m horizontally to the right with a trolley speed of $v_\mathrm{c} = 1\,\mathrm{m}\cdot\mathrm{s}^{-1}$, yielding $\delta t = 3\,\mathrm{s}$.

In the first example, it is assumed that the load has a velocity amplitude of $v_o = 1\,\mathrm{m}\cdot\mathrm{s}^{-1}$. According to equation (28), $\tau = 5.32\,\mathrm{s}$ which, as an input for eq. (25), yields $t_1 = 2.57\,\mathrm{s}$. Fig. 2 shows a numerical simulation of this example where the solid traces represent the position x_m (bottom panel) and the velocity v_m (top panel) of the load while the dotted traces represent the position x_c and velocity v_c of the trolley. To better visualize the initial swing, t_1 begins a full period late, i.e. $t_1 = 8.97\,\mathrm{s} + 2.57\,\mathrm{s} = 11.54\,\mathrm{s}$.

The calculation has been made by numerically solving the differential equation (1) with the Runge-Kutta method. The program is written in Python and is freely available at https://github.com/schlammis/pendulum. The simulation also allows solving the nonlinear differential equation of motion, and the equation when friction is non zero ($c \neq 0$). Both topics would go beyond the scope of this article.

The second example is a case where the initial velocity amplitude of the load $v_o = 5\,\mathrm{m}\cdot\mathrm{s}^{-1}$ is larger than four times the trolley speed. According to eq. (30), $\tau = 17.95\,\mathrm{s}$ which yields $t_1 = 0.74\,\mathrm{s}$. Fig. 3 shows the position and velocity of both the load and the trolley in this case. Similar to the previous example, t_1 begins a full period late at $t_1 = 9.72\,\mathrm{s}$. The wait time between the two movements is twice the pendulum period, $\tau = 2\,T_o = 17.95\,\mathrm{s}$. Due

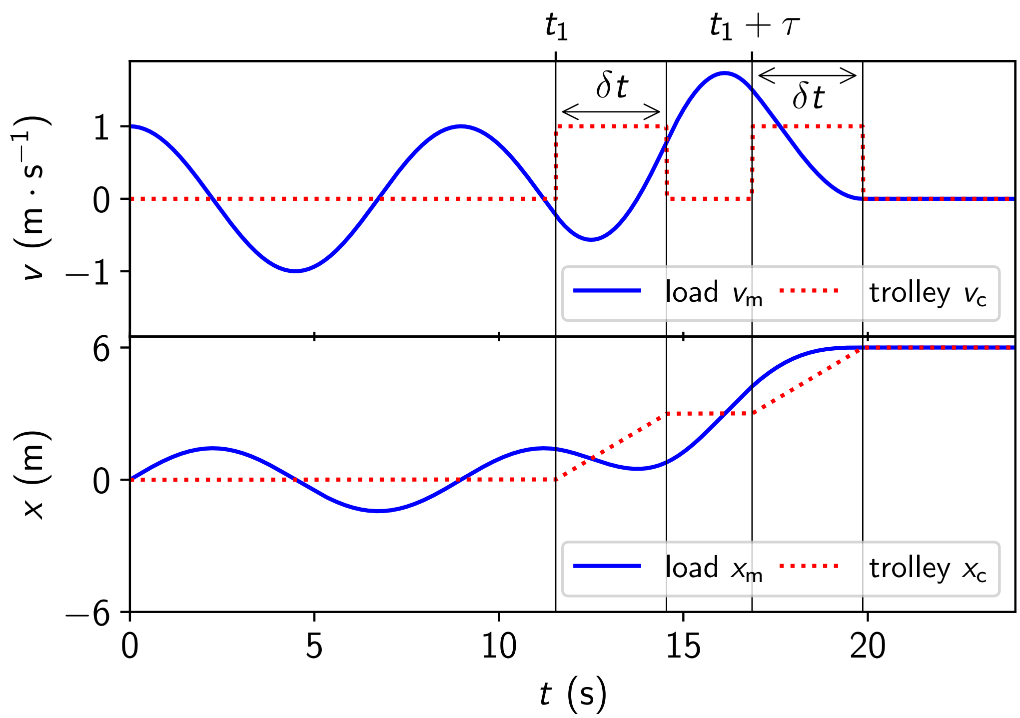

FIG. 2. Damping a load that swings with an initial velocity amplitude $v_{\mathrm{m}}(0) = v_o = 1\,\mathrm{m}\cdot\mathrm{s}^{-1}$. Top: solid blue line indicates the velocity v_{m} of the load and dotted red line the velocity v_{c} of the trolley. Bottom: the solid blue line indicates the position x_{m} of the load and the dotted red line the position x_{c} of the trolley. After a maneuver, or two trolley movements, the load is at rest at the target position.

to the large initial velocity of the load, the motion can only be attenuated and not fully removed after the maneuver.

