# ON THE FAMILY OF ELLIPTIC CURVES $X + 1/X + Y + 1/Y + t = 0$

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**Abstract**

We study various properties of the family of elliptic curves $x+1/x+y+1/y+t=0$, which is isomorphic to the Weierstrass curve

$$E_t : Y^2 = X\left(X^2 + \left(\frac{t^2}{4} - 2\right)X + 1\right).$$

This equation arises from the study of the Mahler measure of polynomials. We show that the rank of $E_t(\overline{\mathbb{Q}}(t))$ is 0 and the torsion subgroup of $E_t(\mathbb{Q}(t))$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Over the rational field $\mathbb{Q}$ we obtain infinite subfamilies of ranks (at least) one and two, and find specific instances of $E_t$ with rank 5 and 6. We also determine all possible torsion subgroups of $E_t(\mathbb{Q})$ and conclude with some results regarding integral points in arithmetic progression on $E_t$.

## 1. Introduction and Main Result

The family of polynomials

$$P_t(x, y) : x + \frac{1}{x} + y + \frac{1}{y} + t,$$

has attracted significant attention. The polynomial $P_t(x, y)$ is well known for those who are familiar with Mahler measure. The (logarithmic) Mahler measure of a non-zero Laurent polynomial, $P(x_1, \ldots, x_n)$, with complex coefficients is defined as

$$m(P) = \int_0^1 \cdots \int_0^1 \log|P(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})| dt_1 \cdots dt_n.$$

In [3], Boyd studied the Mahler measure of several families of polynomials. In particular, he considered the two-variable family

$$P_t(x, y) = x + \frac{1}{x} + y + \frac{1}{y} + t,$$

where $t \in \mathbb{N}$. The zeros of $P_t(x, y)$ correspond, generically, to a curve of genus 1. Let $E_t$ denote the elliptic curve corresponding to the algebraic closure of $P_t(x, y) = 0$. Let us denote $m(t) = m(P_t)$. Boyd computed $m(t)$ for positive integers $t$ less than or equal to 100. He found that

$$m(t) \stackrel{?}{=} r_t L'(E_t, 0),$$

where $r_t$ is a rational number, $L(E, s)$ is the well-known $L$-series, and the question mark stands for an equality that has only been established numerically (typically to at least 50 decimal places).

There are many conjectures (a few of which are now theorems) related to the family $P_t(x, y)$ [8, 22, 23, 24]. For example, in [8] Deninger proved the formula

$$m(x + \frac{1}{x} + y + \frac{1}{y} + 1) \stackrel{?}{=} \frac{15}{4\pi^2} L(E, 2) = L'(E, 0),$$

where the Laurent polynomial defines an elliptic curve $E$ of conductor 15, and $L(E, 2)$ is its $L$-series at $s = 2$.

In this article we study the polynomial $P_t(x, y)$ in a different direction. The algebraic closure of $P_t(x, y) = 0$ is a genus one curve which we denote by $E_t$. Over the years, several authors have given considerable effort to study different families of elliptic curves. See, for example, [10, 13, 14, 15, 30]. First we explore $E_t$ over the function field $\mathbb{Q}(t)$. Then we will study $E_t$ over the rational field $\mathbb{Q}$. We will also try to find high rank curves in the family $\{E_t\}$, as well as explore the integral points on $E_t$.

The organization of this paper is as follows. In Section 2, we will introduce the notion of elliptic surfaces and study the curve $E_t$ over the function field $\mathbb{Q}(t)$. In Section 3, we consider the curve family over $\mathbb{Q}$ and examine the torsion subgroup, before constructing infinite families with ranks (at least) 1 and 2. We also run some experiments to find high rank curves in the family $E_t$. Finally, we explore integral points which are in arithmetic progression in Section 4, and give some directions for future study.

## 2. Elliptic Surfaces

The aim of this section is to study the curve $E_t$ over $\mathbb{Q}(t)$. First we recall some basic notions about elliptic surfaces.

**Definition.** Let $C$ be a smooth, irreducible projective curve over an algebraically closed field $k$. An elliptic surface over $C$ is a pair $(S, f)$, where $S$ is a smooth, irreducible, projective surface over $k$, and $f : S \longrightarrow C$ is a relatively minimal elliptic fibration having a singular fiber and a zero section. We often write $f : S \longrightarrow C$ to denote the elliptic surface $(S, f)$ over $C$.

