

Exact Moments of Mutual Information of Jacobi MIMO Channels in High-SNR Regime

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Abstract—In this paper, we propose an analytical framework to derive all positive integer moments of MIMO mutual information in high-SNR regime. The approach is based on efficient use of the underlying matrix integrals of the high-SNR mutual information. As an example, the framework is applied to the study of Jacobi MIMO channel model relevant to fiber optical and interference-limited multiuser MIMO communications. For such a channel model, we obtain explicit expressions for the exact moments of the mutual information in the high-SNR regime. The derived moments are utilized to construct approximations to the corresponding outage probability. Simulation shows the usefulness of the results in a crucial scenario of low outage probability with finite number of antennas.

I. INTRODUCTION

Mutual information is among the most important quantities in information theory. For Multiple-Input-Multiple-Output (MIMO) communications, the supremum of mutual information provides the fundamental performance measure, the channel capacity. A great effort has been made to understand the statistical behavior of MIMO mutual information of various channel models. Existing knowledge in the literature is, however, mostly limited to either exact mean values (first moments) [1–4] or asymptotic (in channel dimensions) means and variances [5, 6]. The first moment corresponds to the ergodic mutual information, whereas the higher moments are needed to describe the outage probability relevant to slow or block fading scenarios. Another motivation of our study is that the prevalingly adopted asymptotic variances based approximate outage probabilities [5, 6] fail to capture the true one when the number of antennas is small and/or the outage probability is low. An accurate characterization requires the exact higher moments of mutual information that governs the tail behavior of the distribution.

Determining the exact higher moments of MIMO mutual information for an arbitrary Signal-to-Noise Ratio (SNR) is a well-known difficult task. We will show in this paper that progress can be made when assuming the SNR is high. In particular, we propose an analytical framework to obtain the exact moments of any order in the high-SNR regime, which is valid for a family of channel models. The idea comes from the observation that moments of high-SNR mutual information can be efficiently calculated via the underlying matrix integrals. The high-SNR regime provides crucial insights to the behavior

of the MIMO channels. For example, it characterizes the minimum required transmit power, also known as the high-SNR power offset [7].

To demonstrate the usefulness of the proposed framework, we study the mutual information of the Jacobi MIMO channels. The Jacobi MIMO channel is a useful channel model for MIMO optical communications [2, 6] as well as the interference-limited multiuser MIMO [5]. The main result of this paper is the exact yet explicit expressions for the mutual information moments of any order of the Jacobi MIMO channels in the high-SNR regime.

II. PROBLEM FORMULATION

A. MIMO Mutual Information

For a generic MIMO system consisting of n transmit and m receive antennas, the communication channel in between is described by an $m \times n$ random matrix \mathbf{H} . Assuming i.i.d. input across transmit antennas and that the channel matrix \mathbf{H} is only known to the receiver, the mutual information in nats/second/Hz of the MIMO system is [1]

$$I = \ln \det \left(\mathbf{I}_m + r \mathbf{H} \mathbf{H}^\dagger \right) = \sum_{i=1}^m \ln (1 + r \theta_i), \quad (1)$$

where it can be made without loss of generality to first assume that $m \leq n$. Here, $\ln(\cdot)$ is the natural logarithm, $\det(\cdot)$ is the matrix determinant, r is the SNR, and $\theta_m \leq \dots \leq \theta_2 \leq \theta_1$ denote the eigenvalues of the Hermitian matrix $\mathbf{H} \mathbf{H}^\dagger$. In the high-SNR regime, by ignoring the constant \mathbf{I}_m in (1) the mutual information can be approximated by

$$\mathcal{I} = m \ln r + \ln \det \left(\mathbf{H} \mathbf{H}^\dagger \right) \quad (2a)$$

$$= m \ln r + \sum_{i=1}^m \ln \theta_i. \quad (2b)$$

The above approximation becomes exact as the SNR r grows to infinity.

A fundamental information-theoretic quantity of MIMO channels is outage probability, which is the probability that a given rate exceeds the value of the mutual information. For the high-SNR case (2), the channel outage probability $P_{\text{out}}(z)$ as a function of the rate z is defined as

$$P_{\text{out}}(z) = \mathbb{P}(\mathcal{I} < z). \quad (3)$$

B. Jacobi MIMO Channels

As will be seen, the Jacobi MIMO channel is a channel model for both MIMO optical communications [2, 6] and interference-limited multiuser MIMO [5]. However, we will formulate the problem and set up the notations mainly in the context of the former application. The relevance to the latter application will only be briefly mentioned.

