Analytical Modeling of White Space Utilization for a Dynamic Spectrum Access System

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***Abstract*—Dynamic Spectrum Access (DSA) promises to be a shared spectrum technology that can alleviate the artificial spectrum crunch created by the static allocation of spectrum. There have been many studies on DSA systems in the literature. However, most of them are analyzed using simulation studies rather than analytical modeling. Analytical models are useful in evaluating performance of such systems quickly and easily. In this paper, we present an analytical model of an opportunistic DSA system. Using an alternating renewal process to represent primary user traffic of the DSA system and applying theory from survival analysis and stochastic process, we derive an expression to compute the white space utilization (WSU) of a DSA system for a general distribution of idle durations of primary traffic. Taking the exponential distribution as an example, we validate our analytical model by comparing its results to results obtained from two simulation experiments. One experiment uses idle durations generated from pseudorandom variates and the other uses data collected from a real Long-Term Evolution (LTE) system whose idle duration distribution is approximately exponential. Our analytical WSU results match closely with those from the first experiment and match reasonably well with those obtained from the second experiment.**

I. Introduction

Most of the spectrum in the sub 6 GHz frequency has been allocated to incumbents. However, the utilization of spectrum in some of these bands is low. Thus, the static allocation of spectrum has led to inefficient use and artificial shortage of spectrum in the sub 6 GHz band. Dynamic Spectrum Access (DSA) promises to be a shared spectrum paradigm that can make the spectrum usage more efficient and alleviate the artificial shortage. In a DSA system, there are two types of users. The incumbent is the Primary User (PU) of the system and has the higher priority. Secondary Users (SUs) have lower priority and access the channel opportunistically when it is not being used by the PUs. However, the SUs have to vacate the channel as soon as a PU starts to use the channel. There are quite a few methods proposed in the literature to provide access to the channel by an SU [1], [2], [3], [4]. One such method was proposed in [5]. In this scheme, given an SU request for transmission for a duration τ that arrives after an idle period of duration t has elapsed, the DSA system accepts the request if the probability that the idle period will last for another τ units of time is above a predetermined threshold. It uses the theory of *survival analysis* to make the decision whether to accept or reject an SU request. One of the important performance metrics of a DSA system is the proportion of idle duration that is utilized by the SUs commonly referred to as *White Space Utilization* (WSU). In the study presented in [5] we used simulation experiments to calculate WSU of the DSA system based on Long-Term Evolution (LTE) as the PU system.

A. Motivation

Many previous works, including ours [5], have reported performance of their respective DSA system through simulation studies. While those studies have been helpful in understanding different DSA systems and their performance, they lack analytical modeling. In the absence of analytical models, if a DSA service provider wants to evaluate the performance of its system in a given PU system without installing it, then it has to resort to simulation. A DSA service provider, for example, may want to estimate the WSU achievable when operating in a PU system. This information may be useful to the service provider for making various business decisions. If an analytical model is available, then a DSA service provider can quickly and easily get the DSA system performance numbers (e.g., WSU). If a provider has to compare performance of its DSA systems across multiple PU system locations, then obviously using an analytical model can save cost.

Hence, in this work we present an analytical model for the opportunistic DSA scheme proposed in [5], which is based on survival analysis. Our analytical model uses an alternating renewal process to model the PU traffic and uses theory and methods from survival analysis and stochastic processes to come up with WSU for the SUs. So, this can be a very useful tool for the DSA service providers to compare the WSU of SUs operating in different PU systems and pick the best PU system in which to deploy the DSA system. For a given PU system, if there are different traffic patterns at different times of the day, then our analytical model can be used to find out what time of the day gives the best performance in terms of WSU and hence the SU traffic can be accordingly controlled.

II. Related Work

A definition of channel occupancy and methods for measuring it have been studied in [6]. Spectrum occupancy models have been studied in the literature for quite some time. Spectrum occupancy has been modeled using a two state Discrete-Time Markov Chain (DTMC) in [7]. Time-inhomogeneous DTMC models have been used [7] in place of stationary DTMC models to overcome some of the limitation of stationary DTMC. In [8], the authors used a semi-Markov model to represent spectrum occupancy and model idle and busy periods using a general distribution. Some works in the literature have represented spectrum occupancy as an Alternating Renewal Process [8], [4], since it has only two states. Spectrum idle and busy periods have also been modeled using Continuous-Time Markov Chain (CTMC). The studies reported in [9], [10], [11] used semi-Markov CTMC to account for non-exponential distribution of ON and OFF periods of spectrum. A Two-dimensional Markov chain has been used to model adjacent channel occupancy in [12], [13].