Let $k(C)$ denote the function field of the curve $C$. Given an elliptic curve $E$ over $k(C)$, one can associate an elliptic surface $f : \mathcal{E} \longrightarrow C$ with generic fiber $E$, the existence and uniqueness of which is guaranteed by the work of Kodaira and Néron. This elliptic surface is known as the Kodaira-Néron model of the elliptic curve $E$ over $k(C)$.

Given that all the relevant results needed to prove our main theorem are well known, we just give their statements and omit the proofs.

**Theorem 1** ([26, Corollary 2.2])**.** *Let $(S, f)$ be an elliptic surface over $C$. The Néron-Severi group, denoted $NS(S)$, is finitely generated and torsion-free.*

Recall the classical Shioda-Tate formula.

**Theorem 2** ([26, Corollary 5.3])**.** *Let $(S, f)$ be an elliptic surface over $C$. For each point $v$ of $C$ having singular fiber, let $m_v$ denote the number of components of the singular fiber above $v$. Let $E$ denote the generic fiber of $S$. The rank of the Néron-Severi group of $S$, denoted $\rho(S)$, can be obtained from the equality*

$$\rho(S) = \text{rank } E(k(C)) + 2 + \sum_v (m_v - 1),$$

*where the summation ranges over the the points of $C$ under singular fibers.*

We will also need the following lemma.

**Lemma 1** ([28, Theorem IV.8.2] and [27, Corollary 7.5])**.** *Let $E$ be an elliptic curve over $\overline{\mathbb{Q}}(t)$. Let $\Sigma \subset \mathbb{P}^1(\overline{\mathbb{Q}}(t))$ be the set of points of bad reduction of $E$. Let $G(F_v)$ denote the group generated by simple components of the fiber $F_v$ at $v \in \Sigma$. There exists an injective homomorphism*

$$\phi : E(\overline{\mathbb{Q}}(t))_{tors} \longrightarrow \prod_{v \in \Sigma} G(F_v).$$

*If $F_v$ is of multiplicative type $I_n$ in Kodaira notation, the corresponding group is $\mathbb{Z}/n\mathbb{Z}$. If $F_v$ is of additive type $I^*_{2n}$, the group is $(\mathbb{Z}/2\mathbb{Z})^2$.*

We now turn to the main object of our study, the polynomials $P_t(x, y)$. A Weierstrass model for $P_t$ can be given by

$$E_t : Y^2 = X\left(X^2 + \left(\frac{t^2}{4} - 2\right) X + 1\right), \tag{1}$$

where

$$x = \frac{tX - 2Y}{2X(X - 1)},\ y = \frac{tX + 2Y}{2X(X - 1)}.$$

Then $E_t$ is an elliptic curve, provided that $t \neq 0, \pm 4$. The main result of this section is the following theorem.

**Theorem 3.** *Let $E_t$ be an elliptic curve over $\mathbb{Q}(t)$ given by the equation*

$$E_t : y^2 = x(x^2 + (t^2/4 - 2)x + 1).$$

*Then*

*(i) The associated elliptic surface (denoted $\mathcal{E}$) is rational.*

*(ii) The rank of $E_t(\overline{\mathbb{Q}}(t))$ is 0,*

*(iii) The torsion subgroup of $E_t(\mathbb{Q}(t))$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.*

### 2.1. Proof of Theorem 3

In this section, we give the proof of Theorem 3.

*Proof.* The elliptic curve $E_t$ (or equivalently Equation (1)) over $\mathbb{Q}(t)$ can be written in short Weierstrass form as

$$y^2 = x^3 + A(t)x + B(t),$$

where

$$A(t) = -27(t^4 - 16t^2 + 16),$$
$$B(t) = 54(t^2 - 8)(t^4 - 16t^2 - 8).$$

The discriminant is then given by

$$\Delta(t) = t^2(t - 4)(t + 4).$$

We now prove each of the parts of the theorem.

(i) Given an elliptic curve

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)$$

over $\mathbb{Q}(t)$ in long Weierstrass form, we know from [26, Equation 10.14] that if $\deg(a_i(t)) \leq i$ for each $i$, then the associated elliptic surface $\mathcal{E}$ is rational. In our case, since $\deg(A(t)) = 4$ and $\deg(B(t)) = 6$, the underlying elliptic surface is rational.

