# Measurement of sample stage error motions in cone-beam X-ray computed tomography instruments by minimization of reprojection errors 

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#### Abstract

Reconstruction algorithms in X-ray computed tomography assume a particular geometrical alignment of the instrument components: X-ray source, sample stage, and detector. Motion errors and misalignments in the actual instrument contribute to errors in the reconstruction. In previous work, we presented an object-based procedure to measure the instrument geometrical alignment at a single position of the sample rotation stage along its linear translation axes. Here, we present an object-based method to determine error motions of the sample stage manipulator in cone-beam X-ray computed tomography instruments. The proposed method is applied with a reference object comprising calibrated sphere center positions to determine error motions of the stage as it is translated along the nominal magnification axis from the X-ray source towards the detector. Results agree with previous reference measurements performed using laser interferometers and electronic levels, albeit at a loss of sensitivity at lower imaging magnifications when the projected object occupies progressively smaller areas of the detector image. To compensate for the lower sensitivity, we propose a solution based on having a limited set of reference objects of various sizes to be used along the entire magnification range of the sample stage.


## 1. Introduction

Accurate tomographic reconstruction in X-ray computed tomography (CT) relies on accurate knowledge of the instrument geometrical alignment, which is defined by the relative positions and orientations of X-ray source focal spot, sample stage and its axis of rotation, and X-ray detector. The back-projection step in conventional reconstruction assumes an alignment of these instrument components. The actual instrument geometry will deviate from this assumed alignment, hence introducing errors in the reconstruction [1-4]. In previous work [5,6], we describe and implement experimentally a method to measure the 'static' instrument geometry at a given position of the sample stage. This method is based on minimizing reprojection errors, i.e., the difference in observed and modelled two-dimensional (2D) fiducial coordinates in the projections of a calibrated reference object placed on the sample rotation stage. Subsequent instrument adjustment [6] or software correction [7] based on the measured instrument geometry result in significant error reductions. However, when the sample stage is repositioned, error motions mean that the new instrument geometry cannot be simply determined by applying the nominal translations to the previously determined geometry.

The new instrument geometry can be determined by repeating the measurement of the static instrument geometrical alignment (using the
method described in Ref. [5,6]), albeit at the new sample stage position. The robustness of the measurement procedure relies, in part, on the imaged reference object occupying a large portion of the projection while at the same time remaining within the field of view. Thus, reference objects of varying sizes are needed to measure the static instrument geometrical alignment for all stage positions along the nominal magnification axis. Alternatively, reference instruments such as interferometers and electronic levels can be used to map the geometric errors of the manipulator axes. This error map can then be used to determine the equivalent changes in the instrument imaging geometry. Performing a full error mapping using reference measuring instruments (e.g., Ref. [8]) might not be practical for all users of X-ray CT. Here we present a modification of the method described and implemented in Ref. $[5,6]$ to determine the error motions of the sample stage as it is translated along the nominal magnification axis.

## 2. Parameterizing the static instrument geometry

Conventional reconstruction algorithms assume an alignment of the instrument components, whereby the full instrument geometry can be described by the source-to-rotation axis distance (SRD) and the source-to-detector distance (SDD) along a common longitudinal axis connecting the point source, sample stage axis of rotation, and the detector

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Fig. 1. Left: Ideally aligned cone-beam X-ray CT instrument geometry. Right: Actual instrument geometry.
geometrical center (Fig. 1, left). In this alignment, the longitudinal axis orthogonally intersects the sample stage axis of rotation and is perpendicular to the detector plane. As such, the longitudinal axis coincides with the imaging magnification axis, which is defined as the line from the source focal spot that intersects the detector perpendicularly. The detector is a flat-panel array of $N_{\text {row }}$ pixel rows, $u$, and $N_{\text {col }}$ pixel columns, $v$. The detector columns, $v$, in an aligned instrument, are parallel to the sample stage rotation axis, while the detector rows, $u$, follow the right-hand rule with respect to the columns and detector normal. In practice, there are deviations of the instrument geometry from this assumed alignment. To parameterize the actual instrument geometry, a right-handed source-rotation axis coordinate frame \{SRF\} is defined with its origin set at the X-ray focal spot, which is modelled as an infinitesimally small point source. The Y axis is parallel to the sample stage rotation axis and is positive upwards (against gravity). The Z axis is defined by the line from the X-ray focal spot that intersects the sample stage axis of rotation orthogonally; the positive Z axis points away from the sample stage rotation axis. The X axis follows the right-hand rule. Henceforth, the coordinate frame with which geometrical parameters are defined will be denoted in superscript. For a given sample stage position, the actual instrument geometry can be fully parameterized by seven parameters (Fig. 1, right). The intersection of the Z axis with the axis of rotation defines the position $\boldsymbol{R}^{\mathrm{SRF}}=\left(0,0, z_{\mathrm{R}}^{\mathrm{SRF}}\right)$ of the sample stage and the geometrical center of the measurement volume in tomographic reconstruction. The position and orientation of the detector is defined by its own coordinate frame $\{\mathrm{DF}\}$, where the detector X axis coincides with the detector rows, $u$, and the detector Y axis coincides with the detector columns, $v$. The detector Z axis is normal to the plane of the detector and the origin of $\{\mathrm{DF}\}$ is the geometrical center of the detector. In $\{\mathrm{SRF}\}$, the position and orientation of $\{\mathrm{DF}\}$ is given by the three-dimensional coordinate position of the detector geometrical center $\boldsymbol{D}^{\mathrm{SRF}}=\left(x_{\mathrm{D}}^{\mathrm{SRF}}, y_{\mathrm{D}}^{\mathrm{SRF}}, z_{\mathrm{D}}^{\mathrm{SRF}}\right)$, and three extrinsic rotation angles $(\eta, \phi, \theta)$ about axes parallel to the $\mathrm{Z}, \mathrm{Y}$, and X axes of \{SRF\}, respectively, and in this order. The center of rotation for the detector angles is the detector geometrical center. In the presence of detector out of plane rotations $\phi$ and $\theta$, the Z axis of $\{\mathrm{SRF}\}$ is not parallel to the Z axis of the $\{\mathrm{DF}\}$, and therefore not parallel to the imaging magnification axis.

In previous work [5,6], we describe a method to determine the seven geometrical parameters at a given position of the sample stage along the nominal magnification axis. Translation of the sample stage should ideally be free from geometric error motions, i.e., positioning error along the axis of movement, straightness errors along transverse directions, and angular errors (pitch, yaw, and roll). In the presence of such error motions, the instrument geometrical alignment at a different sample stage position cannot be calculated by applying the nominal translations to the previously determined geometrical alignment. Instead, users can perform a full error mapping of the sample stage
motion and apply appropriate transformations to their imaging geometry, e.g., from a lookup table. In the next section, we describe a method based on imaging a reference object to perform such an error mapping for the sample stage motion.

