Units and Constants

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We describe atomic and natural units relevant for the atomic, molecular, and optical physics described in this book. For nonrelativistic models of atoms and molecules absorbing and emitting photons atomic units are appropriate. That is energies and lengths are expressed in terms of the Hartree energy and Bohr radius, respectively. Relativistic models are required for precision quantum electrodynamics-based determinations of energy levels of hydrogen and other light atoms and molecules. Natural units are then most appropriate and energies and length are expressed in the rest energy of the electron and the reduced Compton wavelength, respectively. Finally, we present values for an abbreviated list of fundamental constants taken from the CODATA adjustment of fundamental constants based on data published or made available before end of 2018.

I. INTRODUCTION

In science we communicate measurements of quantities or observables in terms a product of a numerical value and a unit, where the unit is a particular example of the quantity. For example, the spin of an electron is $(1/2)\,\hbar$, where 1/2 is the numerical value and the reduced Planck constant $\hbar$ is the unit or example angular momentum. Equivalently, the spin of the electron is $0.527\,285\ldots \times 10^{-34}$ J s in the International System of Units (SI) [1]. More digits for the numerical value can be easily found now that the Planck constant has been exactly defined in the SI. In fact, seven constants define the SI base units [1]. Their names and values are summarized in Table I.

The example in the previous paragraph shows that some units are more convenient than others. We tend to prefer units where the numerical value of the measured quantity is of order one. Preferred units for a system follow from its Hamiltonian H in a process that makes H as well as all operators and constants appearing in H dimensionless. Operators can correspond to particle spins, positions and momenta but also fields in quantum field theories.

TABLE I. Exact quantities and their symbols, numerical values, and units in the SI.

Quantity	Sym.	Value	Unit
hyperf. transition freq. of ^{133}Cs	$\Delta\nu_{\rm Cs}$	$9\,192\,631\,770$	Hz
speed of light in vacuum	c	$299\,792\,458$	m s^{-1}
Planck constant[a]	h	$6.626\,070\,15 \times 10^{-34}$	J Hz^{-1}
	$\hbar$	$1.054\,571\,817\ldots \times 10^{-34}$	J s
elementary charge	e	$1.602\,176\,634 \times 10^{-19}$	C
Boltzmann constant	k	$1.380\,649 \times 10^{-23}$	J K^{-1}
Avogadro constant	$N_{\rm A}$	$6.022\,140\,76 \times 10^{23}$	mol^{-1}
luminous efficacy	$K_{\rm cd}$	683	lm W^{-1}

[a] The energy of a photon with frequency ν expressed in unit Hz is $E = h\nu$ in J. Unitary time evolution of the state of this photon is given by $\exp(-iEt/\hbar)|\varphi\rangle$, where $|\varphi\rangle$ is the photon state at time $t = 0$ and time is expressed in unit s. The ratio $Et/\hbar$ is a phase.

In this chapter we derive atomic and natural units as they appear in atomic, molecular, and optical physics. For a different view of units in the field of electromagnetism we recommend the appendix on units and dimensions of Ref. [2]. We also present a list of relevant fundamental constants.

II. ATOMIC UNITS

We derive atomic units starting from the nonrelativistic Hamiltonian for an electron with charge $-e$ and mass $m_{\rm e}$ bound to an infinitely heavy point-like source with charge $+Ze$, possibly absorbing and emitting transverse photons. Following Ref. [3] we use the Coulomb gauge and assume that the charges are contained in a large cubic box with periodic boundary conditions and length L on each side. Here, e is the (positive) elementary charge and Z is a positive integer. Hence, we have

$$H = \frac{(\vec{p} + e\vec{A}_\perp(\vec{r}))^2}{2m_{\rm e}} - \frac{1}{4\pi\epsilon_0}\frac{Ze^2}{r} + \sum_j \hbar\omega_j \left[a_j^\dagger a_j + \frac{1}{2}\right], \tag{1}$$

where $\vec{r}$ and $\vec{p}$ are the position and momentum operators of the electron, respectively, and the commutation relations $[r_i, p_j] = i\hbar\delta_{ij}$ for vector components i and j hold. The source is located at $\vec{r} = \vec{0}$, ϵ_0 is the vacuum electric permittivity, and δ_{ij} is the Kronecker delta. For simplicity, we have omitted the Zeeman interaction of the electron spin coupling to a magnetic field.