(ii) From the expression of the discriminant, we see that $E_t$ has singular fibers at the values $t = 0, \pm 4$, and $\infty$. We determine the numbers $m_v$, of irreducible components of the fiber over $v$, from Kodaira types of singular fibers [17, section 4]:

| v | coefficients | | | | |
|---|---|---|---|---|---|
| | $ord_{t=v}(A)$ | $ord_{t=v}(B)$ | $ord_{t=v}(\Delta)$ | Kodaira type | $m_v - 1$ |
| 0 | 0 | 0 | 2 | $I_2$ | 1 |
| $-4$ | 0 | 0 | 1 | $I_1$ | 0 |
| 4 | 0 | 0 | 1 | $I_1$ | 0 |
| $\infty$ | 0 | 0 | 8 | $I_8$ | 7 |

Since $\mathcal{E}$ is a rational surface, we have $\rho(\mathcal{E}) = 10$. Thus by Theorem 2 we obtain

$$10 = \mathrm{rank\ E_k(\overline{\mathbb{Q}}(t)) + 2 + 1 + 0 + 0 + 7},$$

and hence $\mathrm{rank\ E_k(\overline{\mathbb{Q}}(t))} = 0$.

(iii) By Lemma 1 and the table in the proof of (ii) above, we see that the torsion subgroup of $E_t(\overline{\mathbb{Q}}(t))$ is embedded in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$. We have the cyclic subgroup $\{\infty, (0,0), (1, \pm t/2)\}$ on the elliptic curve $E_t$ and $(0,0)$ is the only point of order 2. Thus the possibilities for $E_t(\overline{\mathbb{Q}}(t))_{tors}$ are $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$. Now we claim that there is no point of order 8 on $E_t(\overline{\mathbb{Q}}(t))$. If $P = (x, y)$ is a point of order 8, then $2P = (1, \pm t/2)$. (Note that $P \in E_t(\overline{\mathbb{Q}}(t))$ means that $x, y \in \overline{\mathbb{Q}}(t)$.)

We have $x_{2P} = (x^4 - 2x^2 + 1)/(4x^3 + (t^2 - 8)x^2 + 4x)$. Setting this equal to 1, we have

$$(tx + x^2 - 2x + 1)(tx - x^2 + 2x - 1) = 0.$$

Solving, we get

$$x = \frac{(2+t) \pm \sqrt{t(t+4)}}{2}, \frac{(2-t) \pm \sqrt{t(t-4)}}{2} \notin \overline{\mathbb{Q}}(t).$$

Thus there is no point of order 8 in $E_t(\overline{\mathbb{Q}}(t))$. Hence $E_t(\overline{\mathbb{Q}}(t))_{tors} = \mathbb{Z}/4\mathbb{Z}$. Since all the torsion points of $E_t(\overline{\mathbb{Q}}(t))$ are also defined over $\mathbb{Q}(t)$, it follows that $E_t(\mathbb{Q}(t))_{tors} = \mathbb{Z}/4\mathbb{Z}$.

$\square$

## 3. The Torsion Subgroup over $\mathbb{Q}$

We now turn to examining the curve family $E_t$ over the rationals $\mathbb{Q}$. We start with determining what the possible torsion subgroups are.

**Theorem 4.** *For any value of $t \neq 0, \pm 4$, the torsion subgroup of $E_t(\mathbb{Q})$ is $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, or $\mathbb{Z}/8\mathbb{Z}$.*

*Proof.* By Mazur's theorem, there are only a finite number of possibilities for the torsion subgroup: $\mathbb{Z}/n\mathbb{Z}$, for $n = 1, 2, \ldots, 10$ or $n = 12$, and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ with $n = 1, 2, 3, 4$. As noted earlier, for any value of $t \neq 0, \pm 4$, we have the point $P = (1, t/2)$ which has order 4. Thus, the only possible torsion subgroups are $\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/12\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

The point $2P = (0, 0)$ is of order 2. To have other points of order 2, it is necessary that the $y$-coordinate equal 0 or in other words $x^2 + (t^2/4 - 2)x + 1 = 0$. The discriminant of this quadratic in $x$ is $t^2 - 16$. In order for this to be a square, say $t^2 - 16 = j^2$, we parameterize solutions by setting

$$t = \frac{m^2 + 16}{2m},$$

$$j = \frac{m^2 - 16}{2m}.$$

For any rational value of $m$, if we set $t$ as above, then there will be two additional points of order 2: $(-m^2/16, 0)$, and $(-16/m^2, 0)$. In this case, the torsion group will be $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. The other points of order 4 are the points $(-1, \pm\frac{m^2-16}{4m})$.