## 3. Method

Here we present an object-based method to measure the error motions of the sample stage as it is stepwise translated along the instrument magnification axis. The object-based method consists of acquiring multiple projections of a reference object, having multiple features of known positions, as it is rotated on the sample stage at each stepped position and solving for six error motions of the stage. The reference object comprises a carbon fiber cylinder with $N_{\text {spheres }}=49$ steel spheres affixed to its outer circumference in a dedicated multi-helix arrangement (Fig. 2). The shape of the object and arrangement of spheres can be different, although it has been shown that the cylindrical shape with helically arranged spheres provides the most robust determination of instrument geometry [9]. This particular reference object was developed in collaboration with the University of Padova [10]. Sphere center positions in an object coordinate frame $\{\mathrm{OF}\} \boldsymbol{C}_{i}^{\mathrm{OF}}=\left(x_{\mathrm{C}}^{\mathrm{OF}}, y_{\mathrm{C}}^{\mathrm{OF}}, z_{\mathrm{C}}^{\mathrm{OF}}\right)_{i}$, where $i=1,2, \ldots, N_{\text {spheres }}$, are measured on a coordinate measuring machine (CMM) with a maximum permissible error MPE $=2+\mathrm{L} / 400 \mu \mathrm{~m}$, where L is the measured length in millimeters. The definition of $\{\mathrm{OF}\}$ is arbitrary; nevertheless, here we define $\{O F\}$ such that its Y axis is approximately parallel to the cylindrical axis of the carbon fiber framework, the Z axis is parallel to the vector connecting the cylindrical axis to sphere 1 , and the X axis follows the right hand rule. We apply our method to determine error motions of the sample stage in a Nikon XT H $450^{1}$ X-ray CT instrument as the stage is translated stepwise in 20 mm increments from an initial higher magnification position (i.e., close to the source) to progressively lower magnification positions along the instrument magnification axis (Fig. 3). The SDD specified by the instrument data acquisition program is 1048.21 mm . The initial sample stage position is at the instrument-specified SRD of 364 mm , corresponding to a nominal magnification of approximately $\mathrm{M}=2.88$. The final sample stage position is at the instrument-specified SRD of 724 mm , corresponding to a magnification of approximately $\mathrm{M}=1.45$ and a total nominal travel distance of 360 mm from the initial position.

[^1]

Fig. 2. Error motions are determined from projections acquired of a reference object developed in collaboration with the University of Padova (left), comprising 49 spheres with known center positions in a local object coordinate frame (center and right) [10].

### 3.1. Notations

In the following sections, we describe the parameterization of the instrument geometry and stage error motions in homogeneous coordinates. Point coordinates and directions are represented by 4-element column vectors and transformations are represented by $4 \times 4$ matrices that right multiply into the column vectors.

Translations are denoted $\mathbf{T}(h, k, l)$ where $h k$, and $l$ are the translations along $\mathrm{X}, \mathrm{Y}$, and Z axes, respectively. The translation matrix is given by
$\mathbf{T}(h, k, l)=\left[\begin{array}{llll}1 & 0 & 0 & h \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1\end{array}\right]$
Rotations about the coordinate axes are denoted $\operatorname{Rot}_{\mathrm{X} / \mathrm{Y} / \mathrm{Z}}(\delta)$, where $\delta$ is the angle of rotation and the subscript $\mathrm{X} / \mathrm{Y} / \mathrm{Z}$ denotes the axis about which the rotation is being applied. This notation is used when rotations are applied about the origin of the current coordinate frame, i.e., the coordinate frame of the points being rotated. Rotation matrices about each coordinate axis $\mathrm{X}, \mathrm{Y}$, and Z are as follows:
$\boldsymbol{R o t}_{\mathrm{X}}(\delta)=\left[\begin{array}{cccc}1 & 0 & 0 & 0 \\ 0 & \cos \delta & -\sin \delta & 0 \\ 0 & \sin \delta & \cos \delta & 0 \\ 0 & 0 & 0 & 1\end{array}\right], \boldsymbol{R o t}_{\mathrm{Y}}(\delta)=\left[\begin{array}{cccc}\cos \delta & 0 & \sin \delta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \delta & 0 & \cos \delta & 0 \\ 0 & 0 & 0 & 1\end{array}\right]$
$\operatorname{Rot}_{\mathrm{Z}}(\delta)=\left[\begin{array}{cccc}\cos \delta & -\sin \delta & 0 & 0 \\ \sin \delta & \cos \delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{array}\right]$
Rotations about an arbitrary axis $\widehat{\boldsymbol{e}}=\left(e_{\mathrm{X}}, e_{\mathrm{Y}}, e_{\mathrm{Z}}\right)$ are denoted $\operatorname{Rot}_{\mathbf{e}}(\delta)$ and parameterized using the axis-angle rotation matrix:


Fig. 3. Sample stage error motions are determined from projections of the reference object acquired at 20 mm stepped increments along the instrument longitudinal axis.

Rotations about an arbitrary point $(a, b, c)$ and along the arbitrary axis $\widehat{\boldsymbol{e}}$ are given by the following sequence of translation-rotationtranslation matrices:
$\mathbf{T}(a, b, c) \operatorname{Rot}_{\underset{\mathbf{e}}{ }}(\delta) \mathbf{T}(-a,-b,-c)$
Initial instrument geometry in SRF.
In a first step, we determine the full X-ray CT instrument geometry in \{SRF\} (per section 2) at the initial $k=1$ position of the sample stage. A set of $N_{\alpha}=360$ X-ray projections of the reference object are acquired at equiangular positions $\alpha_{j}=(j-1) \cdot\left(\frac{360}{N_{\alpha}}\right)^{\circ}$, where $j=1,2, \ldots, N_{\alpha}$, as the sample stage performs a full revolution. The number of projections, $N_{\alpha}$, needed to provide robust determination of the instrument geometry can be as few as 30 [6]; here we use a larger number of projections to ensure a large ratio of number of input observations to number of solvable parameters. . For each sphere $i$ and in each acquired projection $j$, we determine the two-dimensional column and row coordinates in the detector plane $\left(u_{\mathrm{obs}}, v_{\mathrm{obs}}\right)_{i, j, k=1}^{\mathrm{DF}}$ of the projected sphere center. These observed center projection coordinates are given by the center of an ellipse fit to the edge of each projected sphere. The center projection estimation method proposed by Deng et al. [11] was used in previous studies $[5,6]$ but was found to provide similar results to the ellipse center; it was therefore not used here. The projection of each sphere is tracked as the sphere performs a full rotation; an automated overlap detection algorithm rejects data points where two or more spheres are overlapping.

The instrument geometry (seven parameters indicated in Fig. 1, right) is determined by applying a Levenberg-Marquardt nonlinear leastsquares algorithm [12] to minimize the reprojection error, i.e., the difference between experimentally observed center projection coordinates $\left(u_{\mathrm{obs}}, v_{\mathrm{obs}}\right)_{i, j, k=1}$ and an equivalent set of modelled center projection coordinates $\left(u_{\text {mod }}, v_{\text {mod }}\right)_{i, j, k=1}$ from all acquired projections of the reference object (see Appendices for details on modeling center projection coordinates). The forward projection model used to generate the modelled center projection coordinates comprises 13 variables $\boldsymbol{w}$ : seven

$$
\boldsymbol{R o t}_{\mathbf{e}}(\delta)=\left[\begin{array}{cccc} 
& & \\
\cos \delta+e_{\mathrm{X}}^{2}(1-\cos \delta) & e_{\mathrm{X}} e_{\mathrm{Y}}(1-\cos \delta)-e_{\mathrm{Z}} \sin \delta & e_{\mathrm{X}} e_{\mathrm{Z}}(1-\cos \delta)+e_{\mathrm{Y}} \sin \delta & 0 \\
e_{\mathrm{X}} e_{\mathrm{Y}}(1-\cos \delta)+e_{\mathrm{Z}} \sin \delta & \cos \delta+e_{\mathrm{Y}}^{2}(1-\cos \delta) & e_{\mathrm{Y}} e_{\mathrm{Z}}(1-\cos \delta)-e_{\mathrm{X}} \sin \delta & 0 \\
e_{\mathrm{X}} e_{\mathrm{Z}}(1-\cos \delta)-e_{\mathrm{Y}} \sin \delta & e_{\mathrm{Y}} e_{\mathrm{Z}}(1-\cos \delta)+e_{\mathrm{X}} \sin \delta & \cos \delta+e_{\mathrm{Z}}^{2}(1-\cos \delta) & 0 \\
0 & 0 & 0 & 1 \\
0 &
\end{array}\right]
$$