The last term of Eq. 1 describes the photon-field Hamiltonian, where operators a_j and $a_j^\dagger$ annihilate and create photons, respectively. For index j photons are specified by polarization $\vec{\epsilon}_j$ and wavevector $\vec{k}_j$. The components of the wavevector are integer multiples of $2\pi/L$.

By construction $\vec{\epsilon}_j$ and $\vec{k}_j$ are perpendicular to each other and the frequency of photon j is $\omega_j = c|k_j|$, where c is the speed of light in vacuum. Operators a_j and $a_j^\dagger$ satisfy the commutation relations $[a_i, a_j^\dagger] = \delta_{ij}$. Finally, the transverse vector field operator $\vec{A}_\perp(\vec{r})$ [4] is given by

$$\vec{A}_\perp(\vec{r}) = \sum_j \sqrt{\frac{\hbar}{2\epsilon_0 \omega_j L^3}} \left[a_j \vec{\epsilon}_j e^{i\vec{k}_j\cdot\vec{r}} + a_j^\dagger \vec{\epsilon}_j e^{-i\vec{k}_j\cdot\vec{r}} \right] . \quad (2)$$

Similar expressions for the transverse electric $\vec{E}_\perp(\vec{r})$ and magnetic $\vec{B}(\vec{r})$ field operators can be written down. The constants ϵ_0, c, and the vacuum magnetic permeability μ_0 are not independent. They satisfy

$$c^2 = \frac{1}{\mu_0 \epsilon_0} . \quad (3)$$

We are now ready to make the Hamiltonian dimensionless. We assume that there exists a convenient length scale a_0 and write for the position operator of the electron $\vec{r} = \vec{x}\,a_0$, where operator $\vec{x}$ is dimensionless. An equally valid interpretation of $\vec{x}\,a_0$ is that $\vec{x}$ is a vector of numerical values and a_0 is the unit for position. The commutation relation for $\vec{r}$ and $\vec{p}$ imply $\vec{p} = -i\hbar\nabla_r$, where ∇_r is the del or nabla vector differential operator in $\vec{r}$, and thus $\vec{p} = -i\nabla_x\,\hbar a_0^{-1}$. The wavevector and frequency of the photon field are made dimensionless with $\vec{k}_j = \vec{q}_j\,a_0^{-1}$ and $\omega_j = |q_j|\,c a_0^{-1}$. The components of $\vec{q}_j$ are integer multiples of $2\pi/\ell$, where $L = \ell\,a_0$. Thus the units of electron momentum, photon wavevector and frequency are $\hbar a_0^{-1}$, a_0^{-1}, and $c a_0^{-1}$, respectively.

To summarize, Eq. 1 becomes

$$H = \frac{\hbar^2}{m_{\mathrm{e}} a_0^2} \frac{1}{2} \left(-i\nabla_x + \frac{e a_0}{\hbar} \vec{A}_\perp(\vec{x}\,a_0) \right)^2 - \frac{e^2}{4\pi\epsilon_0 a_0} \frac{Z}{x} + \hbar c a_0^{-1} \sum_j |q_j| \left[a_j^\dagger a_j + \frac{1}{2} \right] . \quad (4)$$

A comparison of the energies of the first two terms suggests that we choose a_0 such that

$$\frac{\hbar^2}{m_{\mathrm{e}} a_0^2} = \frac{e^2}{4\pi\epsilon_0 a_0} \equiv E_{\mathrm{h}} \quad (5)$$

and thus

$$a_0 = \frac{4\pi\epsilon_0 \hbar^2}{m_{\mathrm{e}} e^2} . \quad (6)$$

The reference quantities E_{h} and a_0 or the atomic units of energy and length are better known as the Hartree energy and the Bohr radius, respectively. The Rydberg frequency cR_∞ is defined by $cR_\infty = E_{\mathrm{h}}/2h$.