We also investigate whether the torsion group can be $\mathbb{Z}/8\mathbb{Z}$ or $\mathbb{Z}/12\mathbb{Z}$. For the $\mathbb{Z}/8\mathbb{Z}$ case, we must solve $2R = (1, t/2)$. Using the formulas for the $x$-coordinate when doubling points, this is equivalent to solving

$$(-x^2 + (t + 2)x - 1)(x^2 + (t - 2)x + 1) = 0.$$

The discriminants of these quadratics are $t^2 + 4t$ and $t^2 - 4t$ respectively. We can parameterize solutions to these discriminants being square by $t = 4/(m^2 - 1)$ and $t = -4/(m^2 - 1)$ respectively. In either case, the points of order 8 are then $((m - 1)/(m + 1), \pm 2m/(m + 1)^2)$ and $((m + 1)/(m - 1), \pm 2m/(m - 1)^2)$.

Finally, in order for $\mathbb{Z}/12\mathbb{Z}$ to be the torsion group, then there must be a rational point of order 3. The 3-torsion polynomial for this curve is

$$3x^4 + (t^2 - 8)x^3 + 6x^2 - 1 = 0.$$

This equation is a genus 1 elliptic curve in the variables $t$ and $x$. We can use a birational transformation to turn it into the Weierstrass equation

$$y^2 = x^3 - x^2 + 4x - 4.$$

This curve has rank 0, and 8 torsion points. Tracing back these eight torsion points does not lead to any rational solutions of the 3-torsion polynomial. Thus, there are no values of $t$ for which $E_t$ has a point of order 3. □

We give concrete examples to show each torsion subgroup is possible. If we let $t = 5$ then the curve $E_5 := y^2 = x^3 + (17/4)x^2 + x$ has torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

The points $(1, \pm 5/2)$ and $(\pm 1, \pm 3/2)$ have order 4, while the points $(0,0), (-1/4, 0)$, and $(-4, 0)$ each have order 2. If instead we let $t = 1/2$, the curve $E_{1/2} : y^2 = x^3 - (31/16)x^2 + x$ has torsion group $\mathbb{Z}/8\mathbb{Z}$. The points $(2, \pm 3/2)$ and $(1/2, \pm 3/8)$ have order 8. The points $(1, \pm 1/4)$ have order 4, while the point $(0,0)$ has order 2. Finally, for $t = 2$, the curve $E_2 := x^3 - x^2 + x$ has torsion subgroup $\mathbb{Z}/4\mathbb{Z}$, being generated by $P = (1, \pm 1)$.

We note that if $(x, y)$ is a point on $E_t$, then $(x, y) + (0, 0)$ results in the point $(1/x, -y/x^2)$.

### 3.1. Infinite Families with Positive Rank

A quick experiment seems to show that the rank of $E_t$ (over $\mathbb{Q}$) is frequently 0 or 1, with the number of curves yielding each rank being about equal. One interesting property we observed is that most of the rational non-torsion points on the positive rank curves (including the generators) seem to have $x$-coordinates which are squares. In a small number of cases, they are negative squares. We are not sure why this is the case.