A prediction scheme for SU spectrum access based on the expected remaining idle time of spectrum is proposed in [2]. Zhao et al. have proposed a Partially-Observable Markov Decision Process (POMDP) based model for SU spectrum access in [1]. In [8], the authors use a two-state semi-Markov process to model PU channel occupancy and try to maximize spectrum opportunities by sensing period adaptation and minimize the delay in finding an available channel. The authors in [3] have proposed a spectrum access scheme for SUs which limits the maximum bound on probability of interference to the primary user (PU). A SU spectrum access scheme based on the residual idle time distribution of PU traffic which is modeled as an Alternating Renewal Process is presented in [4]. In [5], [14], authors proposed a DSA scheme based on survival analysis. The scheme uses a non-parametric estimate of cumulative hazard function to predict remaining idle time. An SU is then given access to the spectrum based on this prediction, subject to the constraint that the probability of the SU successfully finishing the transmission is above a preset threshold. Opportunistic channel access schemes based on the Restless Multiarm Bandit model have been proposed in [15], [16]. Pattern mining of spectrum occupancy data has also been used to provide spectrum opportunity to SUs [17], [18].

III. Analytical Model

A. PU Traffic Process

We assume that there are two categories of users accessing a single communications channel. A user may be either a primary user (PU) or a secondary user (SU). PUs are the incumbent of the channel and have higher priority. The SUs have lower priority and access the channel opportunistically when it is not being used by the PUs. Consider a period of time, $[0, T]$, over which the channel is observed for PU traffic. The channel starts out in an idle state and alternates between idle and busy periods. We record all the successive lengths of idle and busy periods as $\mathbf{I} = (I_1, I_2, \ldots, I_n)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_n)$, respectively. It is assumed that the random vectors $(I_j, B_j), j = 1, 2, \ldots$ are independent with the same joint distribution, having marginal distribution functions $F(x)$ and $G(x)$, respectively and with means μ_{idle} and μ_{busy}.

This PU traffic model generates an alternating renewal process [19]. Let random variables $Y_j = I_j + B_j$ represent interarrival times between renewals (consecutive idle and busy periods). Let $S_n = Y_1 + Y_2 + \cdots + Y_n$ denote the length of time for n renewals to occur. Note that each renewal ends in a busy period. Let $F_Y(t) = F * G(t)$ be the convolution distribution of Y_j. The number of renewals in the time interval $(0, t]$ is given by

$$M_t = \max\{n : S_n \leq t\} \quad t \geq 0 \tag{1}$$

The process $M_t, t \geq 0$ is called an renewal process. The current life of the renewal system at time t, $\gamma_t = t - S_{M_t}$, is the elapsed time since the last busy period. Except in the case of a few distributions, e.g., the exponential, no closed form expression for the distribution $C_t(s)$ of γ_t exists. In the limit, (see [20]) as $t \to \infty$

$$P[\gamma_t > s] \to \frac{1}{\mu_{\text{idle}} + \mu_{\text{busy}}} \int_s^\infty \bar{F}_Y(y)dy \tag{2}$$

where $\bar{F}_Y(s) = 1 - F_Y(s)$. This asymptotic distribution is often used to approximate $\bar{C}_t(s)$. For the special case of exponential I_j and B_j, the current life has distribution function [20]

$$C_t(s) = \begin{cases} 1 - e^{-(\mu_{\text{idle}} + \mu_{\text{busy}})s} & 0 \leq s \leq t \\ 1 & s > t \end{cases}$$

B. SU Channel Access Scheme

Using the above PU traffic model, an SU channel access scheme was presented in [5] that is based on *survival analysis*. When an SU requests τ units of time to transmit during an idle period and the channel has been observed idle for $s \geq 0$ units of time, then the decision to allow the SU to transmit is based on the hazard function of $F(s)$. By definition, the hazard function $h(s)$, measures how likely an idle period of unknown length I will end in the next instance, given it has lasted for s units of time and is given by

$$h(s) = \lim_{ds \to 0} \frac{Pr[s \leq I < s + ds \mid I \geq s]}{ds} = \frac{f(s)}{1 - F(s)} \tag{3}$$

where $f(s) = dF(s)/ds$. The SU is allowed to transmit if the probability that the current idle period I will last for additional duration τ given that it has been idle for duration s (when the SU request arrived) is more than a given threshold p. Thus, the SU is allowed to transmit if the following condition is satisfied.

$$Pr[I \geq s + \tau \mid I \geq s] > p \tag{4}$$

This threshold $p, 0 < p < 1$, is the probability of successful transmission by the SU. It is shown in [5] that

$$P[I \geq s + \tau \mid I \geq s] = \exp(-[H(s + \tau) - H(s)]) \tag{5}$$

where $H(s) = \int_0^s h(t)dt, s \geq 0$ is the cumulative hazard function. Using (4) and (5), it can easily be deduced that an SU is allowed to transmit if the change in the cumulative hazard function over the time period $[s, s + \tau]$, $H(s + \tau) - H(s)$, is below a certain value $\theta = (-\ln p)$, i.e., $H(s + \tau) - H(s) < \theta$. Thus, an SU request at time t is evaluated using the following criteria,

$$\text{Request is} \begin{cases} \text{Denied} & \text{if channel is busy} \\ \text{Granted} & \text{if channel is idle and } H(\gamma_t + \tau) - H(\gamma_t) < \theta \\ \text{Denied} & \text{if channel is idle and } H(\gamma_t + \tau) - H(\gamma_t) \geq \theta \end{cases} \tag{6}$$

In practice, the cumulative hazard function $H(\cdot)$ is estimated from an observed sample of idle time lengths $I_j, j = 1, \ldots, n$. The cumulative hazard function is estimated by

$$H_n(s) = \sum_{j: I_{(j)} \leq s} \frac{1}{n - j + 1} \tag{7}$$

where $I_{(1)} \leq I_{(2)} \leq \cdots \leq I_{(n)}$ are the ordered $I_j, j = 1, \ldots, n$. Since $H_n(s) \to H(s)$, for large n, this estimate of cumulative hazard function $H_n(s)$ is then used in (6) to decide if an SU request should be granted or denied.