Fig. 4. The difference between modelled and observed center projection coordinates, i.e., the reprojection error, is minimized.
instrument parameters $\left(z_{\mathrm{R}}^{\mathrm{SRF}}, x_{\mathrm{D}}^{\mathrm{SRF}}, y_{\mathrm{D}}^{\mathrm{SRF}}, z_{\mathrm{D}}^{\mathrm{SRF}}, \theta, \phi, \eta\right)$ and 6 nuisance parameters describing the position $P^{\mathrm{SRF}}=\left(x_{\mathrm{P}}^{\mathrm{SRF}}, y_{\mathrm{P}}^{\mathrm{SRF}}, z_{\mathrm{P}}^{\mathrm{SRF}}\right)$ of the $\{\mathrm{OF}\}$ origin and orientation $\boldsymbol{\rho}=\left(\rho_{\mathrm{X}}, \rho_{\mathrm{Y}}, \rho_{\mathrm{Z}}\right)$ of the $\{\mathrm{OF}\}$ coordinate axes at the $j=1$ angular position of the stage.

The sphere center coordinates in \{SRF\} at the $j=1$ angular position of the stage $\boldsymbol{C}_{i, j=1, k=1}^{\mathrm{SRF}}$ are given by applying the following transformation to the sphere center coordinates in the object coordinate frame $\boldsymbol{C}_{i}^{\mathrm{OF}}$.

$$
\left[\begin{array}{c}
x_{\mathrm{C}}^{\mathrm{SRF}}  \tag{1}\\
y_{\mathrm{C}}^{\mathrm{SF}} \\
z_{\mathrm{C}}^{\mathrm{SF}} \\
1
\end{array}\right]_{i, j=1, k=1}=\mathbf{T}\left(x_{\mathrm{P}}, y_{\mathrm{P}}, z_{\mathrm{P}}\right) \mathbf{R o t}_{\mathbf{Y}}\left(\rho_{\mathrm{Y}}\right) \mathbf{R o t}_{\mathbf{Z}}\left(\rho_{\mathrm{Z}}\right) \mathbf{R o t}_{\mathbf{x}}\left(\rho_{\mathrm{X}}\right)\left[\begin{array}{c}
x_{\mathrm{C}}^{\mathrm{OF}} \\
y_{\mathrm{C}}^{\mathrm{OF}} \\
z_{\mathrm{C}}^{\mathrm{OF}} \\
1
\end{array}\right]_{i}
$$

The sphere center coordinates in \{SRF\} at an angular position $\alpha_{j}$ of the sample stage $C_{i, j, k=1}^{\mathrm{SRF}}$ are given as follows.

$$
\left[\begin{array}{c}
x_{\mathrm{C}}^{\mathrm{SRF}} \\
y_{\mathrm{C}}^{\mathrm{SRF}} \\
z_{\mathrm{C}}^{\mathrm{SRF}} \\
1
\end{array}\right]=\mathbf{T}\left(0,0, z_{\mathrm{R}}^{\mathrm{SRF}}\right) \operatorname{Rot}_{\mathrm{Y}}\left(\alpha_{j}\right) \mathbf{T}\left(0,0,-z_{\mathrm{R}}^{\mathrm{SRF}}\right)\left[\begin{array}{c}
x_{\mathrm{C}}^{\mathrm{SRF}} \\
y_{\mathrm{C}}^{\mathrm{SRF}} \\
z_{\mathrm{C}}^{\mathrm{SRF}} \\
1
\end{array}\right]_{i, j=1, k=1}
$$

The modelled projection coordinates in \{DF \} for each sphere and at each angular position of the sample stage $\left(u_{\text {mod }}, v_{\text {mod }}\right)_{i, j, k=1}^{\mathrm{DF}}$ are determined from the intersection of rays drawn from the source focal spot through each sphere center in \{SRF\} with the detector plane. This forward projection operation is described Appendix A.

The actual instrument geometry and reference object parameters are given by the set of modelled parameter values $\boldsymbol{w}$ that minimize the reprojection error, i.e. the 'solved' values (Fig. 4). The minimization argument is given by equation (3).
$\min _{\boldsymbol{w}} f(\boldsymbol{w})_{2}^{2}=\min _{\boldsymbol{w}}\left(f_{u, 1}(\boldsymbol{w})^{2}+f_{v, 1}(\boldsymbol{w})^{2}+f_{u, 2}(\boldsymbol{w})^{2}+f_{v, 2}(\boldsymbol{w})^{2}+\ldots+f_{u, \mathrm{Q}}(\boldsymbol{w})^{2}+f_{v, \mathrm{Q}}(\boldsymbol{w})^{2}\right)$
where $f(\boldsymbol{w})=\left[\begin{array}{c}f_{u, 1}(\boldsymbol{w})=\left(u_{\text {mod }}(\boldsymbol{w})-u_{\text {obs }}\right)_{i=1, j=1, k=1} \\ f_{v, 1}(\boldsymbol{w})=\left(v_{\text {mod }}(\boldsymbol{w})-v_{\text {obs }}\right)_{i=1, j=1, k=1} \\ f_{u, 2}(\boldsymbol{w})=\left(u_{\text {mod }}(\boldsymbol{w})-u_{\text {obs }}\right)_{i=1, j=2, k=1} \\ f_{v, 2}(\boldsymbol{w})=\left(v_{\text {mod }}(\boldsymbol{w})-v_{\text {obs }}\right)_{i=1, j=2, k=1} \\ \vdots \\ f_{u, \mathrm{Q}=N_{\text {sphers }} \cdot N_{\alpha}}(\boldsymbol{w})=\left(u_{\text {mod }}(\boldsymbol{w})-u_{\text {obs }}\right)_{)_{i=N_{\text {spheres }}, j=N_{\alpha}, k=1}} \\ f_{v, \mathrm{Q}=N_{\text {spheres }} \cdot N_{\alpha}}(\boldsymbol{w})=\left(v_{\text {mod }}(\boldsymbol{w})-v_{\text {obs }}\right)_{i=N_{\text {sphers }}, j=N_{\alpha}, k=1}\end{array}\right]$
and $f(\boldsymbol{w})_{2}$ denotes the Euclidean norm of $f(\boldsymbol{w})$.

### 3.2. Conversion to detector frame $\{D F\}$

The solved instrument geometry at the $k=1$ manipulator position of the sample stage is converted to a detector frame \{DF\} (Fig. 5). The conversion to $\{\mathrm{DF}\}$ is necessary since $\{\mathrm{SRF}\}$ depends on the position and orientation of the sample stage rotation axis to define the coordinate axes, resulting in a changing coordinate frame with changes in sample stage position and rotation axis orientation.

As introduced earlier, the origin of $\{\mathrm{DF}\}$ is the detector geometrical center $\boldsymbol{D}^{\mathrm{SRF}}=\left(x_{\mathrm{D}}^{\mathrm{SRF}}, y_{\mathrm{D}}^{\mathrm{SRF}}, z_{\mathrm{D}}^{\mathrm{SRF}}\right)$, its X axis is parallel to the detector rows $\widehat{\boldsymbol{u}}^{\mathrm{SRF}}=\left(u_{\mathrm{X}}^{\mathrm{SRF}}, u_{\mathrm{Y}}^{\mathrm{SRF}}, u_{\mathrm{Z}}^{\mathrm{SRF}}\right)$, and its Y axis is parallel to the detector columns $\widehat{\boldsymbol{v}}^{\mathrm{SRF}}=\left(v_{\mathrm{X}}^{\mathrm{SRF}}, v_{\mathrm{Y}}^{\mathrm{SRF}}, v_{\mathrm{Z}}^{\mathrm{SRF}}\right) . \widehat{\boldsymbol{u}}^{\mathrm{SRF}}$ and $\widehat{\boldsymbol{v}}^{\mathrm{SRF}}$ are defined at the end of Appendix A.