To construct a subfamily $E'_c$ of $E_t$ which has positive rank, let $t = c^2 + c - 2$, for $c \neq -3, -2, 1, 2$. Then there is a rational point $R(c) = (c^2, c(c-1)(c^2+2c+2)/2)$ on $E'_c$. We now use [12, Theorem 1.3] to show that rank of $E'_c(\mathbb{Q}(c))$ is exactly one. The theorem deals with elliptic curves $E$ given by $y^2 = x^3 + A(t)x^2 + B(t)x$, where $A, B \in \mathbb{Z}[t]$ with exactly one nontrivial 2-torsion point over $\mathbb{Q}(t)$. If $t_0 \in \mathbb{Q}$ satisfies the condition that for every nonconstant square-free divisor $h$ of $B(t)$ or $A(t)^2 - 4B(t)$ in $\mathbb{Z}[t]$ the rational number $h(t_0)$ is not a square in $\mathbb{Q}$, then the specialized curve $E_{t_0}$ is elliptic and the specialization homomorphism at $t_0$ is injective. If additionally there exist $P_1, \dots, P_r \in E(\mathbb{Q}(t))$ such that $P_1(t_0), \dots, P_r(t_0)$ are the free generators of $E(t_0)(\mathbb{Q})$ then $E(\mathbb{Q}(t))$ and $E(t_0)(\mathbb{Q})$ have the same rank $r$, and $P_1, \dots, P_r$ are the free generators of $E(\mathbb{Q}(t))$.

For $E'_c$ we have $A(c) = (c^4 + 2c^3 - 3c^2 - 4c - 4)/4$ and $B(c) = 1$. We rescale by an isomorphism $(x, y) \to (2^2x, 2^3y)$ so that the coefficients are integers. This yields $A'(c) = c^4 + 2c^3 - 3c^2 - 4c - 4$ and $B'(c) = 16$. Then we calculate

$$A'(c)^2 - 4B'(c) = (c-2)(c+3)(c^2+c+2)(c+2)^2(c-1)^2.$$

There are 31 squarefree factors of this polynomial, which are the various non-trivial products of the factors $c - 2, c + 3, c^2 + c + 2, c + 2$ and $c - 1$. We find that the specialization at $c = 8$ satisfies all the conditions of [12, Theorem 1.3], with all the squarefree factors evaluating to be nonsquare. Also at $c = 8$ the specialized curve $E'_8$ is easily computed to have rank 1, with $R(8) = (64, 2296)$ being a generator [25]. Thus $E'_c$ has rank 1 over $\mathbb{Q}(c)$ and we can conclude that $R(c)$ is its free generator.

It is not difficult to construct other positive rank (infinite) subfamilies of the curves $E_t$. In the next section we show how to do so yielding curves with (at least) two linearly independent points. We note that the record for infinite curve families over $\mathbb{Q}(t)$ with torsion group $\mathbb{Z}/4\mathbb{Z}$ is 5 [10].

### 3.2. A Rank 2 family

In this section we construct another infinite family with positive rank. Again let $c$ be a given rational value, for which we want to have a rational point with $x$-coordinate $c$ be on the curve. Setting $x = c$ in Equation (1), we are led to consider the equation

$$y^2 - (c^2/4)t^2 = c(c-1)^2$$

as a quadratic in $y$ and $t$. We may parameterize the solutions by

$$t = \frac{cm^2 - 4(c-1)^2}{2cm},$$

$$y = \frac{-cm^2 - 4(c-1)^2}{4m}.$$

Thus, given any value of $c$ we may set $t$ by the equation above. The point

$$\left(c, \frac{-cm^2 - 4(c-1)^2}{4m}\right)$$

will then be a rational point on the curve $E_t$.

Assuming that $c \neq \pm 1$, the point will almost assuredly have infinite order. Indeed, let $m = 1$ and $c = 3$. Then $t = -13/6$ and the point $(3, 19/4)$ lies on the curve $E_{-13/6}$ and has infinite order. By Silverman's specialization theorem [29], the rank of this infinite family is (at least) 1 for all but finitely many values of $c$ and $m$.

We can further increase the rank of the family. To do so, we force $3c$ to be an $x$-coordinate of a rational point. The resulting equation which needs to be satisfied is

$$144c^4 + (360m^2 - 576)c^3 + (9m^4 - 144m^2 + 864)c^2 + (-24m^2 - 576)c + 144 = z^2,$$

for some rational $z$. We find a solution is

$$c = \frac{6m^4}{(m^2 - 24)(m^4 - 24m^2 + 72)}.$$

The resulting value of $t$ is

$$t = \frac{m^{12} + 72m^{10} - 6480m^8 + 141696m^6 - 1213056m^4 + 4478976m^2 - 5971968}{6m^5(m^2 - 24)(m^4 - 24m^2 + 72)}.$$

With this value of $t$, there are then rational points with $x$-coordinates $c$ and $3c$. By specialization, we may see they are linearly independent points. Take $m = 3$, and then $t = 2257/1890$ and $c = 18/35$. The two points are $(18/35, 3413/7350)$ and $(54/35, 2797/2450)$. Using SAGE, we compute the determinant of their height pairing matrix, which is $23.3477634284835 \neq 0$. Thus, the points are linearly independent, which shows the rank of this family is (at least) 2.