In atomic units the vector potential is made dimensionless with the choice

$$\vec{A}_\perp(\vec{x}\,a_0) \equiv \vec{\mathcal{A}}_\perp(\vec{x}) \frac{\hbar}{e a_0} \quad (7)$$

so that the Hamiltonian reads

$$H = \left\{ \frac{1}{2} \left(-i\nabla_x + \vec{\mathcal{A}}_\perp(\vec{x}) \right)^2 - \frac{Z}{x} + \sum_j \frac{1}{\alpha} |q_j| \left[a_j^\dagger a_j + \frac{1}{2} \right] \right\} E_{\mathrm{h}} , \quad (8)$$

where the dimensionless fine-structure constant α is

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \quad (9)$$

and $1/\alpha$ is the numerical value of the speed of light in vacuum in atomic units. Specifically, the unit of velocity is $\hbar/m_{\mathrm{e}} a_0$.

III. NATURAL UNITS

Natural units follow from Dirac's relativistic description of the negatively-charged electron bound to an infinitely-heavy point charge in the presence of transverse photons. Thus in Coulomb gauge we have

$$H_{\mathrm{rel}} = \beta m_{\mathrm{e}} c^2 + c\vec{\alpha} \cdot (\vec{p} + e\vec{A}_\perp(\vec{r})) - \mathrm{I}_4 \frac{1}{4\pi\epsilon_0} \frac{Z e^2}{r} + \sum_j \hbar\omega_j \left[a_j^\dagger a_j + \frac{1}{2} \right] , \quad (10)$$

where $\vec{\alpha}$ and β are four dimensionless mutually anti-commuting four-component Dirac matrices with $\alpha_i^2 = \beta^2 = \mathrm{I}_4$ and I_4 is the four-component identity matrix. The definition for the transverse vector potential $\vec{A}_\perp(\vec{r})$ remains that of Eq. 2. For our purpose of defining natural units we can ignore interpretations of the positive and negative energy solutions of the Dirac equation as well as effects such as virtual electron-positron pairs, which will be present in the complete quantum electrodynamic (QED) theory.

This relativistic Hamiltonian can be made dimensionless by assuming length unit $\lambdabar_{\mathrm{C}}$ and $\vec{r} = \vec{x}\,\lambdabar_{\mathrm{C}}$ with similar expressions for box size L, momentum $\vec{p}$, and photon wavevector and frequency $\vec{k}_j$ and ω_j. Then we find

$$H_{\mathrm{rel}} = \beta m_{\mathrm{e}} c^2 + \hbar c \lambdabar_{\mathrm{C}}^{-1}\,\vec{\alpha} \cdot \left(-i\nabla_x + \frac{e\lambdabar_{\mathrm{C}}}{\hbar} \vec{A}_\perp(\vec{x}\,\lambdabar_{\mathrm{C}}) \right) - \frac{e^2}{4\pi\epsilon_0 \lambdabar_{\mathrm{C}}} \frac{Z}{x} + \hbar c \lambdabar_{\mathrm{C}}^{-1} \sum_j |q_j| \left[a_j^\dagger a_j + \frac{1}{2} \right] \quad (11)$$

and, by comparing the energies of the first two terms, choose $\lambdabar_{\mathrm{C}}$ such that

$$m_{\mathrm{e}} c^2 = \hbar c \lambdabar_{\mathrm{C}}^{-1} \quad (12)$$

or

$$\lambdabar_{\mathrm{C}} = \frac{\hbar}{m_{\mathrm{e}} c} . \quad (13)$$

The reference quantities $m_{\mathrm{e}}c^2$ and $\bar{\lambda}_{\mathrm{C}}$ or the natural units of energy and length are the energy equivalent of the electron rest mass and the reduced Compton wavelength, respectively. We also note that $\bar{\lambda}_{\mathrm{C}} = \alpha a_0$ and $E_{\mathrm{h}} = \alpha^2 m_{\mathrm{e}} c^2$.