The Z axis of $\{\mathrm{DF}\}$ is given by the normal to the detector plane, $\widehat{\boldsymbol{n}}^{\mathrm{SRF}}=\left(n_{\mathrm{X}}^{\mathrm{SRF}}, n_{\mathrm{Y}}^{\mathrm{SRF}}, n_{\mathrm{Z}}^{\mathrm{SRF}}\right)$. The X-ray source focal spot position $\boldsymbol{S}^{\mathrm{DF}}=\left(x_{\mathrm{S}}^{\mathrm{DF}}\right.$, $\left.y_{\mathrm{S}}^{\mathrm{DF}}, z_{\mathrm{S}}^{\mathrm{DF}}\right)$ in $\{\mathrm{DF}\}$ is given by equation (4).
$\left[\begin{array}{c}x_{\mathrm{S}}^{\mathrm{DF}} \\ y_{\mathrm{S}}^{\mathrm{DF}} \\ z_{\mathrm{S}}^{\mathrm{DF}} \\ 1\end{array}\right]=\left(\boldsymbol{R o t}_{\mathrm{Z}}(\eta) \boldsymbol{R o t}_{\mathrm{Y}}(\phi) \boldsymbol{R o t}_{\mathrm{X}}(\theta)\right)^{-1} \mathbf{T}\left(-x_{\mathrm{D}}^{\mathrm{SRF}},-y_{\mathrm{D}}^{\mathrm{SRF}},-z_{\mathrm{D}}^{\mathrm{SFF}}\right)\left[\begin{array}{l}0 \\ 0 \\ 0 \\ 1\end{array}\right]$,
Since we ensured that the tube is kept under optimal operating temperatures, in this study we assume a stationary focal spot, i.e., $S^{\mathrm{DF}}$ is constant for all sample stage positions $k$. In future work, we will exploit the parameterization of the focal spot in the detector frame to examine focal spot drift; such drift can then be compensated.


Fig. 5. The instrument geometry measured at the initial sample stage position in the source-rotation axis frame \{SRF\} (left) is converted to a detector-based coordinate frame $\{\mathrm{DF}\}$ (right).

The sample stage is defined by the three-dimensional coordinate position $\boldsymbol{R}_{k=1}^{\mathrm{DF}}=\left(x_{\mathrm{R}}^{\mathrm{DF}}, y_{\mathrm{R}}^{\mathrm{DF}}, z_{\mathrm{R}}^{\mathrm{DF}}\right)_{k=1}$, given in equation (5), while its rotation axis is defined by the unit vector $\widehat{r}_{k=1}^{\mathrm{DF}}=\left(r_{\mathrm{X}}^{\mathrm{DF}}, r_{\mathrm{Y}}^{\mathrm{DF}}, r_{\mathrm{Z}}^{\mathrm{DF}}\right)_{k=1}$, given in equation (6).
$\left[\begin{array}{c}x_{\mathrm{R}}^{\mathrm{DF}} \\ y_{\mathrm{R}}^{\mathrm{DF}} \\ z_{\mathrm{R}}^{\mathrm{DF}} \\ 1\end{array}\right]_{k=1}=\left(\boldsymbol{R o t}_{\mathrm{Z}}(\eta) \operatorname{Rot}_{\mathrm{Y}}(\phi) \mathbf{R o t}_{\mathrm{X}}(\theta)\right)^{-1} \mathbf{T}\left(-x_{\mathrm{D}}^{\mathrm{SRF}},-y_{\mathrm{D}}^{\mathrm{SRF}},-z_{\mathrm{D}}^{\mathrm{SRF}}\right)\left[\begin{array}{c}0 \\ 0 \\ z_{\mathrm{R}}^{\mathrm{SRF}} \\ 1\end{array}\right]$, 5
$\left[\begin{array}{l}r_{\mathrm{X}}^{\mathrm{DF}} \\ r_{\mathrm{Y}}^{\mathrm{DF}} \\ r_{\mathrm{Z}}^{\mathrm{DF}} \\ 1\end{array}\right]=\left(\boldsymbol{\operatorname { R o t }}_{\mathrm{Z}}(\eta) \boldsymbol{\operatorname { R o t }}_{\mathrm{Y}}(\phi) \boldsymbol{\operatorname { R o t }}_{\mathrm{X}}(\theta)\right)^{-1}\left[\begin{array}{l}0 \\ 1 \\ 0 \\ 1\end{array}\right]$,
The sphere center coordinates at $\alpha_{j=1}=0^{\circ}$ in the detector frame
$\left[\begin{array}{c}x_{\mathrm{R}}^{\mathrm{DF}} \\ y_{\mathrm{R}}^{\mathrm{DF}} \\ z_{\mathrm{R}}^{\mathrm{DF}} \\ 1\end{array}\right]_{k}=\mathbf{T}\left(\mathrm{d} x_{\mathrm{R}, k}, \mathrm{~d} y_{\mathrm{R}, k}, \mathrm{~d} z_{\mathrm{R}, k}\right)\left[\begin{array}{c}x_{\mathrm{R}}^{\mathrm{DF}} \\ y_{\mathrm{R}}^{\mathrm{DF}} \\ z_{\mathrm{R}}^{\mathrm{DF}} \\ 1\end{array}\right]_{k=1}$
The axis of rotation at sample stage position step $k$ is denoted by the unit vector $\widehat{\boldsymbol{r}}_{k}^{\mathrm{DF}}=\left(r_{\mathrm{X}}^{\mathrm{DF}}, r_{\mathrm{Y}}^{\mathrm{DF}}, r_{\mathrm{Z}}^{\mathrm{DF}}\right)_{k}$ in equation (9).
$\left[\begin{array}{c}r_{\mathrm{X}}^{\mathrm{DF}} \\ r_{\mathrm{Y}}^{\mathrm{DF}} \\ r_{\mathrm{Z}}^{\mathrm{DF}} \\ 1\end{array}\right]_{k}=\boldsymbol{R o t}_{\mathrm{X}}\left(\gamma_{\mathrm{X}, k}\right) \boldsymbol{\operatorname { R o t }}_{\mathrm{Y}}\left(\gamma_{\mathrm{Y}, k}\right) \boldsymbol{\operatorname { R o t }}_{\mathrm{Z}}\left(\gamma_{\mathrm{Z}, k}\right)\left[\begin{array}{c}r_{\mathrm{X}}^{\mathrm{DF}} \\ r_{\mathrm{Y}}^{\mathrm{DF}} \\ r_{\mathrm{Z}}^{\mathrm{DF}} \\ 1\end{array}\right]_{k=1}$

The sphere center coordinates $\boldsymbol{C}_{i, j=1, k}^{\mathrm{DF}}=\left(x_{\mathrm{C}}^{\mathrm{DF}}, y_{\mathrm{C}}^{\mathrm{DF}}, z_{\mathrm{C}}^{\mathrm{DF}}\right)_{i, j=1, k}$ at the sample stage position step $k$ and at the $j=1$ sample stage angular position are given by equation (10).