The exact same technique may be done more generically, replacing $3c$ by $rc$, for any rational value of $r$. We omit the details.

We now use [12, Theorem 1.3] to show that rank of $E_m(\mathbb{Q}(m))$ is exactly two. As before, we need to scale by an isomorphism so that the coefficients are integers. The resulting coefficients are

$$\begin{aligned} A'(m) = m^{24} &- 144m^{22} + 19872m^{20} - 1686528m^{18} + 78879744m^{16} \\ &- 2170810368m^{14} + 37076963328m^{12} - 403537821696m^{10} \\ &+ 2818207531008m^{8} - 12558905376768m^{6} + 34549889236992m^{4} \\ &- 53496602689536m^{2} + 35664401793024, \end{aligned}$$

and

$$B'(m) = 20736m^{20}(m^2 - 24)^4(m^4 - 24m^2 + 72)^4.$$

Then

$$\begin{aligned} A'^2(m) - 4B'(m) = (m^{12} &- 24m^{11} + 72m^{10} + 1152m^9 - 6480m^8 - 15552m^7 \\ &+ 141696m^6 + 41472m^5 - 1213056m^4 + 4478976m^2 \\ &- 5971968)(m^{12} + 24m^{11} + 72m^{10} - 1152m^9 - 6480m^8 \\ &+ 15552m^7 + 141696m^6 - 41472m^5 - 1213056m^4 + 4478976m^2 \\ &- 5971968)(m^{12} + 72m^{10} - 6480m^8 + 141696m^6 - 1213056m^4 \\ &+ 4478976m^2 - 5971968)^2. \end{aligned}$$

It is easy to obtain the squarefree factors from the formulas above. Specializing at $m = 8/5$ leads to none of them being rational squares. The rank of the specialized curve is 2. Using SAGE, we have checked that the two points with $x$-coordinates $c$ and $3c$ (resulting from $m = 8/5$) are generators. Concretely, these points are $(-9600/89579, 46430503325/64195177928)$ and $(-28800/89579, 148191339735/64195177928)$.

### 3.3. Examples of Elliptic Curves of High Rank

We searched for curves $E_t$ with high rank, and were able to find a single elliptic curve of rank 6 in the family $E_t$, as well as many curves of rank 5. The record for elliptic curves with torsion group $\mathbb{Z}_4$ is 13 [9].

We use the sieving method based on Mestre-Nagao sums ([16], [20]). Let $E/\mathbb{Q}$ be an elliptic curve, and $p$ be a prime. Set $a_p = a_p(E) = p + 1 - |E(\mathbf{F_p})|$. Given a fixed integer $N$, the Mestre-Nagao sum is defined by

$$S(N,E) = \sum_{\text{primes } p\leq N} \left(1 - \frac{p-1}{|E(\mathbf{F_p})|}\right)\log(p)$$
$$= \sum_{\text{primes } p\leq N} \frac{-a_p+2}{p+1-a_p}\log(p).$$

It has been conjectured that in general, larger values of $S(N,E)$ tend to correspond to curves with high rank. Provided $N$ is not too large, $S(N,E)$ can be calculated using SAGE [25].

We performed some experiments and searched the family $E_t$ for integral $t$ with $t \leq 1,000,000$ (without loss of generality we may assume $t > 0$. We found many curves with rank 4. Up to $t < 500,000$, the list for such $t$ includes the following values: 15388, 63404, 63436, 95493, 103437, 107684,120006, 128176, 144231, 182249, 187351, 190381, 207404, 302512, 316863, 324972, 422212, 426404. We also searched using rational values of $t$, where the numerator and denominator were bounded by 5000. We again found many curves with rank 4, with the first three being $t = 101/251, 110/221, 242/279$. In addition, we found some curves with rank 5: $t = 1121/595, 1577/1309$.