From an SU's perspective, the utilization of available idle time, commonly called white space utilization, is an important metric. For the scheme proposed in [5] the proportion of the white space utilized in the time interval $[0, T]$ by a sequence of SU requests that arrive according to a Poisson process, $N(t), t \geq 0$ with intensity λ is determined. An SU requests τ units of channel time to transmit. It is assumed that transmission times requested by SUs vary sufficiently such that τ can be reasonably assumed to be a random variable with distribution function $K(x)$, i.e.,

$$K(x) = P[\tau \leq x] \tag{8}$$

Furthermore, they are assumed to be independent of one another and independent of the alternating renewal process. In this study, we assume that the probability of arrival of SU requests during an ongoing SU transmission is negligible. In the next subsection we derive a closed form approximation for the proportion of the white space utilized by the SUs.

C. White Space Utilization

Let the random variable R_T denote the total amount of time the channel is idle (i.e., no PUs are transmitting) in the interval $[0, T]$ and the random variable W_T denote the total amount of time in R_T when SUs are transmitting. Then the fraction of white space utilized is given by

$$\rho_{ws} = \frac{W_T}{R_T}. \tag{9}$$

To estimate R_T, we use a well-known result from the theory of alternating renewal processes [19],

$$\lim_{T \to \infty} \frac{R_T}{T} = \frac{\mu_{\text{idle}}}{\mu_{\text{idle}} + \mu_{\text{busy}}} = q \tag{10}$$

where q, $0 \leq q \leq 1$, represents the mean proportion of time the system is idle. Thus, the total white space in $[0, T]$ can be approximated by

$$R_T \approx Tq. \tag{11}$$

In addition to R_T, an approximation for W_T is needed. The fraction of white space utilized is then approximated by

$$\hat{\rho}_{ws} = E[W_T]/Tq \tag{12}$$

where $E[\cdot]$ denotes the expectation operator. Next, we need a closed form expression for $E[W_T]$ using the SU channel access scheme presented above.

We use the theory of "thinning" or decomposition of the Poisson SU request arrival process, $N(t)$ having rate λ [21], [22] . When a SU request arrives, the channel is either idle or busy. It is idle with asymptotic probability q, see (10), and busy with asymptotic probability $(1 - q)$. The first thinning occurs here. Requests made when the system is idle form a Poisson process, $N_1(t)$, with rate $\lambda_1 = \lambda q$.

For the second thinning we need the probability that a request is granted for an SU to transmit when the system is idle at time t. Since $H_n(t) \to H(t)$, for large n, the following approximation is used.

$$\begin{aligned} P_t(\theta) &= P[H(\gamma_t + \tau) - H(\gamma_t) < \theta] \\ &= \int_0^\infty P[H(s + \tau) - H(s) < \theta \mid \gamma_t = s] dP[\gamma_t \leq s] \\ &= \int_0^\infty P[H(s + \tau) - H(s) < \theta] dP[\gamma_t \leq s] \\ &= \int_0^\infty L_H(s) c_t(s) ds \end{aligned} \tag{13}$$

where $c_t(s)$ denotes the density function of γ_t and

$$L_H(s) = P[H(s + \tau) - H(s) < \theta] \tag{14}$$

So, an SU request that arrives during an idle period is granted with probability $P_t(\theta)$ or equivalently rejected with probability $(1 - P_t(\theta))$. Thus thinning of $N_1(t)$ occurs here. The sequence of times the SUs are allowed to transmit when the channel is idle forms a nonhomogeneous Poisson process, $N_2(t)$ with rate $\lambda_2(t) = \lambda_1 P_t(\theta) = \lambda q P_t(\theta)$ and probability of m arrivals in $[0, t]$:

$$P[N_2(t) = m] = \frac{e^{-\int_0^t \lambda_2(s) ds} (\int_0^t \lambda_2(s) ds)^m}{m!} \quad m = 0, 1, \ldots \tag{15}$$

$m = 0$ means no request is made and the requested transmission time is defined to be zero when this happens.

Let us say there are exactly m transmission requests granted to the SUs in duration $[0, T], m = 0, 1, 2, \ldots$, i.e., $N_2(T) = m$, occurring at $0 < t_1 < t_2 < \ldots < t_m \leq T$ for $m \geq 1$ (see Figure 1). Then the amount of white space utilized is $(\tau_1 + \cdots + \tau_m)$ if $(t_m + \tau_m) \leq T$ and is $(\tau_1 + \cdots + \tau_{m-1} + (T - t_m))$ if $(t_{m-1} + \tau_{m-1}) < t_m < T < (t_m + \tau_m)$, where $\tau_k, k = 1, \ldots, m$ are requested transmission durations.