The spatial degrees of freedom of the MIMO Rayleigh channels provide the well-known linear capacity scaling law [1] with respect to the number of transceiver antennas. The idea of the MIMO fiber optical channels is to achieve a similar scaling law by also exploiting the spatial degrees of freedom. In particular, multiple spatial transmission within the same fiber is achieved by designing a multi-mode and/or multi-core fiber. As a first step to exploring the spatial diversity, the Jacobi MIMO optical channel has been proposed in [2, 6], which is based on the following assumptions. The propagation through the fiber is considered as lossless such that it is modeled as an $l \times l$ random unitary matrix $\mathbf{U}\mathbf{U}^\dagger = \mathbf{I}_l$, which is also known as the scattering matrix. Assuming n transmitting and m receiving modes with $m \leq n$, the effective MIMO optical channel¹ \mathbf{H} equals the upper left sub-matrix of the scattering matrix $\mathbf{U} = (u_{ij})$ with the condition $l > m + n$, i.e.,

$$\mathbf{H} = (u_{ij})_{i=1, \dots, m; j=1, \dots, n}. \quad (4)$$

Under the above assumptions, the joint eigenvalue density of the hermitianized channel matrix $\mathbf{H}\mathbf{H}^\dagger$ is given by [2, 6]

$$p(\boldsymbol{\theta}) = \frac{1}{c} \prod_{1 \leq i < j \leq m} (\theta_i - \theta_j)^2 \prod_{i=1}^m \theta_i^{\alpha_1} (1 - \theta_i)^{\alpha_2}, \quad (5)$$

where $0 \leq \theta_m \leq \dots \leq \theta_2 \leq \theta_1 \leq 1$ and

$$\alpha_1 = n - m, \quad \alpha_2 = l - m - n.$$

For the above parameters α_1 and α_2 , the resulting normalization constant c is

$$c = \frac{\prod_{i=1}^m \Gamma(i+1) \Gamma(l-n-i+1) \Gamma(n-i+1)}{\Gamma(l-i+1)}.$$

The ensemble (5) is known as the Jacobi ensemble [8] in random matrix theory, and hence the name Jacobi MIMO channels in the communications theory/information theory community. The eigenvalue density of the interference-limited MIMO channel considered in [5] takes the form of (5) with the same parameter $\alpha_1 = n - m$ as the difference between the number of transmit and receive antennas. For this application, the parameter α_2 is now

$$\alpha_2 = kn - m,$$

where k is the number of interferers. For detailed information of the considered interference-limited MIMO channel including its connection to the Jacobi ensemble, we refer interested readers to [5].

¹For a detailed physical interpretation of this channel model, we refer the readers to the excellent discussion in [6].

For the application to MIMO optical communications, the exact ergodic mutual information $\mathbb{E}[\mathcal{I}]$ of the Jacobi MIMO channels has been calculated in [2] by integrating the mutual information (1) over the eigenvalue density (5), whereas an unexplicit and an asymptotic second moment expressions are available in [6]. For the application to interference-limited MIMO channels, the first two asymptotic moments as well as a differential equation for the moments have been derived in [5]. The exact higher moments of the mutual information $\mathbb{E}[\mathcal{I}^k]$, $k = 2, 3, \dots$, which are needed to characterize the outage probability, remain an open problem². Despite the fact that even the exact second moment of the mutual information $\mathbb{E}[\mathcal{I}^2]$ is notoriously difficult to obtain, we will show that all the exact moments of the high-SNR mutual information (2), $\mathbb{E}[\mathcal{I}^k]$, $k = 1, 2, \dots$, can be explicitly calculated. These moments are utilized to construct approximations to the outage probability in the high-SNR regime, which are in fact accurate for moderate SNR values as will be seen.

III. EXACT CUMULANTS OF HIGH-SNR MUTUAL INFORMATION

To compute $\mathbb{E}[\mathcal{I}^k]$, one naturally would like to integrate (2b) over the eigenvalue density (5) rather than to integrate (2a) over the density of the matrix $\mathbf{H}\mathbf{H}^\dagger$. This is because the former integral only involves m variables, whereas the latter involves m^2 variables. Contrary to this intuition, we show that by working with the corresponding matrix integrals the exact moment of any order can be obtained in a straightforward manner. Instead of directly deriving the moments as in [7, 9], another ingredient that leads to our results is the study of the cumulants of \mathcal{I} , which is technically more convenient as will be seen.