V. APPLICATION IN THE LABORATORY

The equations derived in the precious sections do not only apply at a construction site. They can also be helpful in a physics laboratory, especially one involving a harmonic oscillator, which is not so unusual in science and engineering. In the present case, the above-described crane shenanigans are directly applicable to a torsion balance that is used to measure the gravitational constant G.[15]

As shown in fig. (4), the torsion balance is comprised of four cylindrical copper test

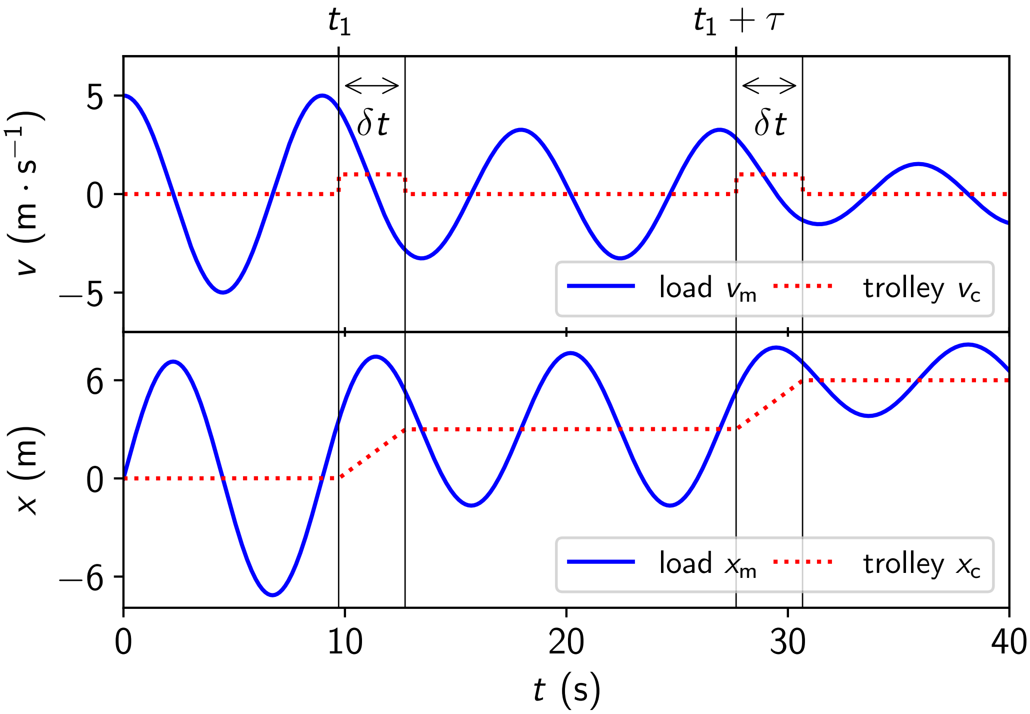

FIG. 3. Damping a load that swings with an initial velocity amplitude $v_{\mathrm{m}}(0) = v_o = 5\,\mathrm{m}\cdot\mathrm{s}^{-1}$. Top: the solid blue line indicates the velocity v_{m} of the load and the dotted red line the velocity v_{c} of the trolley. Bottom: the solid blue line indicates the position x_{m} of the load and the dotted red line the position x_{c} of the trolley. After the maneuver, the load's motion has a significantly attenuated amplitude at the target position.

masses, each of mass approximately 1 kg, resting on a disk suspended from a weak torsion spring. This assembly is inside a vacuum vessel. An autocollimator monitors the small angle φ_{t} the disk makes with respect to a fixed direction via one of four mirrors attached to the disk. The measurement range of the autocollimator is $\pm 0.17°$. A source-mass assembly consisting of four cylindrical source masses on a carousel is located outside the vacuum chamber. The operator can rotate the carousel around the symmetry center to an angle φ_{s} with the help of a stepper motor.

