A New Perspective on an Old Problem: Scattering by a Perfect Electric Conductor

Alex J. Yuffa*, Johannes Markkanen†, Qiang Sun‡, Evert Klaseboe§, and Derek Y. C. Chan‡¶

* National Institute of Standards and Technology, 325 Broadway, Boulder, Colorado 80305, USA

alex.yuffa@nist.gov

† Max Planck Institute for Solar System Research, Justus-von-Liebi-Weg 3, 37077 Göttingen, Germany

‡ University of Melbourne, Parkville 3010 VIC, Australia

§ Institute of High Performance Computing, 1 Fusionopolis Way, Singapore 138632, Singapore

¶ Swinburne University of Technology, Hawthorn 3122 VIC, Australia

***Abstract*—We explore several different singular surface integral equation formulations. These formulations are specifically designed for time-harmonic scattering by a perfect electric conductor and are obtained by choosing non-conventional boundary unknowns.**

***Index Terms*—Boundary element method, electromagnetic scattering, surface integral equations.**

I. Introduction

Electromagnetic scattering by a perfect electric conductor (PEC) is a century old problem. This scattering problem is illustrated in Fig. 1, where $\boldsymbol{N}$ denotes the outward unit normal to the surface of scatterer denoted by Σ. It is well-known that the total electric field satisfies

$$\boldsymbol{N} \times \boldsymbol{E} = 0 \quad \text{on} \quad \Sigma \tag{1}$$

and the total magnetic field satisfies $\boldsymbol{N} \cdot \boldsymbol{H} = 0$ on Σ. In a typical Stratton–Chu integral equation formalism of the above scattering problem [1] the induced surface current density $\boldsymbol{N} \times \boldsymbol{H}$ is related to the induced surface charge density $\boldsymbol{N} \cdot \boldsymbol{E}$ via the continuity equation. This approach yields an integral equation where the surface current density is chosen as the boundary unknown. In this contribution, we explore integral equation formulations with other boundary unknown(s).

Throughout the paper, we assume that all fields are harmonic in time with a suppressed $\exp(-\mathrm{i}\omega t)$ time factor. Furthermore, we use tensor notation with the Einstein summation convention and assume that the incident E-field $\overset{\text{inc}}{\boldsymbol{E}}$ satisfies the vector Helmholtz equation.

II. Integral Equation Formalism

Let the coordinates of a point in the three-dimensional Euclidean space be denoted by Z^i or simply by Z. The covariant ambient basis are then derived from the position vector $\boldsymbol{R}$ via

$$\boldsymbol{Z}_i = \frac{\partial}{\partial Z^i}\boldsymbol{R}(Z) \tag{2}$$

and the covariant metric tensor is given by $Z_{ij} = \boldsymbol{Z}_i \cdot \boldsymbol{Z}_j$.

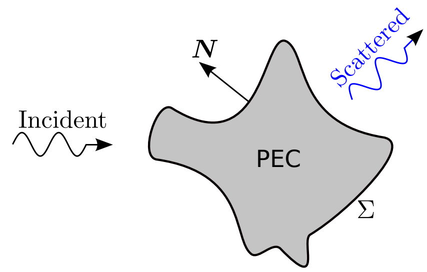

Fig. 1. (Color online) A typical scattering geometry is shown. The scatterer is bounded by the surface Σ with the unit normal vector $\boldsymbol{N}$.

It is well-known that the E-field in a source-free region is divergenceless and satisfies the vector Helmholtz equation. In the tensor notation, the Laplacian may be written as $\nabla^j\nabla_j$ and thus, the vector Helmholtz equation is written as

$$\left(\nabla^j\nabla_j + k^2\right)\boldsymbol{E} = \boldsymbol{0}, \tag{3}$$

where $\nabla^j = Z^{jk}\nabla_k$ and ∇_k denotes the covariant derivative. In order to derive an integral representation of the E-field, we introduce a free-space Green's function G. The Green's function satisfies

$$(\nabla^i\nabla_i + k^2)G = -\delta(\widetilde{\boldsymbol{P}} - \boldsymbol{P}), \tag{4}$$