In natural units the transverse vector potential is not made dimensionless from its appearance in the Hamiltonian, but rather from rewriting its definition in Eq. 2 in terms of the reduced Compton wavelength. In fact, we choose

$$\vec{A}_\perp(\vec{x}\,\bar{\lambda}_{\mathrm{C}}) = \sqrt{4\pi\alpha}\,\vec{\mathcal{A}}_\perp(\vec{x})\,\frac{\hbar}{e\bar{\lambda}_{\mathrm{C}}}\,, \tag{14}$$

where

$$\vec{\mathcal{A}}_\perp(\vec{x}) = \sum_j \sqrt{\frac{1}{2|q_j|\ell^3}}\left[a_j\vec{\epsilon}_j e^{i\vec{q}_j\cdot\vec{x}} + a_j^\dagger\vec{\epsilon}_j e^{-i\vec{q}_j\cdot\vec{x}}\right] \tag{15}$$

by inspection. The transverse vector field $\vec{\mathcal{A}}_\perp(\vec{x})$ only depends on geometric quantities related to the periodic boundary conditions in the cubic box of length $L = \ell\,\bar{\lambda}_{\mathrm{C}}$ and not on α or the charge of the electron. Finally, the Hamiltonian in Eq. 11 reads

$$\begin{aligned} H_{\mathrm{rel}} = \Bigg\{ & \beta + \vec{\alpha}\cdot\left(-i\nabla_x + \sqrt{4\pi\alpha}\vec{\mathcal{A}}_\perp(\vec{x})\right) \\ & -I_4\frac{Z\alpha}{x} + \sum_j |q_j|\left[a_j^\dagger a_j + \frac{1}{2}\right]\Bigg\} m_{\mathrm{e}}c^2\,, \end{aligned} \tag{16}$$

where the fine-structure constant now appears in the strength of the minimal coupling $\vec{\alpha}\cdot\vec{\mathcal{A}}_\perp(\vec{x})$ of the electron with the photons and in the Coulomb term of the Hamiltonian. The numerical value of the speed of light in vacuum is one in natural units. In fact, the unit of velocity is $\hbar/m_{\mathrm{e}}\bar{\lambda}_{\mathrm{C}} = c$.

IV. FUNDAMENTAL CONSTANTS

In Table II we present an abbreviated list of values for fundamental constants based on the 2018 adjustment of fundamental constants as published by the CODATA taskgroup [5]. The table also gives values for atomic and natural units for a range of quantities, such as charge, speed, and time. The two most accurately known constants are the g-factor of a free electron and the Rydberg frequency cR_∞ (or equivalently the Hartree energy) with relative uncertainties of 1.7×10^{-13} and 1.9×10^{-12}, respectively. The least well-known quantity in the table is the proton root-mean-square (rms) charge radius r_{p} with a relative uncertainty of 2.2×10^{-3}. In fact, r_p extracted from precision spectroscopy on the hydrogen atom and on muonic-hydrogen, where the electron is replaced by a muon, are marginally discrepant.

The fine-structure constant α, vacuum electric permittivity ϵ_0, and magnetic permeability μ_0 are dependent. They are related through Eqs. 3 and 9 and, since in the SI the values for $\hbar$, e, and c are exact, one of α, ϵ_0, or μ_0 fixes the other two. In current state-of-the-art experiments that constrain these constants the dimensionless fine-structure constant is measured or most-conveniently extracted. In fact, the CODATA adjustment uses α as an adjusted constant and values for ϵ_0 and μ_0 are derived from Eqs. 3 and 9. Finally, the mass of the electron follows from $m_{\mathrm{e}} = E_{\mathrm{h}}/\alpha^2c^2$. Its relative uncertainty is twice that of α as the relative uncertainty of the Hartree energy is much better known.

TABLE II: Selected constants as well as atomic and natural units based on the 2018 CODATA adjustment of the fundamental constants [5]. The first two columns describe the quantity and its mathematical symbol. The third and fourth columns give its numerical value and unit. For quantity X the number in parenthesis in the numerical value is the combined statistical and systematic one-standard-deviation uncertainty $u(X)$ in the last two digits of the numerical value. Finally, the last column gives the relative standard uncertainty $u_{\mathrm{r}}(X) = u(X)/|X|$. The unit u is the atomic mass unit, one twelfth of the mass of a ^{12}C atom.