Depending on the difference $\varphi_{\mathrm{s}} - \varphi_{\mathrm{t}}$, a gravitational torque n can act on the pendulum. The torque can be written under the form $n = G\Gamma(\varphi_{\mathrm{s}} - \varphi_{\mathrm{t}})$, where $\Gamma(\varphi)$ is a function that depends on the angle difference, but also on the mass distribution of the experiment (itself depending on the various distance of the masses to the rotation center, their height, and densities). Experimentally, numerical integration over the test and source mass volumes is necessary to determine Γ with relative uncertainties of $\sim 1 \times 10^{-6}$.

The experiment is performed as follows. The source masses are moved to the angle where the maximum clockwise torque acts on the pendulum, $\varphi_{\mathrm{s}} = \varphi_{\mathrm{s,max}}$. Due to the gravitational torque, the pendulum oscillates around an equilibrium position $\varphi_{\mathrm{t,max}}$ given by $\kappa(\varphi_{\mathrm{t,max}} - \varphi_{\mathrm{t,0}}) = G\Gamma(\varphi_{\mathrm{s,max}} - \varphi_{\mathrm{t,max}})$. The torque produced by the torsion strip, $\kappa(\varphi_{\mathrm{t,max}} - \varphi_{\mathrm{t,0}})$, counteracts the gravitational torque, and $\varphi_{\mathrm{t,0}}$ denotes the unknown equilibrium position of the pendulum in the absence of external torque. The torsion pendulum is then observed with an autocollimator for a few periods, to determine $\varphi_{\mathrm{t,max}}$, after which the source masses are moved counterclockwise to the position $\varphi_{\mathrm{s,min}}$ where the torque acting on the pendulum is minimal, i.e., maximal counterclockwise torque. The pendulum is observed again, and we obtain $\kappa(\varphi_{\mathrm{t,min}} - \varphi_{\mathrm{t,0}}) = G\Gamma(\varphi_{\mathrm{s,min}} - \varphi_{\mathrm{t,min}})$. The procedure is repeated to achieve a good precision on $\varphi_{\mathrm{t,max}} - \varphi_{\mathrm{t,min}}$. By using the difference, the unknown angle $\varphi_{\mathrm{t,0}}$ drops out. From these measurements, the gravitational constant can be obtained as

$$G = \kappa \frac{\varphi_{\mathrm{t,max}} - \varphi_{\mathrm{t,min}}}{\Gamma(\varphi_{\mathrm{s,max}} - \varphi_{\mathrm{t,max}}) - \Gamma(\varphi_{\mathrm{s,min}} - \varphi_{\mathrm{t,min}})}, \tag{31}$$

where the numerator is obtained from the successive measurements discussed above and the denominator from the numerical integration of Γ. The torsional constant, κ, is obtained by measuring the resonant frequency of the pendulum, f_o, and numerically calculating the pendulum's moment of inertia, I: $\kappa = I(2\pi f_o)^2$. For a more complete description of the setup, see 15.

For our geometry, the source mass positions of $\varphi_{\mathrm{s,max}} = -\varphi_{\mathrm{s,min}} \approx 18°$ produce an angle difference of $\varphi_{\mathrm{t,max}} - \varphi_{\mathrm{t,min}} = 0.00874° = 153\,\mu\mathrm{rad}$. One detail worth mentioning is that the pendulum is constantly swinging. Hence, to determine the equilibrium position of the pendulum with the source masses in either state, a sine function must be fitted to the data obtained with the autocollimator, offset by $\varphi_{\mathrm{t,max}}$ or $\varphi_{\mathrm{t,min}}$. The uncertainty in the equilibrium position is smaller for smaller pendulum amplitude, for, in this case, the nonlinearities of the autocollimator affect the results to a lesser degree.