Since the term $m \ln r$ in (2a) is a constant, we first focus on the statistics of the random variable

$$x = \ln \det(\mathbf{H}\mathbf{H}^\dagger). \quad (6)$$

The cumulant generating function $K(s)$ of x is defined as

$$K(s) = \ln \mathbb{E}[e^{sx}] = \sum_{i=1}^{\infty} \tilde{\kappa}_i \frac{s^i}{i!}, \quad (7)$$

where the i -th cumulant $\tilde{\kappa}_i$ of x is recovered from the generating function as

$$\tilde{\kappa}_i = \left. \frac{d^i}{ds^i} K(s) \right|_{s=0}. \quad (8)$$

Denoting the i -th cumulant of \mathcal{I} by κ_i , we have

$$\kappa_1 = m \ln r + \tilde{\kappa}_1, \quad (9a)$$

$$\kappa_i = \tilde{\kappa}_i, \quad i \geq 2, \quad (9b)$$

²A representation of the outage probability involving nested sums over partitions is also available in [6], which is computationally demanding and provides little insights.

which is obtained by the shift-equivariant and the shift-invariant property for cumulants, respectively. With the knowledge of cumulants of the mutual information \mathcal{I} , the corresponding moments can be determined. Specifically, the i -th moment of \mathcal{I} is written in terms of the first i cumulants as

$$\mathbb{E}[\mathcal{I}^i] = B_i(\kappa_1, \dots, \kappa_i), \quad (10)$$

where B_i is the Bell polynomial [10]. For example, the first five moments of \mathcal{I} are listed below

$$\mathbb{E}[\mathcal{I}] = \kappa_1, \quad (11a)$$

$$\mathbb{E}[\mathcal{I}^2] = \kappa_2 + \kappa_1^2, \quad (11b)$$

$$\mathbb{E}[\mathcal{I}^3] = \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3, \quad (11c)$$

$$\mathbb{E}[\mathcal{I}^4] = \kappa_4 + 4\kappa_3\kappa_1 + 3\kappa_2^2 + 6\kappa_2\kappa_1^2 + \kappa_1^4, \quad (11d)$$

$$\mathbb{E}[\mathcal{I}^5] = \kappa_5 + 5\kappa_4\kappa_1 + 10\kappa_3\kappa_2 + 10\kappa_3\kappa_1^2 + 15\kappa_2^2\kappa_1 + 10\kappa_2\kappa_1^3 + \kappa_1^5. \quad (11e)$$

With the above preparation, we now present the main technical contribution of this paper.

Proposition 1. *The i -th exact cumulant κ_i of the high-SNR mutual information (2) of the Jacobi MIMO channels (4) is given by*

$$\begin{aligned} \kappa_1 &= m \ln r + n\psi_0(n) - l\psi_0(l) - (n-m)\psi_0(n-m) \\ &\quad + (l-m)\psi_0(l-m), \quad i=1, \end{aligned}$$

$$\begin{aligned} \kappa_i &= n\psi_{i-1}(n) - l\psi_{i-1}(l) - (n-m)\psi_{i-1}(n-m) \\ &\quad + (l-m)\psi_{i-1}(l-m) + (i-1)(\psi_{i-2}(n) - \\ &\quad \psi_{i-2}(l) - \psi_{i-2}(n-m) + \psi_{i-2}(l-m)), \quad i \geq 2, \end{aligned}$$

where

$$\psi_i(z) = \frac{\partial^{i+1} \ln \Gamma(z)}{\partial z^{i+1}} = (-1)^{i+1} i! \sum_{k=0}^{\infty} \frac{1}{(k+z)^{i+1}} \quad (13)$$

are the polygamma functions [11].

The proof of Proposition 1 is in the Appendix. A similar, yet unsimplified, expression of κ_1 in Proposition 1 has been obtained in the context of mean power offset of interference-limited MIMO channels [7, Eq. (78)]. In such a setting, our derived higher cumulants will be useful to study the distribution of power offset for nonergodic channels.

Note that in the cases $i = 0, 1$, the polygamma functions (13) of positive integer argument can be reduced to finite sums as [11]

$$\psi_0(l) = -\gamma + \sum_{k=1}^{l-1} \frac{1}{k}, \quad (14a)$$

$$\psi_1(l) = \frac{\pi^2}{6} - \sum_{k=1}^{l-1} \frac{1}{k^2}, \quad (14b)$$

which is known as the digamma function and the trigamma function, respectively, and $\gamma \approx 0.5772$ is Euler's constant.