The average amount of white space utilized is

$$\begin{aligned} E[W_T] &= \sum_{m=0}^{\infty} E[W_T \mid N_2(T) = m] P[N_2(T) = m] \\ &= \sum_{m=1}^{\infty} E[(\sum_{i=1}^{m-1} \tau_i) + J_m \mid N_2(T) = m] P[N_2(T) = m] \\ &= \sum_{m=1}^{\infty} \Big[(m-1) E[\tau] P[N_2(T) = m] + \\ &\quad E[J_m \mid N_2(T) = m] P[N_2(T) = m] \Big] \end{aligned} \tag{16}$$

where

$$J_m = \begin{cases} \tau_m & \text{if } t_m + \tau_m < T \\ T - t_m & \text{otherwise} \end{cases} \tag{17}$$

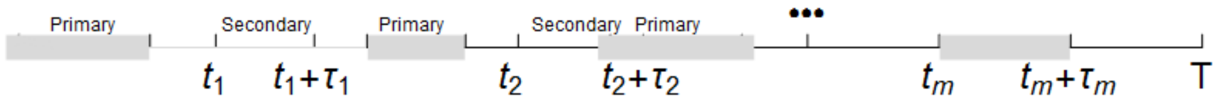

Fig. 1. Schematic of exactly m successful transmissions by SUs. The shaded regions represent PU transmissions. $t_i, i = 1, \ldots, m$ denote all the request times distributed according to the nonhomogeneous Poisson process $N_2(t)$

Now

$$\begin{aligned} &E[J_m \mid N_2(T) = m] \\ &= E[J_m I[t_m + \tau_m < T] \mid N_2(T) = m] + \\ &\quad E[J_m I[t_m + \tau_m \geq T] \mid N_2(T) = m] \\ &= E[\tau_m I[t_m + \tau_m < T] \mid N_2(T) = m] + \\ &\quad E[(T - t_m) I[t_m + \tau_m \geq T] \mid N_2(T) = m] \end{aligned} \quad (18)$$

To complete this computation, we need the following result about nonhomogeneous Poisson processes [19].

Theorem: If $\{N_2(t), t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function $\lambda_2(t)$, then given $N_2(T) = m, T \geq 0$, its m arrival times, $t_1, t_2, \ldots, t_m$ have the same distribution as the order statistics from a sample of m independent and identically distributed random variables having distribution function, $F_2(x)$ given by

$$F_2(x) = \begin{cases} \frac{\Lambda_2(x)}{\Lambda_2(T)} & x \leq T \\ 1 & x > T \end{cases} \quad (19)$$

where $\Lambda_2(x) = \int_0^x \lambda_2(s) ds, 0 \leq x \leq T$. Thus, given $N_2(T) = m$, t_m is distributed as $\max\{t_1^*, t_2^*, \ldots, t_m^*\}$ where the t_i^* are the order statistics from a sample with distribution function $F_2(x)$ and furthermore

$$P[t_m \leq x] = P[\max\{t_1^*, t_2^*, \ldots, t_m^*\} \leq x] = P[t_m^* \leq x] = F_2^m(x), \quad 0 \leq x \leq T \quad (20)$$

This is analogous to a well known result for homogeneous Poisson processes which says given, $N(t) = m$, the m occurrence times are distributed as the order statistics from a uniform $(0, t]$ distribution [19].

Using the Theorem, the two terms in (18) can be evaluated by conditioning on τ_m as follows.

$$\begin{aligned} &E[\tau_m I[t_m + \tau_m < T] \mid N_2(T) = m] = \\ &\quad E[\tau_m I[t_m^* < (T - \tau_m)]] = \\ &\quad \int_0^\infty \tau F_2^m(T - \tau) dK(\tau) \end{aligned} \quad (21)$$

and,

$$\begin{aligned} &E[(T - t_m) I[t_m + \tau_m \geq T \mid N_2(T) = m] = \\ &\quad E[(T - t_m^*) I[t_m^* + \tau_m \geq T]] = \\ &\quad \int_{\tau=0}^\infty E[(T - t_m^*) I[t_m^* + \tau \geq T]] dK(\tau) = \\ &\quad \int_{\tau=0}^\infty \int_{t=0}^\infty (T - t) I[t \geq (T - \tau)] dF_2^m(t) dK(\tau) = \\ &\quad \int_{\tau=0}^\infty \int_{t=(T-\tau)}^T (T - t) dF_2^m(t) dK(\tau) \end{aligned} \quad (22)$$

where $F_2(t)$ is given by (19).