The differential equation describing the system is

$$\ddot{\varphi}_{\mathrm{t}} + \omega_o^2 \varphi_{\mathrm{t}} = \frac{n\,(\varphi_{\mathrm{s}} - \varphi_{\mathrm{t}})}{I}, \tag{32}$$

where $n\,(\varphi_{\mathrm{s}} - \varphi_{\mathrm{t}}) = G\Gamma(\varphi_{\mathrm{s}} - \varphi_{\mathrm{t}})$, is the gravitational torque that is applied by the operator via moving the source mass assembly to φ_s. Note that the omission of damping is justified

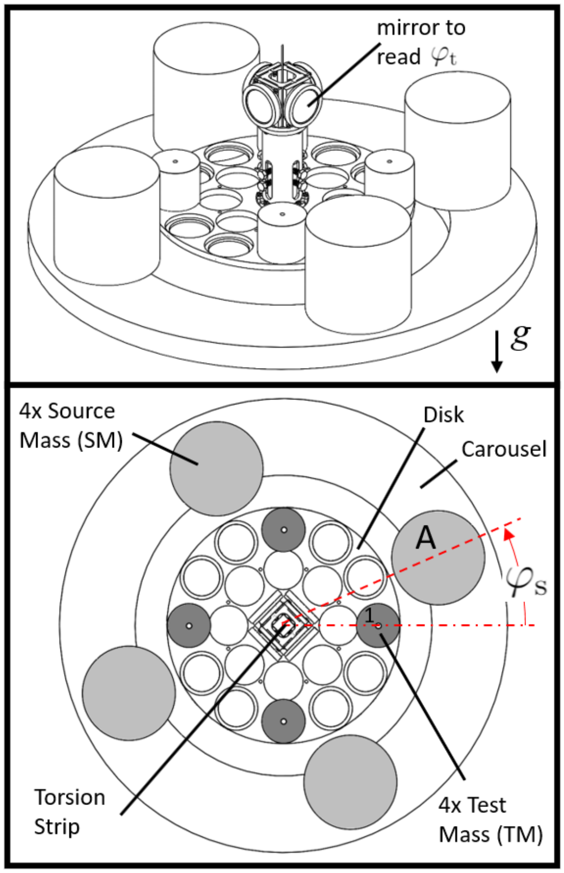

FIG. 4. Top: Three-dimensional model of the torsion balance. Bottom: Top view of the system. The torsion balance is composed of a disk suspended by a thin torsion strip, inside a vacuum chamber (omitted for clarity), on which four cylindrical test masses are placed at 90° from one another. Outside the chamber are four cylindrical source masses on a carousel that can rotate about the strip center. The angle of the torsion balance is measured from the dash-dotted line to the test mass labelled 1. This angle is very small, $|\varphi_{\mathrm{t}}| << 0.01°$. The angle of the source mass assembly φ_{s} is measured from the dash-dotted line to the source mass labelled A. The gravitational torque depends on $\varphi_{\mathrm{s}} - \varphi_{\mathrm{t}} \approx \varphi_{\mathrm{s}}$ and is maximal for $\varphi_{\mathrm{s}} = \varphi_{\mathrm{s,max}}$ with $\varphi_{\mathrm{s,max}} \approx 18°$.

here since the quality factor of the pendulum is very large ($\approx 100\,000$). Here, $\varphi_{\mathrm{s}} >> \varphi_{\mathrm{t}}$, hence, $n\left(\varphi_{\mathrm{s}} - \varphi_{\mathrm{t}}\right) \approx n\left(\varphi_{\mathrm{s}}\right)$. In the Laplace domain, the response function is

$$\frac{\Phi_t(s)}{N(s)} = \frac{1}{I}\frac{1}{s^2 + \omega_o^2}, \tag{33}$$

which closely resembles eq. (4).

The same crane operator is in charge of maneuvering the source masses, and, like before, only two movements per maneuver are allowed. The task is to optimize the timing of both movements, given by the parameters t_1, δt, and τ to damp the pendulum as much as possible. One difference here is that the torque does not change linearly with respect to the source mass position but depends on $n_o \sin\big((\varphi_\mathrm{s}/\varphi_\mathrm{s,max})(\pi/2)\big)$, where $n_o = 1.56 \times 10^{-8}\,\mathrm{N \cdot m}$.