It is worth mentioning that the proposed idea of using matrix-variate integrals to derive cumulants of MIMO mutual information can be equally applied to study other MIMO

channel models. This includes the MIMO Rayleigh channels with³ and without⁴ antenna correlation as well as recent popular MIMO product channels [3,4]. Results on all the integer moments of the above mentioned channel models in the high-SNR regime will be reported separately.

IV. OUTAGE PROBABILITY IN THE HIGH-SNR REGIME

With the exact cumulants in Proposition 1 and the cumulant-moment relations (10), (11), closed-form moment-based approximations to the outage probability (3) can now be constructed. Moment-based approximation is a useful tool in situations when the exact distribution is intractable whereas the moments are available. The basic idea of moment-based approximation is to match the moments and support of an unknown distribution by an elementary distribution and the associated orthogonal polynomials [14, 15].

For convenience, in the simulation we construct the approximative outage probability via the moments of the random variable $-x$ in (6). Since $-x \in [0, \infty)$ has the same support as a gamma random variable, the gamma distribution and the associated Laguerre polynomials are chosen. The resulting approximative distribution function⁵ $F_q(z) = \mathbb{P}(-x < z)$ by matching the first q moments of $-x$ can be read off from [14, Eq. (2.7.27)] as

$$F_q(z) \approx \frac{\gamma(\alpha, z/\beta)}{\Gamma(\alpha)} + \epsilon(z),$$

where

$$\epsilon(z) = \sum_{i=3}^q w_i \sum_{j=0}^i \frac{(-1)^j \Gamma(\alpha+i)}{(i-j)! j!} \frac{\gamma(\alpha+j, z/\beta)}{\Gamma(\alpha+j)}$$

with

$$w_i = \sum_{l=0}^i (-1)^l \frac{\Gamma(i+1) \mathbb{E}[(-x)^l]}{(i-l)! \Gamma(\alpha+l) \beta^l} \quad (15)$$

and $\gamma(a, b) = \int_0^b t^{a-1} e^{-t} dt$ denotes the lower incomplete gamma function. The parameters

$$\alpha = \frac{\mathbb{E}^2[-x]}{\mathbb{E}[x^2] - \mathbb{E}^2[-x]}, \quad \beta = \frac{\mathbb{E}[x^2] - \mathbb{E}^2[-x]}{\mathbb{E}[-x]}$$

are obtained by matching the first two moments of $-x$ to a gamma random variable having a density $p(y|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta}$, $y \in [0, \infty)$. Finally, by the relation (2a), an approximation to the outage probability (3) is obtained as

$$P_{\text{out}}(z) = 1 - F_q(m \ln r - z). \quad (16)$$

As the number of moments involved in (15) increases, the accuracy of the approximation (16) is expected to improve.

Fig. 1 shows the outage probability of the high-SNR mutual information \mathcal{I} assuming the channel dimensions $m = 4$ and $n = 6$, where different dimensions of the scattering matrices

³The corresponding first two moments and some bounds on the outage probability have been derived in [13]

⁴The corresponding first and second moment has been obtained in [7, Eq. (15)] and [9, Eq. (105)], respectively.

⁵Note that the dependence on q is through the summation index in $\epsilon(z)$.

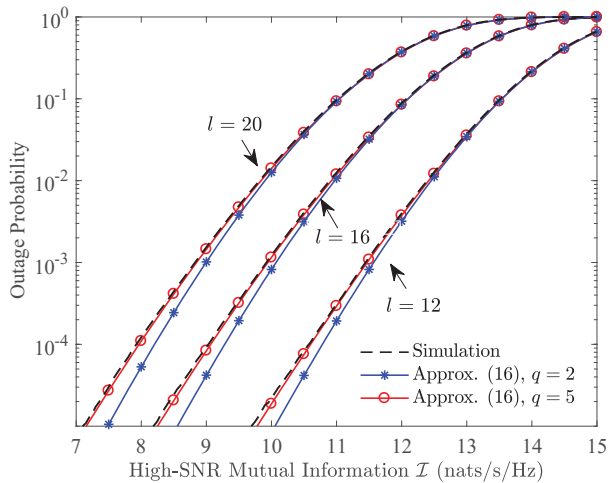


Fig. 1. Outage probability of high-SNR mutual information (2) for different l with $m = 4$, $n = 6$, and $r = 20$ dB.