Using (21) and (22) in (16) we have

$$\begin{aligned} E[W_T] = &E[\tau] \sum_{m=1}^\infty (m - 1) P[N_2(T) = m] + \\ &\sum_{m=1}^\infty \int_0^\infty \tau F_2^m(T - \tau) dK(\tau) P[N_2(T) = m] + \\ &\sum_{m=1}^\infty \int_0^\infty \Big[\int_{T-\tau}^T (T - t) dF_2^m(t)\Big] dK(\tau) P[N_2(T) = m] \end{aligned} \quad (23)$$

$$= E_1 + E_2 + E_3$$

where E_1, E_2 and E_3 are the three terms in (23) respectively.

We know that for the nonhomogeneous Poisson process $N_2(T)$,

$$P[N_2(T) = m] = e^{-\Lambda_2(T)} \frac{(\Lambda_2(T))^m}{m!} \quad m = 0, 1, \ldots \quad (24)$$

So,

$$\begin{aligned} E_1 &= E[\tau] \Big[\sum_{m=1}^\infty (m - 1) e^{-\Lambda_2(T)} \frac{(\Lambda_2(T))^m}{m!}\Big] \\ &= E[\tau] e^{-\Lambda_2(T)} \Big[\sum_{m=1}^\infty (m - 1) \frac{(\Lambda_2(T))^m}{m!}\Big] \\ &= E[\tau] e^{-\Lambda_2(T)} \Big[\sum_{m=1}^\infty m \frac{(\Lambda_2(T))^m}{m!} - \sum_{m=1}^\infty \frac{(\Lambda_2(T))^m}{m!}\Big] \\ &= E[\tau] (\Lambda_2(T) - 1 + e^{-\Lambda_2(T)}) \end{aligned} \quad (25)$$

Note that we have used the power series definition of exponential function in the derivation of (25). Using (15) and (19) in the expression for E_2, we have

$$\begin{aligned}E_2 &= \int_0^\infty \tau\Big[\sum_{m=1}^{\infty} F_2^m(T-\tau)e^{-\Lambda_2(T)}\frac{(\Lambda_2(T))^m}{m!}\Big]dK(\tau)\\ &= e^{-\Lambda_2(T)}\int_0^\infty \tau(e^{F_2(T-\tau)\Lambda_2(T)}-1)dK(\tau)\\ &= e^{-\Lambda_2(T)}\int_0^\infty \tau(e^{\Lambda_2(T-\tau)}-1)dK(\tau) \qquad (26)\end{aligned}$$

To compute E_3, note that $dF_2^m(t)/dt = mf_2(t)F_2^{m-1}(t)$ and $f_2(t)\Lambda_2(T) = \lambda_2(t)$.

$$\begin{aligned}E_3 &= \int_0^\infty \Big[\int_{T-\tau}^{T}(T-t)\sum_{m=1}^{\infty} mf_2(t)F_2^{m-1}(t)\\ &\qquad e^{-\Lambda_2(T)}\frac{(\Lambda_2(T))^m}{m!}dt\Big]dK(\tau)\\ &= e^{-\Lambda_2(T)}\int_0^\infty \Big[\int_{T-\tau}^{T}(T-t)f_2(t)\Big(\sum_{m=1}^{\infty} mF_2(t)^{m-1}\\ &\qquad \frac{(\Lambda_2(T))^m}{m!}\Big)dt\Big]dK(\tau)\\ &= e^{-\Lambda_2(T)}\int_0^\infty \Big[\int_{T-\tau}^{T}(T-t)f_2(t)\Lambda_2(T)e^{F_2(t)\Lambda_2(T)}\\ &\qquad dt\Big]dK(\tau)\\ &= e^{-\Lambda_2(T)}\int_0^\infty \Big[\int_{T-\tau}^{T}(T-t)\lambda_2(t)e^{\Lambda_2(t)}dt\Big]dK(\tau)\\ &= e^{-\Lambda_2(T)}\Big[\int_0^\infty \Big(\int_{T-\tau}^{T}T\lambda_2(t)e^{\Lambda_2(t)}dt\Big)dK(\tau)-\\ &\qquad \int_0^\infty \Big(\int_{T-\tau}^{T}t\lambda_2(t)e^{\Lambda_2(t)}dt\Big)dK(\tau)\Big] \qquad (27)\end{aligned}$$

By a change of variables in the first integral and an integration by parts in the second, we get

$$\begin{aligned}E_3 &= e^{-\Lambda_2(T)}\Big[\int_0^\infty \Big(T(e^{\Lambda_2(T)}-e^{\Lambda_2(T-\tau)})-\\ &\Big(Te^{\Lambda_2(T)}-(T-\tau)e^{\Lambda_2(T-\tau)}+\int_{T-\tau}^{T}e^{\Lambda_2(t)}dt\Big)\Big)dK(\tau)\Big]\\ &= e^{-\Lambda_2(T)}\Big[\int_0^\infty \Big(-\tau e^{\Lambda_2(T-\tau)}-\int_{T-\tau}^{T}e^{\Lambda_2(t)}dt\Big)dK(\tau)\Big]\\ &= \int_0^\infty \Big[-\tau e^{\Lambda_2(T-\tau)-\Lambda_2(T)}-\\ &\qquad \int_{T-\tau}^{T}e^{\Lambda_2(t)-\Lambda_2(T)}dt\Big]dK(\tau) \qquad (28)\end{aligned}$$