Note that this expression is the first order expansion if the masses were point masses. It is a reasonable accurate approximation for the cylindrical masses used in the experiment. The problem can then be solved with a method similar to what has been described for the crane. The exact differential equation of motion and their solutions are beyond the scope of this text but can be found on the GitHub page mentioned earlier. Spoiler alert: the crane operator's trick similar to the one described in Sec. III but adapted to torques can be written in the Laplace domain

$$N(s) = \frac{n_o}{2}\left(\frac{e^{-st_1}}{s} - \frac{se^{-st_1}}{s^2+\nu^2} - \frac{\nu e^{-st_2}}{s^2+\nu^2} + \frac{\nu e^{-st_3}}{s^2+\nu^2} - \frac{se^{-st_4}}{s^2+\nu^2} + \frac{e^{-st_4}}{s}\right), \tag{34}$$

with $\nu = \pi/(2\delta t)$, $t_2 = t_1 + \delta t$, $t_3 = t_2 + \tau$, and $t_4 = t_3 + \delta t$.

Fig. (5) shows the pendulum angular position φ_t as a function of time. When the source mass is set at $+\varphi_\mathrm{s,max}$, the corresponding φ_s and φ_t curves are in magenta, whereas when it is set at $-\varphi_\mathrm{s,max}$, the corresponding φ_s and φ_t curves are in cyan. The pendulum response to the source mass maneuvers are plotted with thin dotted lines. To understand the source mass motion plotted in Fig. (5), recall that there is a maximum amplitude that can be damped in one maneuver, see eq (29). For the torsion pendulum, this amplitude corresponds to 146 µrad. To achieve this reduction, the wait time between the two moves in one maneuver is $\tau - \delta t = 107\,\mathrm{s}$. This wait time is long, almost a full period. Alternatively, by setting $\tau = \delta t$, the pendulum swing amplitude is reduced by 137 µrad instead of 146 µrad. So, the amplitude reduction per maneuver is worse, but the average reduction per unit time is greater. Hence, for amplitude larger than 137 µrad, τ is set to zero. For smaller amplitudes, the ideal τ is calculated. In the latter case, the wait time between two moves is clearly visible in the last two maneuvers in Fig. (5). More information on this topic can be found in the supplemental information.

At the beginning of the data shown in Fig. (5), the pendulum had an amplitude of 1830 µrad. After 15 maneuvers the amplitude has been reduced to 1.5 µrad, which shows how effective the crane operator's trick is. However, an attenuation of 99.9% seems to be the limit of this method for two main reasons. (1) there is some variation in δt, the time it takes to perform one movement and (2) the torque on the pendulum is not exactly proportional to the sine of the angular position of the source masses as mentioned earlier. Other than that, this work would likely get the nod of approval from the Society of Meticulous Crane Operators.

From the physics point of view, the damping of the torsion pendulum is desirable because it ultimately reduces the uncertainty of the angular position during the measurement. The values of $\varphi_{\mathrm{t,max}}$ and $\varphi_{\mathrm{t,min}}$ are determined by measuring the pendulum motion over one period. Fitting a large amplitude curve to determine the equilibrium position is more prone to systematic effects caused by nonlinearities of the autocollimator. Here, the authors employ the crane operator's tricks to achieve some serious damping: The amplitude of the oscillations in φ_{t} after an hour of such maneuvers is about 1200 times smaller than the initial amplitude. In contrast, a wait time of the order of months would be necessary before the oscillations naturally decay via dissipation in the torsion strip to the same levels if no trick is applied.

VI. CONCLUSION

In this paper, we have taken a light-hearted look at a practical application of Newtonian mechanics, using Laplace transformation. We started with a simple example. The trolley of a crane starts moving with constant velocity. We calculated the effect this impetus has on a classical harmonic oscillator: the load suspended from that trolley. The translation property of the Laplace transformation allows us to easily consider the effect of two such impulses in opposite directions, i.e., starting and stopping the trolley. With a few trigonometric manipulations, we showed that if the two impulses are separated in time by an integer number of pendulum periods, no energy is added or subtracted to the pendulum motion of the load. In the classroom, the formalism can be shown on the whiteboard, with a simulation using the code provided on GitHub, and experimentally by moving the top of a plumb bob by hand. The technique can be extended to include the initial swinging condition of the pendulum. We use a variation of the crane operator's trick on a torsion balance in our current research, determining the gravitational constant.