$$
\left[\begin{array}{c}
x_{\mathrm{C}}^{\mathrm{DF}} \\
y_{\mathrm{C}}^{\mathrm{DF}} \\
z_{\mathrm{C}}^{\mathrm{DF}} \\
1
\end{array}\right]_{i, j=1, k}=\mathbf{T}\left(x_{\mathrm{R}, k}^{\mathrm{DF}}, y_{\mathrm{R}, k}^{\mathrm{DF}}, z_{\mathrm{R}, k}^{\mathrm{DF}}\right) \boldsymbol{\operatorname { R o t }}_{\mathrm{X}}\left(\gamma_{\mathrm{X}, k}\right) \boldsymbol{R o t}_{\mathrm{Y}}\left(\gamma_{\mathrm{Y}, k}\right) \boldsymbol{R o t}_{\mathrm{Z}}\left(\gamma_{\mathrm{Z}, k}\right) \mathbf{T}\left(-x_{\mathrm{R}, k}^{\mathrm{DF}},-y_{\mathrm{R}, k}^{\mathrm{DF}},-z_{\mathrm{R}, k}^{\mathrm{DF}}\right)\left[\begin{array}{c}
x_{\mathrm{C}}^{\mathrm{DF}} \\
y_{\mathrm{C}}^{\mathrm{DF}} \\
z_{\mathrm{C}}^{\mathrm{DF}} \\
1
\end{array}\right]_{i, j=1, k=1}
$$

$\boldsymbol{C}_{i, j=1}^{\mathrm{DF}}=\left(x_{\mathrm{C}}^{\mathrm{DF}}, y_{\mathrm{C}}^{\mathrm{DF}}, z_{\mathrm{C}}^{\mathrm{DF}}\right)_{i, j=1, k=1}$ are given by equation (7).

$$
\left[\begin{array}{c}
x_{\mathrm{C}}^{\mathrm{DF}} \\
y_{\mathrm{C}}^{\mathrm{DF}} \\
z_{\mathrm{C}}^{\mathrm{DF}} \\
1
\end{array}\right]_{i, j=1, k=1}=\left(\boldsymbol{\operatorname { R o t }}_{\mathrm{Z}}(\eta) \boldsymbol{\operatorname { R o t }}_{\mathrm{Y}}(\phi) \boldsymbol{R o t}_{\mathrm{X}}(\theta)\right)^{-1} \mathbf{T}\left(-x_{\mathrm{D}}^{\mathrm{SRF}},-y_{\mathrm{D}}^{\mathrm{SRF}},-z_{\mathrm{D}}^{\mathrm{SRF}}\right)\left[\begin{array}{c}
x_{\mathrm{C}}^{\mathrm{SRF}} \\
y_{\mathrm{C}}^{\mathrm{SRF}} \\
z_{\mathrm{C}}^{\mathrm{SRF}} \\
1
\end{array}\right]_{i, j=1, k=1}
$$

### 3.3. Sample stage translation and rotation

Each time the sample stage is moved to a new manipulator position $k=2,3, \ldots, 19,360$ X-ray projections of the reference object are acquired as it is rotated on the sample stage. The observed center projection coordinates from this new set of X-ray projections $\left(u_{\mathrm{obs}}, v_{\mathrm{obs}}\right)_{i, j, k}^{\mathrm{DF}}$ are determined per the method described previously. A modified forward projection model is used to generate an equivalent set of modelled center projection coordinates $\left(u_{\text {mod }}, v_{\text {mod }}\right)_{i, j, k}^{\mathrm{DF}}$. The modified forward projection model parameterizes displacements of the sample stage $\mathbf{d} \boldsymbol{R}_{k}=\left(\mathrm{d} x_{\mathrm{R}}, \mathrm{d} y_{\mathrm{R}}, \mathrm{d} z_{\mathrm{R}}\right)_{k}$ and the change in orientation of the rotation axis as defined by three extrinsic rotations $\gamma_{k}=\left(\gamma_{\mathrm{X}}, \gamma_{\mathrm{Y}}, \gamma_{\mathrm{Z}}\right)_{k}$ in this order (Fig. 6).

The position of the sample stage $\boldsymbol{R}_{k}^{\mathrm{DF}}=\left(x_{\mathrm{R}}^{\mathrm{DF}}, y_{\mathrm{R}}^{\mathrm{DF}}, z_{\mathrm{R}}^{\mathrm{DF}}\right)_{k}$ at its position step $k$ and in $\{\mathrm{DF}\}$ is given by equation (8).

The sphere center coordinates $\boldsymbol{C}_{i, j, k}^{\mathrm{DF}}=\left(x_{\mathrm{C}}^{\mathrm{DF}}, y_{\mathrm{C}}^{\mathrm{DF}}, z_{\mathrm{C}}^{\mathrm{DF}}\right)_{i, j, k}$ at an angular position $\alpha_{j}$ of the sample stage are given by equation (11), where $\boldsymbol{\operatorname { R o t }}_{\boldsymbol{r}_{\boldsymbol{k}}}\left(\alpha_{j}\right)$ is a rotation $\alpha_{j}$ about $\widehat{\boldsymbol{r}}_{k}^{\mathrm{DF}}$.

$$
\left[\begin{array}{c}
x_{\mathrm{C}}^{\mathrm{DF}} \\
y_{\mathrm{C}}^{\mathrm{DF}} \\
z_{\mathrm{C}}^{\mathrm{DF}} \\
1
\end{array}\right]_{i, j, k}=\mathbf{T}\left(x_{\mathrm{R}, k}^{\mathrm{DF}}, y_{\mathrm{R}, k}^{\mathrm{DF}}, z_{\mathrm{R}, k}^{\mathrm{DF}}\right) \mathbf{R o t}_{\boldsymbol{r}_{k}}\left(\alpha_{j}\right) \mathbf{T}\left(-x_{\mathrm{R}, k}^{\mathrm{DF}},-y_{\mathrm{R}, k}^{\mathrm{DF}},-z_{\mathrm{R}, k}^{\mathrm{DF}}\right)\left[\begin{array}{c}
x_{\mathrm{C}}^{\mathrm{DF}} \\
y_{\mathrm{C}}^{\mathrm{DF}} \\
z_{\mathrm{C}}^{\mathrm{DF}} \\
1
\end{array}\right]_{i, j=1, k}
$$

.11
The modelled projection coordinates $\left(u_{\text {mod }}, v_{\text {mod }}\right)_{i, j, k}$ for each sphere, $i$, at each angular position of the sample stage, $j$, and at each sample stage position step, $k$, are determined by finding the intersections with the detector plane of rays from the source position through each sphere center coordinate. The forward projection operation in the detector frame is described in Appendix B.