where $\delta(\widetilde{\boldsymbol{P}} - \boldsymbol{P})$ denotes the Dirac delta function and the position vectors of a source point and a field point are given by $\boldsymbol{P} = \boldsymbol{R}(Z)$ and $\widetilde{\boldsymbol{P}} = \boldsymbol{R}(\widetilde{Z})$, respectively. To obtain an integral representation of the E-field, we multiply (3) by $G(\widetilde{\boldsymbol{P}}, \boldsymbol{P})$ and (4) by $\boldsymbol{E}(Z)$, then take the difference between the two equations. After integrating the resultant equation via Gauss's theorem, we obtain

$$\overset{\text{inc}}{\boldsymbol{E}}(\widetilde{Z}) - \int_\Sigma \left[G\frac{\partial \boldsymbol{E}}{\partial N} - \boldsymbol{E}\frac{\partial G}{\partial N}\right]\mathrm{d}S = \boldsymbol{E}(\widetilde{Z}), \quad \widetilde{Z} \notin \text{PEC}, \tag{5}$$

where $\partial/\partial N = N^i\nabla_i$ denotes the normal derivative with respect to the source coordinates. Finally, taking the limit as $\widetilde{Z}$ approaches the surface and accounting for the Green's function singular nature [2], we obtain

$$\overset{\text{inc}}{\boldsymbol{E}} - \fint_{\Sigma} \left[G \frac{\partial \boldsymbol{E}}{\partial N} - (\boldsymbol{N} \cdot \boldsymbol{E}) \boldsymbol{N} \frac{\partial G}{\partial N} \right] \mathrm{d}S = \frac{1}{2} (\boldsymbol{N} \cdot \boldsymbol{E}) \boldsymbol{N}, \quad (6)$$

where $\fint$ denotes the Cauchy principal value integral and the tangential components of the E-field vanish because of the boundary condition (1). Notice that (6) contains *three* scalar equations. Thus, if we choose $\boldsymbol{N} \cdot \boldsymbol{E}$ (surface charge) and $\partial \boldsymbol{E} / \partial N$ as the boundary unknowns, then we will have *more* unknowns than equations. To remedy this situation, either the number of equations must be increased by one or the number of unknowns must be decreased by one. We discuss both of these approaches below.

A. Increasing the Number of Equations

We can increase the number of equations by noticing that if the E-field is divergenceless, then $\boldsymbol{R} \cdot \boldsymbol{E}$ satisfies the *scalar* Helmholtz equation [3], [4]. Thus, after applying Green's second identity and taking the limit as $\widetilde{Z}$ approaches the surface, we obtain

$$\boldsymbol{R} \cdot \overset{\text{inc}}{\boldsymbol{E}} - \fint_{\Sigma} \left[G \frac{\partial (\boldsymbol{R} \cdot \boldsymbol{E})}{\partial N} - (\boldsymbol{R} \cdot \boldsymbol{E}) \frac{\partial G}{\partial N} \right] \mathrm{d}S = \frac{1}{2} (\boldsymbol{R} \cdot \boldsymbol{E}) . \quad (7)$$

The normal derivative of $\boldsymbol{R} \cdot \boldsymbol{E}$ can be related to the surface charge and the normal derivative of the E-field; namely,

$$\begin{aligned} N^i \nabla_i (\boldsymbol{R} \cdot \boldsymbol{E}) &= N^i (\nabla_i \boldsymbol{R}) \cdot \boldsymbol{E} + \boldsymbol{R} \cdot \left(N^i \nabla_i \boldsymbol{E} \right) \\ &= \boldsymbol{N} \cdot \boldsymbol{E} + \boldsymbol{R} \cdot \frac{\partial \boldsymbol{E}}{\partial N}. \end{aligned} \quad (8)$$

Substituting (8) into (7) and using (1) yields the desired additional equation; namely,

$$\begin{aligned} \fint_{\Sigma} \left[G \left(\boldsymbol{N} \cdot \boldsymbol{E} + \boldsymbol{R} \cdot \frac{\partial \boldsymbol{E}}{\partial N} \right) - (\boldsymbol{R} \cdot \boldsymbol{N}) (\boldsymbol{N} \cdot \boldsymbol{E}) \frac{\partial G}{\partial N} \right] \mathrm{d}S \\ = \boldsymbol{R} \cdot \overset{\text{inc}}{\boldsymbol{E}} - \frac{1}{2} (\boldsymbol{R} \cdot \boldsymbol{N}) (\boldsymbol{N} \cdot \boldsymbol{E}) . \end{aligned} \quad (9)$$

Notice that (II-A) contains *one* scalar equation and thus, together with (6), we have *four* scalar equations that we can solve for the *four* scalar unknowns; namely, $\boldsymbol{N} \cdot \boldsymbol{E}$ and $\partial \boldsymbol{E} / \partial N$.