We believe the ideas presented in this paper can be valuable for the classroom. First, the math is not too complicated and can be followed on the whiteboard. Second, the freely

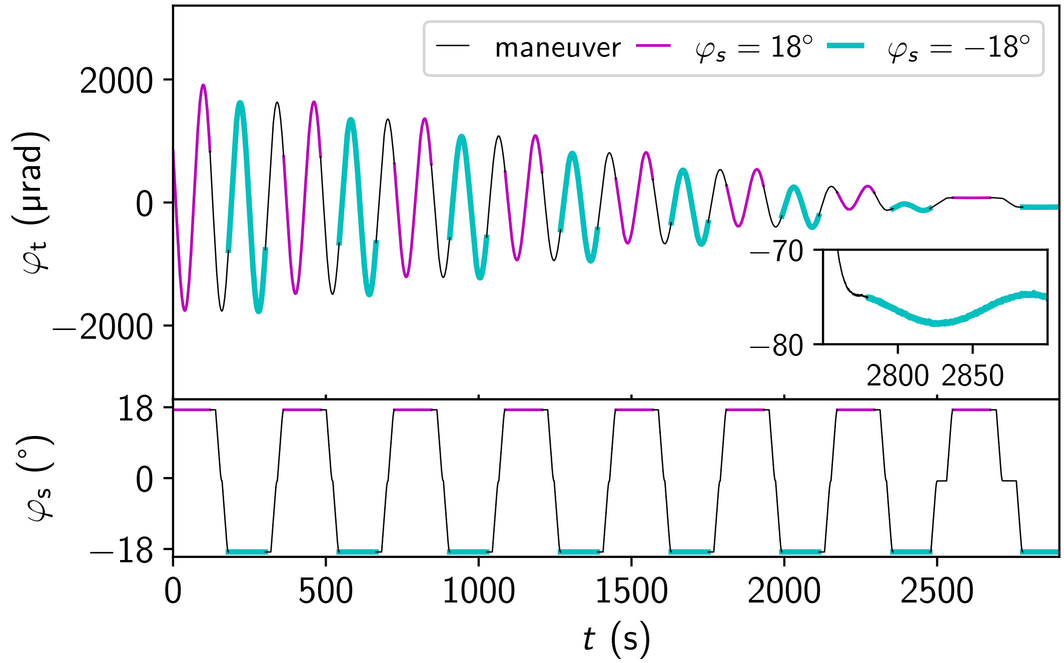

FIG. 5. Angular positions of the torsion pendulum (φ_t, top graph) and of the source mass assembly (φ_s, bottom graph) as a function of time. In both graphs, three different line thicknesses are used: The one-period-long data sets indicated with medium and thick lines were taken with the source-mass assembly at $\varphi_s = 18°$ and $\varphi_s = -18°$, respectively. These data sets are used to determine G. In between the two extreme states, the source masses are moved according to the procedure described in the text. The data taken during the maneuver (not used to determine G) are shown as thin lines. The damping of the oscillatory motion of φ_t can be seen. The inset in the top graph shows the pendulum motion after a little more than an hour. The torsional amplitude is 1.6 µrad, more than 1000 times smaller than at the beginning of the experiment.

available python code can be downloaded, and students and teachers alike can play with different parameters, including damping. Finally, the motivated reader may endeavor to build an actual experimental realization of the systems discussed here – either a crane or a torsion balance. The topic can be approached from many different angles and is therefore fun and educational.

Applying the techniques discussed above, we found it satisfying that the torsion pendulum follows precisely the simple, albeit uncommon, mathematical formalism. We hope that the students and their teachers can find the same satisfaction in this system, even if only by simulation. This satisfaction can inspire a lifelong passion for understanding the world using mathematical and physical reasoning.