A Levenberg-Marquardt non-linear least squares algorithm is applied to minimize the reprojection error between observed and modelled center projection coordinates for each sample stage position. The actual sample stage displacements and rotations are given by the corresponding modelled values that provide the minimized reprojection error. The minimization argument is given by equation (3) albeit the solvable



Fig. 6. We parameterize translations of the sample stage $\mathrm{d} x_{\mathrm{R}, k}, \mathrm{~d} y_{\mathrm{R}, \mathrm{k}}, \mathrm{d} z_{\mathrm{R}, k}$ and changes in orientation of the axis of rotation as defined by three extrinsic rotation angles $\gamma_{\mathrm{X}, k}, \gamma_{\mathrm{Y}, k}, \gamma_{\mathrm{Z}, k}$ as the sample stage is moved from its initial position $k=1$ to a new position $k$.
parameters are $w=\left(\mathrm{d} x_{\mathrm{R}, k}, \quad \mathrm{~d} y_{\mathrm{R}, k}, \quad \mathrm{~d} z_{\mathrm{R}, k}, \gamma_{\mathrm{X}, k}, \gamma_{\mathrm{Y}, k}, \gamma_{\mathrm{Z}, k}\right)$ for each stepped position $k$ of the sample stage. We perform the same measurements of sample stage error motions using only the first X-ray projection, i.e., $j=$ 1 , and compare the results to the results from $j=1,2, \ldots, N_{\alpha}=360$. Z positioning errors are given by the difference between measured $\mathrm{d} z_{\mathrm{R}, k}$ and the corresponding SRD readings from the instrument acquisition software. X and Y straightness errors are given by the residuals in linear least-squares regression of $\mathrm{d} y_{\mathrm{R}, k}$ and $\mathrm{d} z_{\mathrm{R}, k}$ over the entire range of travel, respectively. Pitch, yaw, and roll are given by $\gamma_{\mathrm{X}, \mathrm{k}} \gamma_{\mathrm{Y}, \mathrm{k}}$, and $\gamma_{\mathrm{Z}, \mathrm{k}}$, respectively.

## 4Results

We compare the error motions measured with the object-based method to reference measurements made by laser interferometer and electronic level, published in Ref. [13]. The sign convention of object-based results is adapted to correspond to the coordinate frame used in instrument-based measurements. We expect additional discrepancies as a result of the separation in time between datasets. However, the object-based method provides results that agree with the instrument-based measurements despite these differences.

The measured error motions are compared to instrument-based reference measurements in Figs. 7-12. Object-based results from 360 projections of the object as the stage performs a full rotation are denoted


Fig. 7. Z positioning errors of sample stage translation.


Fig. 8. Straightness error of sample stage translation in $X$ direction.
by blue circular markers ('multi', as in multiple projections) while the results from the first projection are denoted by orange markers ('single'); reference measurements are given by the solid black line ('reference'). Deviations between object-based results and reference measurements are given by the triangular markers in the corresponding delta $(\Delta)$ plots


Fig. 9. Straightness of sample stage translation in Y direction.


Fig. 10. Pitch error of sample stage translation (about $X$ axis). Units of ordinates are arcseconds.
below. The object-based results from a single projection generally agreed with the results from multiple projections, further supporting time savings of the object-based procedure. The trends observed from instrument-based measurements are also captured in the object-based measurements.


Fig. 11. Stage roll (about Z). Units of ordinates are arcseconds.


Fig. 12. Stage yaw (about Y). Units of ordinates are arcminutes.

### 4.1Positioning errors

The object-based Z positioning errors capture the instrument-based trend of increasing positioning errors as the stage is moved closer to the detector (Fig. 7). In this case, a positive positioning error corresponds to the sample stage travelling less than its indexed displacement. The deviation of object-based results from reference measurements increases with increasing travel distance of the stage. At the nominal stage SRD of 724 mm , the reference positioning error is $493 \mu \mathrm{~m}$, while the object-based positioning errors are $+581 \mu \mathrm{~m}$ and $+603 \mu \mathrm{~m}$ for multiprojection and single-projection measurements, respectively. This behavior is consistent with the results presented by Ferrucci et al. [5], in which the measurement of $Z$ positions by minimization of reprojection errors is consistently skewed positively, i.e., towards the source. Furthermore, the sensitivity of object-based measurements using the same object is expected to decrease with decreasing magnification as the projection of the object occupies smaller regions of the detector field of view.

### 4.2. Straightness errors

The parabolic curves in the reference $X$ and $Y$ straightness errors (Figs. 8 and 9, respectively) are captured by object-based results. Absolute deviations between object-based and reference measurements were within $10 \mu \mathrm{~m}$ and generally increase with increasing SRD, further supporting the decreasing sensitivity of object-based measurements at decreasing magnifications.

### 4.3Angular error motions

Deviations of object-based pitch (Fig. 10) and roll (Fig. 11) measurements were within approximately 10 arcseconds of reference measurements. There is a large discrepancy between object-based yaw measurements and reference values (Fig. 12), on the order of $\pm 10$ arcminutes. We believe this discrepancy is due to the fact that the sample stage performs a full revolution about an axis parallel to the yaw axis with each new position; random angular indexing errors of the sample stage rotation axis upon returning to the zero position are absorbed in the measurement of $\gamma_{\mathrm{Y}}$. In future work, the stage error motions will be determined using single X-ray projections at each stage position without effecting stage rotations about its axis.

## 5Conclusion

The object-based method for measuring stage error motions has been described and shown to agree with reference measurements performed using laser interferometers and electronic levels. The sensitivity of object-based measurements decreases with decreasing object magnification as can be expected due to the reduced coverage of the projected object on the detector field of view. Adequate object-based measurement of sample stage error motions therefore relies on the use of multiple objects of varying sizes to ensure sufficient coverage on the detector. The benefit of an object-based method is that the results are presented in an imaging coordinate frame, as opposed to the mechanical frame used when measuring error motions with separate instrumentation. This means that the measured error motions can be readily applied to adapt the back-projection geometry in tomographic reconstruction, without the need for a laborious registration procedure between imaging and mechanical coordinate frames. The performance when using only a single image presents an opportunity for extending this objectbased method for pose determination in more complex acquisition architectures, such as robot-based CT. Future work includes applying a set of reference objects of varying sizes to measure the sample stage error motions over a longer range of longitudinal (Z) positions.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
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## Appendix C. Supplementary data

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## Appendix A. Forward projection in source-rotation axis frame \{SRF\}

Each ray is defined by the parametric equation of a line:
$\left[\begin{array}{l}x \\ y \\ z \\ 1\end{array}\right]=\left[\begin{array}{c}x_{\mathrm{o}} \\ y_{\mathrm{o}} \\ z_{0} \\ 1\end{array}\right]+t\left[\begin{array}{c}x_{1}-x_{\mathrm{o}} \\ y_{1}-y_{\mathrm{o}} \\ z_{1}-z_{0} \\ 1\end{array}\right]$
where $(x, y, z)$ is any point on the ray, $\left(x_{0}, y_{0}, z_{0}\right)$ denotes the origin of the ray and $\left(x_{1}, y_{1}, z_{1}\right)$ denotes a second point along the ray in the positive direction of the parameter $t$. Here, the origin of each ray is given by the source focal spot position $(0,0,0)$ and the second point corresponds to each sphere center coordinate. Thus, the ray from the source to the center of each sphere $i$ at rotation position $j$ is given by equation (A2).
$\left[\begin{array}{c}x \\ y \\ z \\ 1\end{array}\right]=t_{i, j, k=1}\left[\begin{array}{c}x_{\mathrm{C}}^{\mathrm{SRF}} \\ y_{\mathrm{C}}^{\mathrm{SRF}} \\ z_{\mathrm{C}}^{\mathrm{SRF}} \\ 1\end{array}\right]_{i, j, k=1}$.
We then determine the intersection of each ray with the detector. We define the detector plane by the geometrical center $\boldsymbol{D}^{\mathrm{SRF}}=\left(x_{\mathrm{D}}^{\mathrm{SRF}}, y_{\mathrm{D}}^{\mathrm{SRF}}, z_{\mathrm{D}}^{\mathrm{SRF}}\right)$ and the detector normal $\widehat{\boldsymbol{n}}^{\mathrm{SRF}}=\left(n_{\mathrm{X}}^{\mathrm{SRF}}, n_{\mathrm{Y}}^{\mathrm{SRF}}, n_{\mathrm{Z}}^{\mathrm{SRF}}\right)$ as follows.