Quantity	Symbol	Numerical value	Unit	Relative std. uncert. u_{r}
		General		
vacuum magnetic permeability $4\pi\alpha\hbar/e^2c$	μ_0	$1.256\,637\,062\,12(19)\times10^{-6}$	N A^{-2}	1.5×10^{-10}
$\mu_0/(4\pi\times10^{-7})$		$1.000\,000\,000\,55(15)$	N A^{-2}	1.5×10^{-10}
vacuum electric permittivity $1/\mu_0c^2$	ϵ_0	$8.854\,187\,8128(13)\times10^{-12}$	F m^{-1}	1.5×10^{-10}
fine-structure constant $e^2/4\pi\epsilon_0\hbar c$	α	$7.297\,352\,5693(11)\times10^{-3}$		1.5×10^{-10}
inverse fine-structure constant	α^{-1}	$137.035\,999\,084(21)$		1.5×10^{-10}
Rydberg frequency $\alpha^2m_{\mathrm{e}}c^2/2h = E_{\mathrm{h}}/2h$	cR_∞	$3.289\,841\,960\,2508(64)\times10^{15}$	Hz	1.9×10^{-12}
energy equivalent	$hc\,R_\infty$	$2.179\,872\,361\,1035(42)\times10^{-18}$	J	1.9×10^{-12}
		$13.605\,693\,122\,994(26)$	eV	1.9×10^{-12}
Rydberg constant	R_∞	$10\,973\,731.568\,160(21)$	[m^{-1}][1]	1.9×10^{-12}
Bohr magneton $e\hbar/2m_{\mathrm{e}}$	μ_{B}	$9.274\,010\,0783(28)\times10^{-24}$	J T^{-1}	3.0×10^{-10}
	μ_{B}/h	$1.399\,624\,493\,61(42)\times10^{10}$	Hz T^{-1}	3.0×10^{-10}
nuclear magneton $e\hbar/2m_{\mathrm{p}}$	μ_{N}	$5.050\,783\,7461(15)\times10^{-27}$	J T^{-1}	3.1×10^{-10}
	μ_{N}/h	$7.622\,593\,2291(23)$	MHz T^{-1}	3.1×10^{-10}

