

Accurate Monte Carlo Uncertainty Analysis for Multiple Measurements of Microwave Systems

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Abstract—Uncertainty analysis of microwave electronic measurements enables the quantification of device performance and aides in the development of robust technology. The Monte Carlo method is commonly used to attain accurate uncertainty analyses for complicated nonlinear systems. Combining multiple similar measurements, each with a Monte Carlo uncertainty analysis, allows one to incorporate the uncertainty given by their spread. In this paper, we compare two Monte Carlo sampling methods, illustrate that one method reduces the bias of averaged quantities, show how this impacts computed uncertainties, and highlight microwave applications for which this corrected method can be applied.

Keywords—Measurement uncertainty, Monte Carlo methods, Microwave technology.

I. INTRODUCTION

Uncertainty analysis quantifies the precision of measurements and facilitates accurate characterization of systems; Here we focus on applications to microwave engineering. The Monte Carlo method is a key component of uncertainty analysis [1] and is primarily used to propagate uncertainties through nonlinear transformations. It has been used by several national metrology institutes [2], [3] to accurately determine uncertainty in high-frequency electronic measurements and systems. At the Communications Technology Lab at the National Institute of Standards and Technology (NIST) we perform Monte Carlo analyses for many of our measurements including VNA calibrations [4], electro-optic sampling [5], as well as derived communication metrics including error vector magnitude (EVM) [6].

Repeating equivalent measurements gives insight into variability, noise, etc., allowing a more complete characterization of the measurement setup and environment. Incorporating the spread of these measurements in uncertainty analyses often increases the estimate of uncertainty in the measurement. This can account for differing or changing environments (including temperature or humidity fluctuations), lack of precise control in the measurement (including connection repeatability [7], cable bending [6], probe placement or other spatial positioning), reproducibility using different components [8] or other unknown random processes (*e.g.* unknown radio-frequency interference in the laboratory). Thus it is often of interest to combine multiple measurements, each with their own errors, to determine the mean measurement response and its associated variability.

For Monte Carlo uncertainty analyses, this requires combining multiple Monte Carlo samples, one for each measurement, to calculate an estimate of the mean response and its variance as represented by a further Monte Carlo sample. Frey *et al.* [9] analyzed two Monte Carlo sampling methods applied to this situation, determined that one of the variants is typically more accurate, and recommended its use in the NIST Microwave Uncertainty Framework (MUF), a software tool for representing, propagating, and reporting uncertainty in microwave measurement systems.

In this paper we illustrate the trade-offs of these Monte Carlo sampling techniques, verify their implementation in the MUF, and discuss the relevance to microwave systems. We begin with a review of covariance-based uncertainty analysis and the Monte Carlo method.

II. UNCERTAINTY ANALYSIS

Covariance-based uncertainty analysis [10] preserves correlations across records (*e.g.* time records or frequency data) through transformations. This allows the accurate propagation of uncertainties in measurements through to derived data, including corrected device responses, modulated signals, or system-level metrics. Linear transformations, such as the Fourier transform, can use sensitivity analysis [11] to propagate these uncertainties. However, sensitivity analysis assumes that the transformation is linear and reduces to a local approximation for nonlinear transformations.

A. The Monte Carlo Method

The typical method for propagating uncertainties through nonlinear transformations is the Monte Carlo method. This method tracks the statistical biases introduced by nonlinearities, which are not captured by a sensitivity analysis. These biases can include differences of the center (mean), spread (variance) and shape of the probability distribution of the transformed data.

Here we represent a quantity with known uncertainty as a random variable S which is characterized by a nominal value, s_{nom} , and a probability distribution. The Monte Carlo method draws random realizations of S from the probability distribution, attaining a sample of size Q as $\{s_q\}_{q=1}^Q$.

Repeated measurements of a quantity can give an estimate of its uncertainty. Taking multiple, J , measurements of a

random variable X gives a sample $\{x_j\}_{j=1}^J$ of its distribution. These data can be propagated through transformations to obtain uncertainty in derived measurements. One relevant transformation for high-frequency measurements takes a form similar to

$$Y = F(X, S) \quad (1)$$

which maps the random variables X and S into another random variable Y . Equations of the form Eq. (1) can represent a correction or calibration of measured data X using a standard S which is only known up to some uncertainty. The Monte Carlo method maps the nominal values

$$y_{j,\text{nom}} = F(x_j, s_{\text{nom}}) \quad (2)$$

and Monte Carlo samples

$$y_{j,q} = F(x_j, s_q) \quad (3)$$

for $j = 1, \dots, J$ and $q = 1, \dots, Q$ through this transformation. Thus, for each j we have a complete Monte Carlo uncertainty analysis of y_j : a nominal value $y_{j,\text{nom}}$ and a Monte Carlo sample $\{y_{j,q}\}_{q=1}^Q$. The y_j incorporate the uncertainty due to the uncertainty in S , however the spread of the x_j has not been included in these individual quantities.