$$
\left[\begin{array}{c}
n_{\mathrm{X}}^{\mathrm{SRF}} \\
n_{\mathrm{Y}}^{\mathrm{SRF}} \\
n_{\mathrm{Z}}^{\mathrm{SRF}} \\
1
\end{array}\right] \cdot\left(\left[\begin{array}{c}
x \\
y \\
z \\
1
\end{array}\right]\left[\begin{array}{c}
x_{\mathrm{D}}^{\mathrm{SRF}} \\
y_{\mathrm{D}}^{\mathrm{SRF}} \\
z_{\mathrm{D}}^{\mathrm{SRF}} \\
1
\end{array}\right]\right)=0
$$

where $(x, y, z)$ is any point on the plane. The normal to the detector in the presence of angular misalignments is given by equation (A4).
$\left[\begin{array}{c}n_{\mathrm{X}}^{\mathrm{SRF}} \\ n_{\mathrm{Y}}^{\mathrm{SRF}} \\ n_{\mathrm{Z}}^{\text {SRF }} \\ 1\end{array}\right]=\boldsymbol{\operatorname { R o t }}_{\mathrm{Z}}(\eta) \boldsymbol{\operatorname { R o t }}_{\mathrm{Y}}(\phi) \boldsymbol{\operatorname { R o t }}_{\mathrm{X}}(\theta)\left[\begin{array}{l}0 \\ 0 \\ 1 \\ 1\end{array}\right]$.
The intersection of each ray with the detector plane is determined by substituting equation (A2) into equation (A3),
$\left[\begin{array}{c}n_{\mathrm{X}}^{\mathrm{SRF}} \\ n_{\mathrm{Y}}^{\mathrm{SRF}} \\ n_{\mathrm{Z}}^{\mathrm{SRF}} \\ 1\end{array}\right] \cdot\left(t_{i, j, k=1}\left[\begin{array}{c}x_{\mathrm{C}}^{\mathrm{SRF}} \\ y_{\mathrm{C}}^{\mathrm{SRF}} \\ z_{\mathrm{C}}^{\mathrm{SRF}} \\ 1\end{array}\right]_{i, j, k=1}\right)=0$,
and solving for $t_{i, j, k=1}$ (equation (A6)).
$t_{i, j, k=1}=\left[\begin{array}{c}n_{\mathrm{X}}^{\mathrm{SRF}} \\ n_{\mathrm{Y}}^{\mathrm{SRF}} \\ n_{\mathrm{Z}}^{\mathrm{SRF}} \\ 1\end{array}\right]\left[\begin{array}{c}n_{\mathrm{X}}^{\mathrm{SRF}} \\ n_{\mathrm{Y}}^{\mathrm{SRF}} \\ n_{\mathrm{Z}}^{\mathrm{SRF}} \\ 1\end{array}\right] \cdot\left[\begin{array}{c}x_{\mathrm{C}}^{\mathrm{SRF}} \\ y_{\mathrm{C}}^{\mathrm{SRF}} \\ z_{\mathrm{C}}^{\mathrm{SRF}} \\ 1\end{array}\right]_{i, j, k=1}$,

Applying the solved $t_{i, j, k=1}$ again to equation A2 provides the intersection point in the SRF: $\left(x_{\mathrm{int}}^{\mathrm{SRF}}, y_{\mathrm{int}}^{\mathrm{SRF}}, z_{\mathrm{int}}^{\mathrm{SRF}}\right)_{i, j, k=1}$
$\left[\begin{array}{c}x_{\mathrm{int}}^{\mathrm{SRF}} \\ y_{\mathrm{int}}^{\mathrm{SRF}} \\ z_{\mathrm{int}}^{\mathrm{SRF}} \\ 1\end{array}\right]_{i, j, k=1}=\left[\begin{array}{c}n_{\mathrm{X}}^{\mathrm{SRF}} \\ n_{\mathrm{Y}}^{\mathrm{SRF}} \\ n_{\mathrm{Z}}^{\mathrm{SRF}} \\ 1\end{array}\right]_{1}\left[\begin{array}{c}n_{\mathrm{X}}^{\mathrm{SRF}} \\ n_{\mathrm{Y}}^{\mathrm{SRF}} \\ n_{\mathrm{Z}}^{\mathrm{SRF}} \\ 1\end{array}\right]_{i, j, k=1}\left[\begin{array}{c}x_{\mathrm{C}}^{\mathrm{SRF}} \\ y_{\mathrm{C}}^{\mathrm{SRF}} \\ z_{\mathrm{C}}^{\mathrm{SRF}} \\ 1\end{array}\right]_{i, j, k=1}\left[\begin{array}{c}x_{\mathrm{C}}^{\mathrm{SRF}} \\ y_{\mathrm{C}}^{\mathrm{SRF}} \\ z_{\mathrm{C}}^{\mathrm{SRF}} \\ 1\end{array}\right]$

The corresponding image coordinates $\left(u_{\text {mod }}, v_{\text {mod }}\right)_{i, j, k=1}^{\mathrm{DF}}$ of each intersection point are given by equations A8 and A9.
$\left[u_{\mathrm{mod}}\right]_{i, j, k=1}^{\mathrm{DF}}=\left[\begin{array}{c}u_{\mathrm{X}}^{\mathrm{SRF}} \\ u_{\mathrm{Y}}^{\mathrm{SRF}} \\ u_{\mathrm{Z}}^{\mathrm{SRF}} \\ 1\end{array}\right] \cdot \mathbf{T}\left(-x_{\mathrm{D}},-y_{\mathrm{D}},-z_{\mathrm{D}}\right)\left[\begin{array}{c}x_{\mathrm{int}}^{\mathrm{SRF}} \\ y_{\mathrm{int}}^{\mathrm{SRF}} \\ z_{\mathrm{int}}^{\mathrm{SRF}} \\ 1\end{array}\right]_{i, j, k=1}$
$\left[v_{\mathrm{mod}}\right]_{i, j, k=1}^{\mathrm{DF}}=\left[\begin{array}{c}v_{\mathrm{X}}^{\mathrm{SRF}} \\ v_{\mathrm{Y}}^{\mathrm{SRF}} \\ v_{\mathrm{Z}}^{\mathrm{SRF}} \\ 1\end{array}\right] \cdot \mathbf{T}\left(-x_{\mathrm{D}},-y_{\mathrm{D}},-z_{\mathrm{D}}\right)\left[\begin{array}{c}x_{\mathrm{int}}^{\mathrm{SRF}} \\ y_{\mathrm{int}}^{\mathrm{SF}} \\ z_{\mathrm{int}}^{\mathrm{SF}} \\ 1\end{array}\right]_{i, j, k=1}$
Where $\left[\begin{array}{c}u_{\mathrm{X}}^{\mathrm{SRF}} \\ u_{\mathrm{Y}}^{\mathrm{SRF}} \\ u_{\mathrm{Z}}^{\mathrm{SRF}} \\ 1\end{array}\right]=\boldsymbol{\operatorname { R o t }}_{\mathrm{Z}}(\eta) \boldsymbol{R o t}_{\mathrm{Y}}(\phi) \operatorname{Rot}_{\mathrm{X}}(\theta)\left[\begin{array}{c}1 \\ 0 \\ 0 \\ 1\end{array}\right]$ and $\left[\begin{array}{c}v_{\mathrm{X}}^{\mathrm{SRF}} \\ v_{\mathrm{Y}}^{\mathrm{SRF}} \\ v_{\mathrm{Z}}^{\mathrm{SRF}} \\ 1\end{array}\right]=\boldsymbol{R o t}_{\mathrm{Z}}(\eta) \boldsymbol{R o t}_{\mathrm{Y}}(\phi) \boldsymbol{R o t}_{\mathrm{X}}(\theta)\left[\begin{array}{c}0 \\ 1 \\ 0 \\ 1\end{array}\right]$.
The image coordinates can be represented in the units of $\{\mathrm{SRF}\}$ or in pixel coordinates, in which case $\left(u_{\text {mod }}, v_{\text {mod }}\right)_{i, j, k=1}^{\mathrm{DF}}$ in equations A8 and A9 would be divided by the corresponding pixel side length.