TABLE II: *(Continued).*

Quantity	Symbol	Numerical value	Unit	Relative std. uncert. u_r
Electron, e				
electron mass	m_e	$9.109\,383\,7015(28) \times 10^{-31}$	kg	3.0×10^{-10}
		$5.485\,799\,090\,65(16) \times 10^{-4}$	u	2.9×10^{-11}
energy equivalent	$m_\mathrm{e}c^2$	$8.187\,105\,7769(25) \times 10^{-14}$	J	3.0×10^{-10}
		$0.510\,998\,950\,00(15)$	MeV	3.0×10^{-10}
electron-muon mass ratio	m_e/m_μ	$4.836\,331\,69(11) \times 10^{-3}$		2.2×10^{-8}
reduced Compton wavelength $\hbar/m_\mathrm{e}c = \alpha a_0$	$\lambda\!\!\!^{-}_\mathrm{C}$	$3.861\,592\,6796(12) \times 10^{-13}$	m	3.0×10^{-10}
Compton wavelength	λ_C	$2.426\,310\,238\,67(73) \times 10^{-12}$	[m][1]	3.0×10^{-10}
classical electron radius $\alpha^2 a_0$	r_e	$2.817\,940\,3262(13) \times 10^{-15}$	m	4.5×10^{-10}
electron magnetic moment	μ_e	$-9.284\,764\,7043(28) \times 10^{-24}$	J T^{-1}	3.0×10^{-10}
to Bohr magneton ratio	$\mu_\mathrm{e}/\mu_\mathrm{B}$	$-1.001\,159\,652\,181\,28(18)$		1.7×10^{-13}
to nuclear magneton ratio	$\mu_\mathrm{e}/\mu_\mathrm{N}$	$-1838.281\,971\,88(11)$		6.0×10^{-11}
electron magnetic moment anomaly $\lvert\mu_\mathrm{e}\rvert/\mu_\mathrm{B} - 1$	a_e	$1.159\,652\,181\,28(18) \times 10^{-3}$		1.5×10^{-10}
electron g-factor $-2(1 + a_\mathrm{e})$	g_e	$-2.002\,319\,304\,362\,56(35)$		1.7×10^{-13}
electron-proton magnetic moment ratio	$\mu_\mathrm{e}/\mu_\mathrm{p}$	$-658.210\,687\,89(20)$		3.0×10^{-10}
Proton, p				
proton mass	m_p	$1.672\,621\,923\,69(51) \times 10^{-27}$	kg	3.1×10^{-10}
		$1.007\,276\,466\,621(53)$	u	5.3×10^{-11}
energy equivalent	$m_\mathrm{p}c^2$	$1.503\,277\,615\,98(46) \times 10^{-10}$	J	3.1×10^{-10}
		$938.272\,088\,16(29)$	MeV	3.1×10^{-10}
proton-electron mass ratio	$m_\mathrm{p}/m_\mathrm{e}$	$1836.152\,673\,43(11)$		6.0×10^{-11}
proton rms charge radius	r_p	$8.414(19) \times 10^{-16}$	m	2.2×10^{-3}
proton magnetic moment	μ_p	$1.410\,606\,797\,36(60) \times 10^{-26}$	J T^{-1}	4.2×10^{-10}
to Bohr magneton ratio	$\mu_\mathrm{p}/\mu_\mathrm{B}$	$1.521\,032\,202\,30(46) \times 10^{-3}$		3.0×10^{-10}
to nuclear magneton ratio	$\mu_\mathrm{p}/\mu_\mathrm{N}$	$2.792\,847\,344\,63(82)$		2.9×10^{-10}
proton g-factor $2\mu_\mathrm{p}/\mu_\mathrm{N}$	g_p	$5.585\,694\,6893(16)$		2.9×10^{-10}
Atomic units (a.u.)				
a.u. of charge	e	$1.602\,176\,634 \times 10^{-19}$	C	exact
a.u. of mass	m_e	$9.109\,383\,7015(28) \times 10^{-31}$	kg	3.0×10^{-10}
a.u. of action	$\hbar$	$1.054\,571\,817\ldots \times 10^{-34}$	J s	exact
a.u. of length: Bohr radius (bohr) $\hbar/\alpha m_\mathrm{e}c$	a_0	$5.291\,772\,109\,03(80) \times 10^{-11}$	m	1.5×10^{-10}
a.u. of energy: Hartree energy (hartree) $\alpha^2 m_\mathrm{e}c^2 = e^2/4\pi\epsilon_0 a_0 = 2hc\,R_\infty$	E_h	$4.359\,744\,722\,2071(85) \times 10^{-18}$	J	1.9×10^{-12}
a.u. of time	$\hbar/E_\mathrm{h}$	$2.418\,884\,326\,5857(47) \times 10^{-17}$	s	1.9×10^{-12}
a.u. of force	E_h/a_0	$8.238\,723\,4983(12) \times 10^{-8}$	N	1.5×10^{-10}
a.u. of velocity: αc	$a_0 E_\mathrm{h}/\hbar$	$2.187\,691\,263\,64(33) \times 10^{6}$	m s^{-1}	1.5×10^{-10}
a.u. of momentum	$\hbar/a_0$	$1.992\,851\,914\,10(30) \times 10^{-24}$	kg m s^{-1}	1.5×10^{-10}
a.u. of current	$eE_\mathrm{h}/\hbar$	$6.623\,618\,237\,510(13) \times 10^{-3}$	A	1.9×10^{-12}
a.u. of charge density	e/a_0^3	$1.081\,202\,384\,57(49) \times 10^{12}$	C m^{-3}	4.5×10^{-10}
a.u. of electric potential	E_h/e	$27.211\,386\,245\,988(53)$	V	1.9×10^{-12}
a.u. of electric field	E_h/ea_0	$5.142\,206\,747\,63(78) \times 10^{11}$	V m^{-1}	1.5×10^{-10}
a.u. of electric dipole moment	ea_0	$8.478\,353\,6255(13) \times 10^{-30}$	C m	1.5×10^{-10}
a.u. of electric quadrupole moment	ea_0^2	$4.486\,551\,5246(14) \times 10^{-40}$	C m^2	3.0×10^{-10}
a.u. of electric polarizability	$e^2a_0^2/E_\mathrm{h}$	$1.648\,777\,274\,36(50) \times 10^{-41}$	C^2 m^2 J^{-1}	3.0×10^{-10}
a.u. of magnetic flux density	$\hbar/ea_0^2$	$2.350\,517\,567\,58(71) \times 10^{5}$	T	3.0×10^{-10}
a.u. of magnetic dipole moment: $2\mu_\mathrm{B}$	$\hbar e/m_\mathrm{e}$	$1.854\,802\,015\,66(56) \times 10^{-23}$	J T^{-1}	3.0×10^{-10}
a.u. of magnetizability	$e^2a_0^2/m_\mathrm{e}$	$7.891\,036\,6008(48) \times 10^{-29}$	J T^{-2}	6.0×10^{-10}
a.u. of permittivity	e^2/a_0E_h	$1.112\,650\,055\,45(17) \times 10^{-10}$	F m^{-1}	1.5×10^{-10}
Natural units (n.u.)				
n.u. of velocity	c	$299\,792\,458$	m s^{-1}	exact
n.u. of action	$\hbar$	$1.054\,571\,817\ldots \times 10^{-34}$	J s	exact
		$6.582\,119\,569\ldots \times 10^{-16}$	eV s	exact
	$\hbar c$	$197.326\,980\,4\ldots$	MeV fm	exact
n.u. of mass	m_e	$9.109\,383\,7015(28) \times 10^{-31}$	kg	3.0×10^{-10}
n.u. of energy	$m_\mathrm{e}c^2$	$8.187\,105\,7769(25) \times 10^{-14}$	J	3.0×10^{-10}