Thus, from (23)

$$\begin{aligned}E[W_T] &= E[\tau](\Lambda_2(T)-1+e^{\Lambda_2(T)})+\\ &e^{-\Lambda_2(T)}\int_0^\infty \tau(e^{\Lambda_2(T-\tau)}-1)dK(\tau)\\ &+\int_0^\infty \Big[-\tau e^{\Lambda_2(T-\tau)-\Lambda_2(T)}-\\ &\qquad \int_{T-\tau}^{T}e^{\Lambda_2(t)-\Lambda_2(T)}dt\Big]dK(\tau) \qquad (29)\end{aligned}$$

In this study, we take the simple case of SUs sending requests to transmit for a constant duration, i.e., τ is a constant. Then the expression for $E[W_T]$ reduces to

$$\begin{aligned}E[W_T] &= \tau(\Lambda_2(T)-1+e^{-\Lambda_2(T)})+\tau e^{-\Lambda_2(T)}(e^{\Lambda_2(T-\tau)}-1)\\ &\quad -\tau e^{\Lambda_2(T-\tau)-\Lambda_2(T)}-\int_{T-\tau}^{T}e^{\Lambda_2(t)-\Lambda_2(T)}dt\\ &= \tau(\Lambda_2(T)-1)-\int_{T-\tau}^{T}e^{\Lambda_2(t)-\Lambda_2(T)}dt \qquad (30)\end{aligned}$$

D. Exponential Idle Times Distribution

In this section we take the exponential distribution as an example distribution for idle times (of PU traffic) to illustrate our analytical model. We then compare simulation based results with the results from our analytical model to validate the correctness of our model.

In this case we assume $I_j \sim Exp(\alpha)$ and $B_j \sim Exp(\beta)$ where $\mu_{idle} = 1/\alpha$ and $\mu_{busy} = 1/\beta$ and the SU requests arrive as per a Poisson process with parameter, λ. Then,

$$q = \frac{1/\alpha}{1/\alpha + 1/\beta} \qquad (31)$$

and $R_T \approx Tq$ (see (11)). In the exponential case, the hazard function of the idle time distribution $F(x)$ is a constant, viz, $h(t) = \alpha$ and thus $H(t+\tau) - H(t) = \tau\alpha$. The criteria for granting a SU request becomes

$$\text{Request is}\begin{cases}\text{Denied} & \text{if channel is busy}\\ \text{Granted} & \text{if channel is idle and } \tau\alpha < \theta\\ \text{Denied} & \text{if channel is idle and } \tau\alpha \geq \theta\end{cases} \qquad (32)$$

α can be estimated as $1/\bar{I}_n$, where $\bar{I}_n = (I_1+I_2+\cdots+I_n)/n$ is the sample means estimate of μ_{idle} and n denotes the sample size of observed values. Hence, the SU request grant criteria becomes

$$\text{Request is}\begin{cases}\text{Denied} & \text{if channel is busy}\\ \text{Granted} & \text{if channel is idle and } \frac{\tau}{\theta} < \bar{I}_n\\ \text{Denied} & \text{if channel is idle and } \frac{\tau}{\theta} \geq \bar{I}_n\end{cases} \qquad (33)$$

In this case

$$P_t(\theta) = p(\theta) = P[I_1 + \cdots + I_n > n\frac{\tau}{\theta}] \qquad (34)$$

When the I_j's are exponential, $I_1 + \cdots + I_n$ has a gamma distribution [19, pp. 35] and

$$p(\theta) = \sum_{j=0}^{n-1} e^{-\alpha n\tau/\theta}(\alpha n\tau/\theta)^j/j! \qquad (35)$$

In this case, $N_2(t)$ is a homogeneous Poisson process, with parameter $\lambda_2 = q\lambda p(\theta)$ and $\Lambda_2(t) = \lambda_2 t$. Using these values in (30) we have

$$E[W_T] = \tau\lambda_2 T - \tau - \frac{1}{\lambda_2}(1 - e^{-\lambda_2\tau}) \qquad (36)$$