## Appendix B. Forward projection in Detector Frame \{DF $\}$

We determine the center projection coordinates $\left(u_{\text {mod }}, v_{\text {mod }}\right)_{i, j, k}^{\mathrm{DF}}$ using the same procedure outlined in Appendix A, albeit in the detector frame and applying appropriate substitutions. Each ray from the source focal spot to each sphere center is parameterized in equation B1.
$\left[\begin{array}{c}x \\ y \\ z \\ 1\end{array}\right]=\left[\begin{array}{c}x_{\mathrm{S}}^{\mathrm{DF}} \\ y_{\mathrm{S}}^{\mathrm{DF}} \\ z_{\mathrm{S}}^{\mathrm{DF}} \\ 1\end{array}\right]+t_{i, j, k}\left(\left[\begin{array}{c}x_{\mathrm{C}}^{\mathrm{DF}} \\ y_{\mathrm{C}}^{\mathrm{DF}} \\ z_{\mathrm{C}}^{\mathrm{DF}} \\ 1\end{array}\right]\left[\begin{array}{c}x_{\mathrm{C}}^{\mathrm{DF}} \\ y_{\mathrm{C}}^{\mathrm{DF}} \\ z_{\mathrm{CF}}^{\mathrm{DF}} \\ 1\end{array}\right]_{i, j, k}\right)$
The detector plane is parameterized in equation B2.
$\left[\begin{array}{c}n_{\mathrm{X}}^{\mathrm{DF}} \\ n_{\mathrm{Y}}^{\mathrm{DF}} \\ n_{\mathrm{Z}}^{\mathrm{DF}} \\ 1\end{array}\right] \cdot\left(\left[\begin{array}{c}x \\ y \\ z \\ 1\end{array}\right]\left[\begin{array}{c}x_{\mathrm{D}}^{\mathrm{DF}} \\ y_{\mathrm{D}}^{\mathrm{DF}} \\ z_{\mathrm{D}}^{\mathrm{DF}} \\ 1\end{array}\right]\right)=0$,
Since $\left(n_{\mathrm{X}}^{\mathrm{DF}}, n_{\mathrm{Y}}^{\mathrm{DF}}, n_{\mathrm{Z}}^{\mathrm{DF}}\right)=(0,0,1)$ and $\left(x_{\mathrm{D}}^{\mathrm{DF}}, y_{\mathrm{D}}^{\mathrm{DF}}, z_{\mathrm{D}}^{\mathrm{DF}}\right)=(0,0,0)$,
$\left[\begin{array}{l}0 \\ 0 \\ 1 \\ 1\end{array}\right] \cdot\left[\begin{array}{l}x \\ y \\ z \\ 1\end{array}\right]=0$ i.e., the detector plane is given by $\mathrm{z}=0$.
Substituting equation (B1) into equation (B3),
$\left[\begin{array}{l}0 \\ 0 \\ 1 \\ 1\end{array}\right] \cdot\left(\left[\begin{array}{c}x_{\mathrm{S}}^{\mathrm{DF}} \\ y_{\mathrm{S}}^{\mathrm{DF}} \\ z_{\mathrm{S}}^{\mathrm{DF}} \\ 1\end{array}\right]+t_{i, j, k}\right)=0$,
Solving for $t_{i, j, k}$,
$t_{i, j, k}=-\left[\begin{array}{l}0 \\ 0 \\ 1 \\ 1\end{array}\right]\left[\begin{array}{l}0 \\ 0 \\ 1 \\ 1\end{array}\right] \cdot\left(\left[\begin{array}{c}x_{\mathrm{C}}^{\mathrm{DF}} \\ y_{\mathrm{C}}^{\mathrm{DF}} \\ z_{\mathrm{C}}^{\mathrm{DF}} \\ 1\end{array}\right]\left[\begin{array}{c}x_{\mathrm{C}}^{\mathrm{DF}} \\ y_{\mathrm{C}}^{\mathrm{DF}} \\ z_{\mathrm{C}}^{\mathrm{DF}} \\ 1\end{array}\right]_{i, j, k}\right)$,
Applying the solved $t_{i, j, k}$ to equation (B1) provides the intersection point $\left(x_{\mathrm{int}}^{\mathrm{DF}}, y_{\mathrm{int}}^{\mathrm{DF}}, 0\right)_{i, j, k}$ in $\{\mathrm{DF}\}$ :
$\left[\begin{array}{c}x_{\mathrm{int}}^{\mathrm{DF}} \\ y_{\mathrm{int}}^{\mathrm{DF}} \\ 0 \\ 1\end{array}\right]_{i, j, k}=\left[\begin{array}{c}x_{\mathrm{S}}^{\mathrm{DF}} \\ y_{\mathrm{S}}^{\mathrm{DF}} \\ z_{\mathrm{S}}^{\mathrm{DF}} \\ 1\end{array}\right]-\left[\begin{array}{c}0 \\ 0 \\ 1 \\ 1\end{array}\right]\left[\begin{array}{c}0 \\ 0 \\ 1 \\ 1\end{array}\right] \cdot\left(\left[\begin{array}{c}x_{\mathrm{C}}^{\mathrm{DF}} \\ y_{\mathrm{C}}^{\mathrm{DF}} \\ z_{\mathrm{C}}^{\mathrm{DF}} \\ 1\end{array}\right]\left[\begin{array}{c}x_{\mathrm{C}}^{\mathrm{DF}} \\ y_{\mathrm{C}}^{\mathrm{DF}} \\ z_{\mathrm{C}}^{\mathrm{DF}} \\ 1\end{array}\right]_{i, j, k}\right)\left(\left[\begin{array}{c}x_{\mathrm{C}}^{\mathrm{DF}} \\ y_{\mathrm{C}}^{\mathrm{DF}} \\ z_{\mathrm{C}}^{\mathrm{DF}} \\ 1\end{array}\right]\left[\begin{array}{c}x_{\mathrm{C}}^{\mathrm{DF}} \\ y_{\mathrm{C}}^{\mathrm{DF}} \\ z_{\mathrm{C}}^{\mathrm{DF}} \\ 1\end{array}\right]_{i, j, k}\right)$.

The corresponding image coordinates $\left(u_{\text {mod }}, \nu_{\text {mod }}\right)_{i, j, k}^{\mathrm{DF}}$ of each intersection point are given by equation (B7) and B8.
$\left[u_{\mathrm{mod}}\right]_{i, j, k}^{\mathrm{DF}}=\left[x_{\mathrm{int}}^{\mathrm{DF}}\right]_{i, j, k}$
$\left[v_{\mathrm{mod}}\right]_{i, j, k}^{\mathrm{DF}}=\left[y_{\mathrm{int}}^{\mathrm{DF}}\right]_{i, j, k}$
The image coordinates can be represented in the units of $\{\mathrm{DF}\}$ or in pixel coordinates, in which case $\left(u_{\text {mod }}, v_{\text {mod }}\right)_{i, j, k}^{\mathrm{DF}}$ in equation (B7) and B8 would be divided by the corresponding pixel side length.

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