In order to calculate the mean and the uncertainty of the mean of the y_j , we need to combine the y_j to capture the uncertainty due to the spread of the x_j . That is, we want to combine the multiple Monte Carlo samples to produce a new Monte Carlo sample which estimates the mean of these data and the uncertainty in the mean. However, as the random variable S is shared across all of the J measurements the values s_q are perfectly correlated across all of the $y_{j,q}$. In combining these data, care must be taken to preserve this correlation.

B. The Microwave Uncertainty Framework

The NIST MUF is a tool that performs covariance-based and Monte Carlo uncertainty analyses for high-frequency electronic measurements. It supports many operations used in microwave and millimeter wave applications including calibrations, de-embedding, transformations between the time and frequency domains, and the evaluation of system metrics such as EVM. The MUF propagates uncertainties in parameters through such operations obtaining uncertainty in derived quantities. The specified uncertainty in the parameters is typically given by an *a priori* model of the uncertainty in these values. Examples for which a model can be provided include uncertainties in the length of a transmission line or in the alignment of a waveguide. However, there are some uncertainty mechanisms which are difficult or impossible to model. Often these uncertainties are characterized using multiple measurements. Therefore, the process of estimating uncertainties from multiple measurements is an essential issue for users of the MUF and is an important feature that should be handled accurately.

III. MONTE CARLO SAMPLING METHODS

Frey *et al.* [9] presented two procedures for combining multiple Monte Carlo samples to estimate the uncertainty of the mean of multiple measurements with uncertainty while preserving these correlations. The first method corresponds to the original MUF formulation, denoted here as ‘‘Original’’, which estimates the variance of the multiple measurements using their nominal values, the $y_{j,\text{nom}}$. This intuitive approach uses the spread of measurements as transformed using the nominal value of any quantities with uncertainty (the s_{nom} values) as in Eq. (2). The other method, denoted here as the ‘‘Alternative’’ method, estimates this variance using the averages of the J Monte Carlo samples,

$$\bar{y}_{j,\cdot} = \frac{1}{Q} \sum_{q=1}^Q y_{j,q}.$$

In [9] it was shown that both methods produce Monte Carlo samples with unbiased means but biased variances. Monte Carlo samples with an unbiased mean will have an average value that, when averaged over many equivalent experiments, will converge to the true value. The biased variance of the Monte Carlo sample implies that the expected value of the variance of Monte Carlo samples will not approach the true variance. That is, these Monte Carlo samples will not yield the correct value of uncertainty even when averaged over many equivalent experiments. The bias of the Original method was shown to be Ψ/J while the bias of the Alternative method is $\Phi/(JQ)$ where both Ψ and Φ depend upon the function F as well as the distributions of X and S (for more details please see [9]). Although there are trade-offs between these approaches, an analysis of these biases, the variability of the underlying statistic, and the order of the parameters J and Q in [9] yielded a recommendation to use the Alternative method to combine the Monte Carlo samples.

IV. EXAMPLES

We apply these two methods to two examples where we can analytically calculate the variance and corresponding bias terms. Using these closed-form expressions, we analyze the performance of each Monte Carlo sampling method. Each of these examples, like the examples in [9], amount to taking J independent samples of X , Q samples of the systematic error S that are shared across all J samples, and propagating these samples through a nonlinear function, as in Eq. (3).