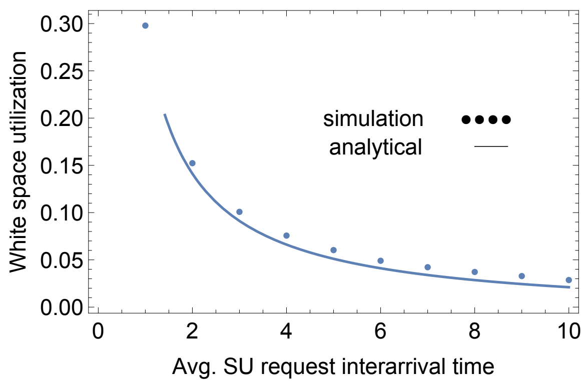

Fig. 2. Comparison of WSU computed using analytical model and simulation

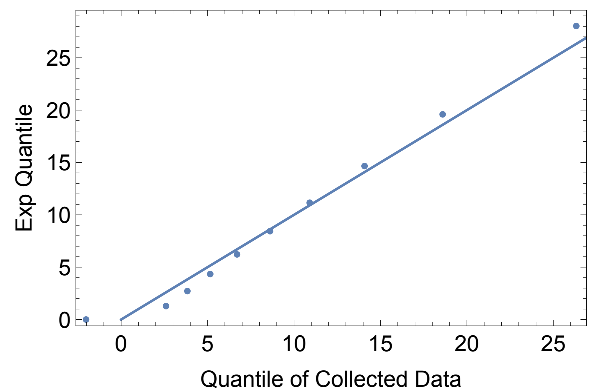

Fig. 3. Q-Q plot of the collected channel 15 data

Hence, using (12) an approximation of white space utilization is obtained as

$$\hat{\rho}_{ws} = \frac{\tau\lambda_2 T - \tau - \frac{1}{\lambda_2}(1 - e^{-\lambda_2 \tau})}{Tq}$$
$$= \frac{\tau}{q}\lambda_2 - \frac{1}{Tq}(\tau + \frac{1}{\lambda_2}(1 - e^{-\lambda_2\tau})) \quad (37)$$

IV. Experiments and Results

To validate our analytical formulation we ran two types of experiments as follows. In the first set of experiments we simulate an alternating renewal process which represents PU traffic. The idle and busy durations of the PU traffic were generated using pseudorandom exponential variates whose means were 10 and 5 units respectively, i.e., $\mu_{idle} = 10$ and $\mu_{busy} = 5$. Then a sequence of SU request arrival times are generated with pseudorandom exponential interarrival times. The requested transmission duration (τ) was set to 0.3 units and the probability of successful transmission (p) was set to 0.9. The simulation was run for a duration $T = 10\,000$ units. An SU request is granted or denied based on the criteria outlined in (32). If a request is granted, then τ units are added to the white space utilized. Figure 2 plots approximation of WSU as per our analytical model (ρ_{ws}) and the simulated WSU as a function of the mean interarrival times of SU requests. The results for the analytical model match very closely with those from the simulation.

In the second set of experiments, we used the I/Q samples collected at a location in the metro Philadelphia area on the CityScape spectrum monitoring system [23] on September 9th, 2017 at around 17:55 hours for a duration of 30 minutes. After carrying out additional processing of the collected I/Q samples and applying the noise threshold to which the CityScape Software Defined Radios (SDRs) were calibrated, the dataset was converted into binary occupancy sequences. We chose channels 15 and 16 for our simulation. Some important statistics of collected data for those two channels are listed in Table I. The first step to run our simulation was to estimate the cumulative hazard of the given channel using (7) over the collected idle time duration data of the channel. We then simulated arrival of SU requests to transmit for 1 ms ($\tau = 1$ ms) as per a Poisson process. The request is granted or denied as per (6) using $p = 0.9$. Each successful SU transmission is counted towards total white space usage of the SU. Then the WSU of the channel is computed as the ratio of total white space usage of the SU to total idle duration of the channel. The histograms of idle duration of channels 15 and 16 are shown in Figure 4 and Figure 5 respectively. The two histograms look approximately exponentially distributed, so we used a *quantile-quantile* (Q-Q) plot using the data from channel 15 to verify this visual observation. For the Q-Q plot of channel 15 shown in Figure 3, a kernel density estimate was used to smooth the idle duration data. Then, the $k^{th}, k = 1, 2, \ldots, 10$, deciles (10-quantiles) of the kernel density were calculated and plotted against the same deciles for the exponential distribution with parameter λ_{data}. The mean of the kernel density is $1/\lambda_{data}$. If the idle time data are generated from this exponential distribution, then these pairs of quantiles should follow the $45°$ line. The plot in Fig 3 indicates that the distribution of the data is close to an exponential. Using the mean idle time of the channels as the parameter for exponential distribution, we then use (37) to calculate analytical WSU. Figure 6 and Figure 7 compare the WSU obtained by simulation and by the analytical method for channels 15 and 16 respectively. For both the channels the difference between the analytical and simulation results is insignificant for an SU request inter-arrival time of 5 ms or more. During simulation, some SU requests fall into the current SU transmission interval, in which case that request is ignored and one or more exponential inter-arrival times are generated until an SU request arrives after the current SU transmission ends. As the request inter-arrival time become smaller, more SU requests fall within an SU transmission duration. Since this phenomenon is not accounted for in the analytical model, the model allows SU requests that arrive during an ongoing SU transmission to be admitted. Thus, the WSU of the analytical model is higher than the WSU of simulation when the SU request inter-arrival time is small. Since idle lengths of the collected data only approximate an exponential distribution, this also contributes to the discrepancy between analytical and simulation WSU.

channel number	mean idle duration (μ_{idle}) (ms)	mean busy duration (μ_{busy}) (ms)	total idle duration (ms)	observed duration (T) (ms)
15	11.175	1.1527	1 631 690	180 000
16	11.341	1.1454	1 634 880	180 000