Appendix A: Derivation of the differential equation

The horizontal position of the trolley is given by a twice differentiable function $x_{\mathrm{c}}(t)$. The vertical position of the trolley, $y_{\mathrm{c}} = l$, and the length of the rope l are fixed. Then, the coordinates of the load are given by

$$x_{\mathrm{m}} = x_{\mathrm{c}} + l\sin\theta \quad \text{and} \quad y_{\mathrm{m}} = l - l\cos\theta. \tag{A1}$$

The corresponding velocities are

$$\dot{x}_{\mathrm{m}} = \dot{x_{\mathrm{c}}} + l\dot{\theta}\cos\theta \quad \text{and} \quad \dot{y}_{\mathrm{m}} = l\dot{\theta}\sin\theta. \tag{A2}$$

The Lagrangian is,

$$L = T - V = \frac{m}{2}\left(\dot{x}_{\mathrm{c}}^2 + 2l\dot{x}_{\mathrm{c}}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right) + mgl\cos\theta. \tag{A3}$$

The exact Lagrange equation for θ is

$$l\ddot{\theta} + \ddot{x}_{\mathrm{c}}\cos\theta + g\sin\theta = 0. \tag{A4}$$

To first order, we approximate $\cos\theta \approx 1$ and $\theta \approx \sin\theta = (x_{\mathrm{m}} - x_{\mathrm{c}})/l$. With $g/l = \omega_o^2$, we obtain

$$\ddot{x}_{\mathrm{m}} + \omega_o^2 x_{\mathrm{m}} = \omega_o^2 x_{\mathrm{c}}, \tag{A5}$$

which is given in the main text as eq. (2).

Appendix B: Laplace transformations used in this article

name	$f(t)$	$F(s)$
impulse	$\delta(t)$	1
unit step	$u(t)$	s^{-1}
unit ramp	$u(t)\,t$	s^{-2}
unit acceleration	$u(t)\,\frac{1}{2}t^2$	s^{-3}
sine for $t > 0$	$u(t)\,\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
cosine for $t > 0$	$u(t)\,\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
translation in time	$f(t-t_o)$	$\exp(-t_o s)F(s)$
derivative	$\frac{\mathrm{d}f(t)}{\mathrm{d}t}$	$sF(s)$
integral	$\int_0^t f(\tau)\mathrm{d}\tau$	$s^{-1}F(s)$

* stephan.schlamminger@nist.gov

1 W.J. O'Connor, "A Gantry Crane Problem Solved," *ASME. J. Dyn. Sys., Meas., Control* **125**(4), 569–576 (2004).

2 M. Hicar *et al.*, "Robust Crane Control," *Acta Polytechnica Hungarica* **3**(2) (2006).

3 S. Garrido *et al.*, "Anti-Swinging Input Shaping Control of an Automatic Construction Crane," *IEEE Trans. Autom. Sci. Eng.* **5**(3), 549-557 (2008).

4 R.A. Nelson and M.G. Olsson, "The pendulum —- Rich physics from a simple system," *Am. J. Phys.* **54**, 112 (1975).

5 P.L. Tea and H. Falk, "Pumping on a Swing," *Am. J. Phys.* **36**, 1165 (1968).

6 W. B. Case, "The Pumping of a Swing from the Standing Position," *Am. J. Phys.* **64**, 215 (1996).

7 P. Glendinning, "Adaptive resonance and pumping a swing," *Eur. J. Phys.* **41**, 025006 (1996).

8 F.S. Crawford, "Damping of a simple pendulum," *Am. J. Phys.* **43**, 276 (1975).

9 P.T. Squire, "Pendulum damping," *Am. J. Phys.* **54**, 276 (1975).

10 J.C. Simbach and J. Priest, "Another look at a damped physical pendulum," *Am. J. Phys.* **73**, 1079 (2005).

11 G.D. Quiroga and P.A. Ospina-Henao , "Dynamics of damped oscillations: physical pendulum," *Eur. J. Phys.* **38**, 065005 (2017).

[12] R.C. Hudson and C.R. Finfgeld, "Laplace Transform Solution for the Oscillator Damped by Dry Friction," *Am. J. Phys.* **39**, 568 (1971).

[13] G.B. Arfken, H.J. Weber, F.E, Harris, "Mathematical Methods for Physicists – A comprehensive guide," Waltham, MA , *Elsevier* 2013.

[14] Note that all equations in this article can be made valid for all times, but that would require multiplying some terms with $u(t-\tau)$ or similar expressions. Since we are only interested in the end state, we decided to omit these terms and add a disclaimer to describe when the equations are valid.

[15] T. Quinn *et al.*, "The BIPM measurements of the Newtonian constant of gravitation, G," *Phil. Trans. R. Soc. A* **372** 20140032 (2014).