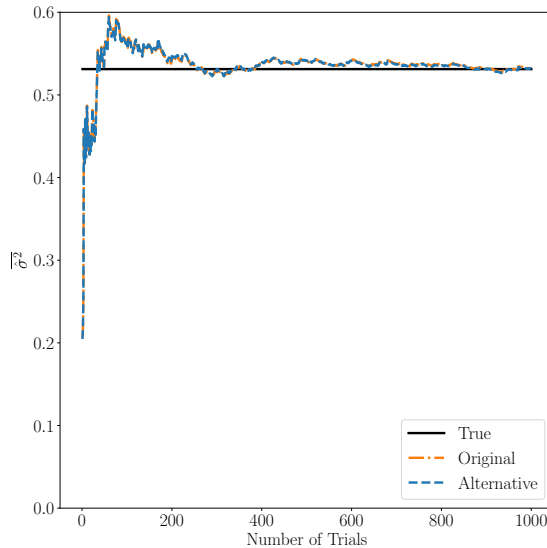
These examples each take $X_j \sim N(\mu, \sigma_x^2)$ and $S_q \sim N(0, \sigma_s^2)$ and only differ in the nonlinear functions. The first example is a quadratic

$$F(X_j, S_q) = (X_j + S_q)^2 \quad (4)$$

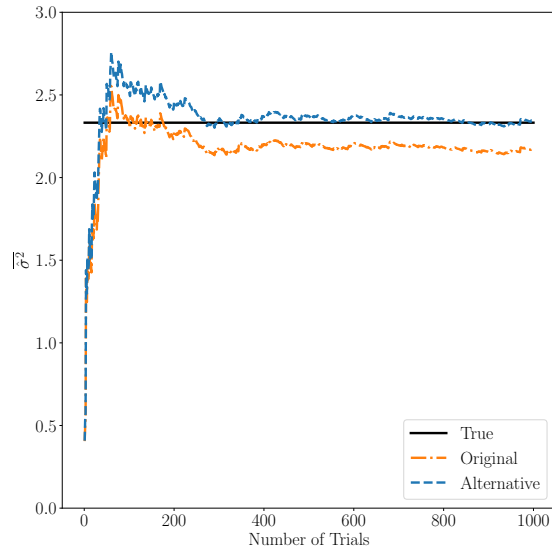
while the second example is a cubic

$$F(X_j, S_q) = (X_j + S_q)^3. \quad (5)$$

These simple polynomial examples allow us to calculate the true mean and variance of the data and also derive closed-form expressions for the the biases (ψ/J and $\phi/(JQ)$) as in [9] for



(a) Quadratic Example



(b) Cubic Example

Fig. 1. The sample variance cumulatively averaged over trials for each of the sampling methods applied to the quadratic, Eq. (4), and cubic, Eq. (5), examples. The Original and Alternative methods produced little or no bias for the quadratic example (a). The Original method had a pronounced bias for the cubic example (b), while the Alternative method showed little bias.

Table 1. Analytically calculated values of the mean, variance and the biases of each method for the quadratic and cubic examples.

	Mean	Variance	Original Bias	Alternative Bias
Quadratic	0.7500	0.5313	0.0000	6.250×10^{-4}
Cubic	0.8750	2.332	-0.1446	6.016×10^{-3}

each method. We use these analytic expressions to examine the effect of these biases for the examples and use these relations to verify the numerical implementation in the MUF.

A. Analytic Results

For each example, Eqs. (4) and (5), we choose $\mu = 0.5$, $\sigma_x = 0.5$, $\sigma_s = 0.5$, $J = 4$, and $Q = 100^\ddagger$, evaluate the closed-form expressions for these parameters, and tabulate the analytically computed values of the true mean and variance as well as the biases of the variance estimates for each sampling method in Table 1. Here we see that the Original method produces no bias of the variance in the quadratic example but a significant bias in cubic example while the Alternative method produces relative biases of $\approx .1\%$ for both cases.

B. Microwave Uncertainty Framework Numerical Results

We now turn to numerical experiments to verify the implementation in the MUF. Using the parameters above (for each example and sampling method) we run 1,000 independent trials and compute the sample mean and variance of the resulting Monte Carlo samples of each trial. Figure 1 shows the value of the sample variance of the Monte Carlo samples produced by the Original and Alternative methods cumulatively averaged over trials — we see that this statistic

[‡]We chose $Q = 100$ for computational convenience and this was sufficient for illustration purposes for these simple polynomial examples. In general, we recommend using 10,000 Monte Carlo samples or more.

Table 2. The averaged mean and variance along with the variance of these quantities as computed from 1,000 simulated Monte Carlo trials.

		Mean	Var.	Var. of Mean	Var. of Var.
Quadratic	Original	0.7515	0.5331	0.3193	0.3892
	Alternative	0.7518	0.5324	0.3194	0.3902
Cubic	Original	0.8755	2.170	0.6646	2.296
	Alternative	0.8745	2.342	0.6661	2.508

begins to converge around 400 trials. Both methods approach the true variance for the quadratic case in Fig. 1a, while the variance of the Original method is clearly biased for the cubic example in Fig. 1b.