TABLE I
STATISTICS OF COLLECTED DATA

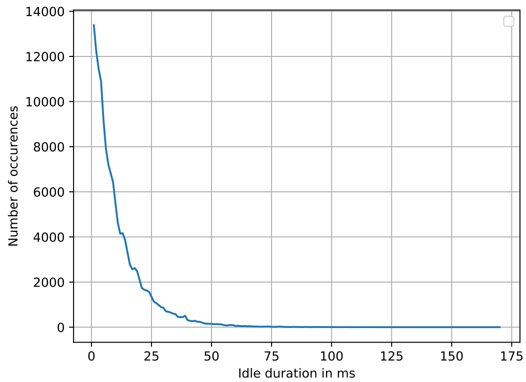

Fig. 4. Idle Duration Histogram of Channel 15 of Collected Data

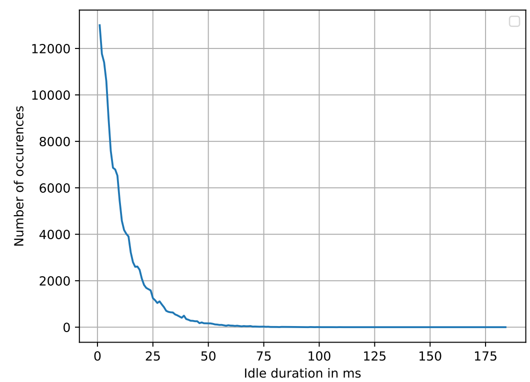

Fig. 5. Idle Duration Histogram of Channel 16 of Collected Data

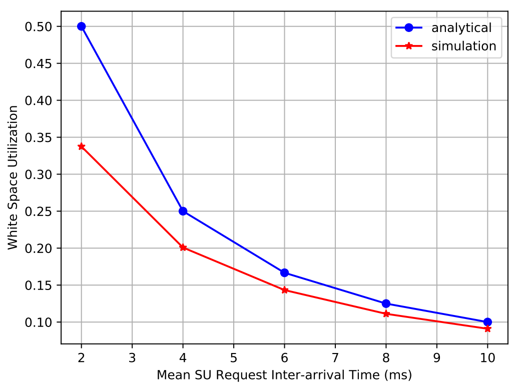

Fig. 6. WSU vs Mean SU Request Inter-arrival time for Channel 15 ($\tau = 1$ ms)

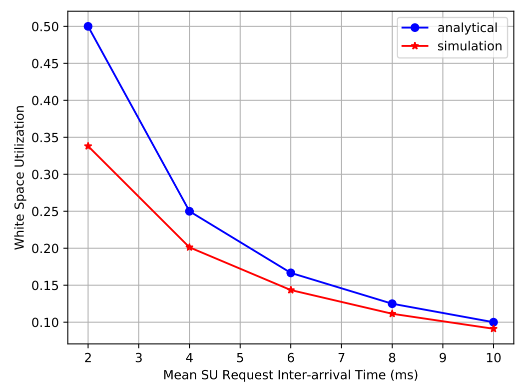

Fig. 7. WSU vs Mean SU Request Inter-arrival time for Channel 16 ($\tau = 1$ ms)

V. DISCUSSION

We presented an analytical model for the opportunistic DSA scheme proposed in [5] that is based on survival analysis. The PU traffic was modeled as an alternating renewal process. Applying theory and methods from survival analysis and stochastic process, we derived an expression to compute the WSU of the DSA system for any general distribution of idle duration of PU traffic. We used an exponential distribution of idle duration as an example to validate our model. The WSU computed using our analytical model was compared against the WSU obtained from two simulation experiments. In one simulation experiment the idle durations were generated from pseudorandom variates whereas in the other the idle durations were taken from data collected from a real LTE system whose idle durations are approximately exponentially distributed. Our analytical results closely match the results from the first experiment and match reasonably well with those obtained from the second experiment.

In this work, we have modeled PU Idle and Busy periods as stationary distributions. In practice, it may not be the case. One way to address this issue is to assume that PU traffic distributions do not change appreciably during a certain period of a day and therefore have different PU traffic distibutions at different time of the day. Based on the time of the day corresponding PU traffic distributions can be used in our analytical model to compute WSU.

In our analytical formulation we assumed that the probability of arrival of SU requests during an ongoing SU transmission is negligible. While this assumption is valid for high SU request inter-arrival times, it does not hold for low inter-arrival times. Hence, we would like to update our model to remove this assumption so that it rejects SU requests arriving during an ongoing SU transmission. We plan to derive analytical expression to compute the WSU for a few other distributions of idle durations. The Generalized Pareto Distribution (GPD) has been used to model the idle and busy durations of a channel [24], [25]. We will take up GPD as the next example distrubtion. Apart from WSU, the Probability of Interference (PoI) is also an important performance metric for DSA systems. Hence we plan to develop an analytical model of PoI for the DSA system. Also, we will validate the model using SU distributions other than the exponential, potentially including mixture distributions.

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