The above coordinate invariant singular integral equation formulation is analogous to the formulation presented in [3], [4]. However, in [3], [4] the singular behavior of the Green's function was removed via a clever subtraction of an auxiliary function.

B. Decreasing the Number of Unknowns

We can decrease the number of unknowns by relating the normal component of the normal derivative of the E-field to the surface charge on Σ. The desired relationship is

$$\boldsymbol{N} \cdot \frac{\partial \boldsymbol{E}}{\partial N} = (\boldsymbol{N} \cdot \boldsymbol{E}) W, \quad (10)$$

where W is the mean curvature. The derivation of (10) is built upon the work presented in [5], [6] and is outside of the scope of this paper. To apply (10) to (6), we decompose the normal derivative of the E-field into normal and tangential parts, i.e.,

$$\frac{\partial \boldsymbol{E}}{\partial N} = \left(\boldsymbol{N} \cdot \frac{\partial \boldsymbol{E}}{\partial N} \right) \boldsymbol{N} + \left(\boldsymbol{S}^{\alpha} \cdot \frac{\partial \boldsymbol{E}}{\partial N} \right) \boldsymbol{S}_{\alpha}, \quad (11)$$

where $\boldsymbol{S}_{\alpha=1,2}$ are the surface covariant basis, i.e., two vectors tangential to Σ. Putting (10) into (11) and the resultant into (6) yields

$$\begin{aligned} \fint_{\Sigma} \left\{ \left[WG - \frac{\partial G}{\partial N} \right] (\boldsymbol{N} \cdot \boldsymbol{E}) \boldsymbol{N} + G \left(\boldsymbol{S}^{\alpha} \cdot \frac{\partial \boldsymbol{E}}{\partial N} \right) \boldsymbol{S}_{\alpha} \right\} \mathrm{d}S \\ = \overset{\text{inc}}{\boldsymbol{E}} - \frac{1}{2} (\boldsymbol{N} \cdot \boldsymbol{E}) \boldsymbol{N}. \end{aligned} \quad (12)$$

Notice that (II-B) contains *three* scalar equations and *three* scalar unknowns; namely, the surface charge $\boldsymbol{N} \cdot \boldsymbol{E}$ and the tangential components of the normal derivative of the E-field, $(\boldsymbol{S}^{\alpha} \cdot \frac{\partial \boldsymbol{E}}{\partial N})_{\alpha=1,2}$. Thus, we can obtain the boundary unknowns directly from (II-B).

If the mean curvature is different from zero, i.e., $W \neq 0$, then another formulation is possible. Putting (10) directly into (6) yields

$$\begin{aligned} \fint_{\Sigma} \left\{ \left[G - \frac{1}{W} \frac{\partial G}{\partial N} \right] \left(\boldsymbol{N} \cdot \frac{\partial \boldsymbol{E}}{\partial N} \right) \boldsymbol{N} + G \left(\boldsymbol{S}^{\alpha} \cdot \frac{\partial \boldsymbol{E}}{\partial N} \right) \boldsymbol{S}_{\alpha} \right\} \mathrm{d}S \\ = \overset{\text{inc}}{\boldsymbol{E}} - \frac{1}{2W} \left(\boldsymbol{N} \cdot \frac{\partial \boldsymbol{E}}{\partial N} \right) \boldsymbol{N}. \end{aligned} \quad (13)$$

In (II-B), we can choose the normal and tangential components of the normal derivative of the E-field as the boundary unknowns. In other words, after discretization we can numerically solve (II-B) for $\boldsymbol{N} \cdot \partial \boldsymbol{E} / \partial N$ and $(\boldsymbol{S}^{\alpha} \cdot \partial \boldsymbol{E} / \partial N)_{\alpha=1,2}$ via well-known methods such as the Galerkin's method [7] or the Nyström method [8].

III. Conclusions

We derived three alternative singular surface integral equation formulations for electromagnetic scattering by a perfect electric conductor. These formulations were obtained by choosing non-conventional boundary unknowns. We showed that the number of the scalar integral equations needed in each formulation depends on the choice of the boundary unknowns. These formulations show an interesting interplay between the normal component of the E-field and its the normal derivative.

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