Histograms of the sample variance, along with the true and averaged values, are shown in Fig. 2. Here we see the center of this statistic (mean of the variance) as well as the spread (variance of the variance) and we tabulate these values in Table 2 along with the corresponding values of the mean.

V. DISCUSSION

Table 2 shows that both methods produce Monte Carlo samples whose mean on average had little deviation from the analytic value. For the quadratic example, the Original method produced Monte Carlo samples whose average variance produced a value close to the analytically calculated value verifying that there is no significant bias in this case. Similarly, the Alternative scheme produced an average variance very close to the predicted value in agreement with the calculated bias shown in Table 1. However, in the cubic case the averaged variance produced by the Original method differed from the true value by -0.1620 which is very close to the analytically calculated bias -0.1446 (from Table 1). The Alternative method differed by 0.010 a bias an order of magnitude smaller than the Original method.

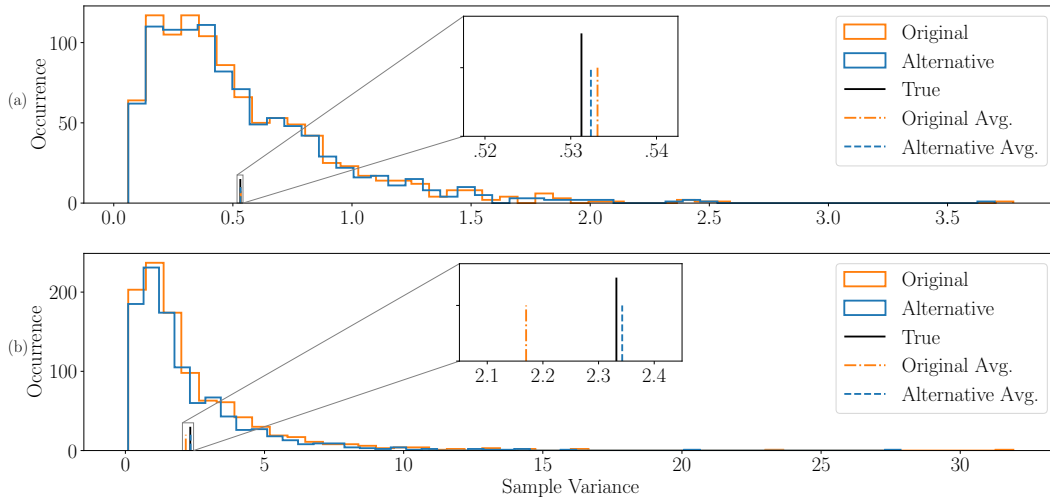


Fig. 2. Histograms of the sample variance using 1,000 trials for both sampling methods. The inset images show small or no significant biases for the quadratic example (a) but a significant bias for the Original method in the cubic example (b).

These results support the intuition for choosing the Alternative sampling method over the Original. Although the Original method is unbiased for some specific examples, such as the quadratic example above, it can produce significant bias in relevant transformations. The Alternative method is biased for both of our examples but, as this bias decreases with Monte Carlo sample size Q , this bias can be quite small even for the limited Monte Carlo sample size used here. Using the Monte Carlo averaged values to calculate the variance, as in the Alternative method, can yield more accurate uncertainty analyses when applying nonlinear transformations. As many microwave systems employ such nonlinear calculations (EVM, bit error rate), this more accurate method will improve the uncertainty analysis of such systems.

This numerical study also verified that the MUF implementation of the Original and Alternative methods agree with the analytically calculated results derived from [9].

VI. CONCLUSION

We applied the two Monte Carlo sampling methods outlined in [9] to two polynomial examples. These simple examples allowed us to derive closed-form expressions for the true variance and the bias of each method, examine the effect of bias in these methods, and verify the implementations in the MUF. The Original method, which estimates the variance of multiple measurements using the corresponding nominal values, can produce significant bias in the estimate of variance. The Alternative method, which estimates the variance of multiple measurements using the averages of the Monte Carlo samples, produces small biases for these examples, even for the small Monte Carlo sample size considered here. As the Alternative sampling method produces more accurate results for these examples and the numerical simulations verified this implementation in the MUF, the MUF will use the Alternative

sampling method moving forward, providing more accurate uncertainty analyses for microwave measurement systems.

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