In addition, we also searched the family $E_t$, with $t = c^2+c-2$. As we saw in Section 3.1, this ensures the rank is at least 1. We found a large number of curves of rank 5, and one curve with rank 6. The following values of $c$ all yield rank 5 curves: $c =$ 27/80, 52/85, 59/268, 75/208, 151/120, 157/280, 235/133, 265/272, 327/55, 381/56, 415/73, 442/159, 507/136, 540/37, 575/147, 598/37, 607/204, 635/46, 655/72, 659/22, 676/119, 687/35, 697/273, 699/143, 717/50, 736, 103, 745/99, 761/17, 791/55, 813/49, 830/49, 831/220, 885/259, 901/31, 915/161, 934/209, 958/261, 968/119, 974/77, 1027/168, 1051/177, 1055/296, 1091/280,157/323, 172/363,190/451, 242/345. The rank 6 curve is $c = 1079/231$, yielding $t = 13899/13157$.

We attempted to find high rank curves in the family of Section 3.2, but the coefficients quickly grow too large to efficiently compute the ranks while doing any kind of extensive search.

## 4. Directions for Future Work

Finding points in arithmetic progressions on the curves is one of the fascinating problem of Diophantine equations. There are several papers dedicated to this problem [1, 2, 4, 5, 6, 7, 19, 18, 31]. One direction for future work is to consider arithmetic progressions on $E_t$. We say three points $P_1, P_2, P_3$ lying on $E_t$ are in arithmetic progressions if their $x$ coordinates (or their $y$ coordinates) are in arithmetic progressions. We know that $T_1 = (1, -\frac{t}{2}), T_2 = (0,0), T_3 = (1, \frac{t}{2})$ are torsion points on $E_t(\mathbb{Q})$. One can obviously see that the $y$- coordinates of $T_1, T_2, T_3$ are in arithmetic progression. Likewise, any point $(x, y)$ with $y \neq 0$ will lead to a length 3 progression in the $y$-coordinates: $(x, y), (0, 0), (x, -y)$.

Besides these trivial cases, we can try to create a progression using two of the torsion points. As a first case, consider using the $x$-coordinates of $T_1$ and $T_2$; we have $x = 0$ and 1. To obtain a longer progression, we thus need to check the cases $x = -1$ and $x = 2$. For each case, we are led to a quadratic equation which can be parameterized leading to infinitely many values of $k$ for which the particular value of $x$ is the $x$-coordinate of a rational point. For example, for $x = 2$ we need $k^2 + 2$ to be square. Setting $k = (m^2 - 2)/2m$ will always make $k^2 + 2$ a square. Thus, there are infinitely many values of $k$ which lead to a progression with $x = 0, 1, 2$. The same is true for the case $x = -1$. It is possible to try an extend this further to a length 4 progression, but using the parameterization we are then led to a quartic equation needing to be square. In every case, this is an elliptic curve for which we can compute the rational points. In no instance does the progression extend further for the many examples we tried.

If we examine the $y$-coordinates, we have $0, \pm t/2$ for the coordinates of the $T_i$. If we look for a point with $y$-coordinate $t$ we are led to needing a root of the equation $x^3 + (t^2/4 - 2)x^2 + x = t^2$, an elliptic curve. This curve only has only the trivial point $(0, 0)$, hence we do not have such a progression. We similarly check for points with $y$-coordinates $t/4$ and $\pm 3t/2$, but have the same outcome.

In this work, we studied the elliptic curves defined by the polynomials $P_t(x, y)$. One could further explore the integral points of $E_t$, including properties like whether they are in arithmetic progressions or even in geometric progression. Possible questions to be answered are listed below.

- **Question 1.** Is there a non-trivial arithmetic progression of either the $x-$ or $y$-coordinates of $\{P_1, P_2, P_3\}$ on the curve $E_t$ when one of the points $P_i$ is equal to the torsion point $T_2 = (0, 0)$, and the other two points are non-torsion?

- **Question 2.** Does there exists three non-torsion points whose $x$ or $y$-coordinates are in arithmetic progression?

In the future we will try to answer these questions. It would also be interesting to continue to find high rank curves in the family $E_t$, or find specific curves with rank higher than 6.

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