TABLE II: *(Continued)*.

Quantity	Symbol	Numerical value	Unit	Relative std. uncert. u_r
		$0.510\,998\,950\,00(15)$	MeV	3.0×10^{-10}
n.u. of momentum	$m_\mathrm{e}c$	$2.730\,924\,530\,75(82) \times 10^{-22}$	kg m s^{-1}	3.0×10^{-10}
		$0.510\,998\,950\,00(15)$	MeV/c	3.0×10^{-10}
n.u. of length: $\hbar/m_\mathrm{e}c$	$ƛ_\mathrm{C}$	$3.861\,592\,6796(12) \times 10^{-13}$	m	3.0×10^{-10}
n.u. of time	$\hbar/m_\mathrm{e}c^2$	$1.288\,088\,668\,19(39) \times 10^{-21}$	s	3.0×10^{-10}

[1] The full description of m^{-1} is cycles or periods per meter and that of m is meter per cycle (m/cycle). The scientific community is aware of the implied use of these units. It traces back to the conventions for phase and angle and the use of unit Hz versus cycles/s. No solution has been agreed upon.

[1] The official description of the International System of Units can be found at the Bureau International des Poids et Mesures (BIPM) website https://www.bipm.org/en/measurement-units.

[2] J. D. Jackson, *Classical electrodynamics*, (John Wiley & Sons, Inc., New York-London, 1962).

[3] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions* (John Wiley & Sons, Inc., New York-London, 1992).

[4] A transverse vector field $\vec{F}_\perp(\vec{r})$ in position space $\vec{r}$ is a field such that its Fourier representation $\vec{\mathcal{F}}_\perp(\vec{k})$ is perpendicular to $\vec{k}$ for all wavevectors $\vec{k}$. Magnetic fields are transverse vector fields in any gauge and subscript $\perp$ is often dropped.

[5] CODATA Internationally recommended values of the Fundamental Physical Constants found at https://physics.nist.gov/constants.