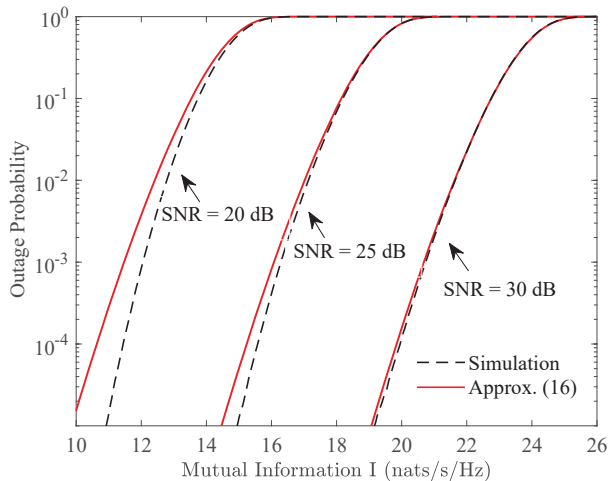


Fig. 2. Outage probability of mutual information (1) for different SNR values with $q = 5$, $l = 12$, $m = 4$, and $n = 6$.

$l = 12, 16$, and 20 are considered. The numerical simulations are compared with the moment-based approximative outage probability (16), where the number of moments considered are $q = 2$ and $q = 5$. We see that the outage probability decreases as the number of untapped channels $l - m - n$ decreases. This phenomenon is also observed in [6]. As expected, it is seen that the accuracy of the proposed approximation (16) improves especially in the tails as the number of moments used increases from $q = 2$ to $q = 5$.

The impact of finite SNR on the accuracy of the approximation (16) is evaluated in Fig. 2, where the number of moments used is $q = 5$ and the channel dimensions are $l = 12$, $m = 4$, and $n = 6$. Despite the asymptotic nature of the approximation, we see that it is already reasonably accurate for not-so-high SNR for outage probability as low as 10^{-4} .

V. CONCLUSION

We study the mutual information of the Jacobi MIMO channels, which is a realistic model for optical and interference-limited multiuser communications. The corresponding exact moments for such a channel model in the high-SNR regime are derived. The results are possible by making use of the matrix integrals involving the density of the high-SNR mutual information. Approximations to the outage probability are constructed based on the obtained exact moments. Simulation demonstrates the accuracy of the results in practical scenarios of low outage probability and finite number of antennas.

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APPENDIX

Before proving Proposition 1, we need the following lemma on the finite sums of polygamma functions.

Lemma 1. For a positive integer l , we have

$$\sum_{k=1}^n \psi_0(k+l) = (n+l)\psi_0(n+l) - l\psi_0(l) - n, \quad (17a)$$

$$\sum_{k=1}^n \psi_i(k+l) = (n+l)\psi_i(n+l) - l\psi_i(l) + i(\psi_{i-1}(n+l) - \psi_{i-1}(l)), \quad i \geq 1. \quad (17b)$$

Proof: By the definition of digamma function (14a), we have

$$\psi_0(k+l) = \psi_0(l) + \sum_{i=0}^{k-1} \frac{1}{l+i}, \quad (18)$$

which gives

$$\begin{aligned} & \sum_{k=1}^n \psi_0(k+l) \\ &= n\psi_0(l) + \sum_{k=1}^n \sum_{i=0}^{k-1} \frac{1}{l+i} \\ &= n\psi_0(l) + \sum_{i=0}^{n-1} \sum_{k=i+1}^n \frac{1}{l+i} \\ &= n\psi_0(l) + n \sum_{i=0}^{n-1} \frac{1}{l+i} - \sum_{i=1}^{n-1} \frac{i}{l+i} \\ &= n\psi_0(l) + n(\psi_0(n+l) - \psi_0(l)) - \sum_{i=1}^{n-1} \frac{l+i-l}{l+i} \\ &= n\psi_0(n+l) + l \sum_{i=1}^{n-1} \frac{1}{l+i} - n + 1 \\ &= n\psi_0(n+l) + l(\psi_0(n+l) - \psi_0(l+1)) - n + 1 \\ &= (n+l)\psi_0(n+l) - l\psi_0(l) - n. \end{aligned}$$

This proves (17a). To show (17b), by the series representation of polygamma function (13), one has

$$\psi_i(k+l) = \psi_i(l) + (-1)^i i! \sum_{j=0}^{k-1} \frac{1}{(l+j)^{i+1}}, \quad (19)$$

which similarly gives

$$\begin{aligned} & \sum_{k=1}^n \psi_i(k+l) = n\psi_i(l) + (-1)^i i! \sum_{j=0}^{n-1} \sum_{k=j+1}^n \frac{1}{(l+j)^{i+1}} \\ &= n\psi_i(l) + (-1)^i i! n \sum_{j=0}^{n-1} \frac{1}{(l+j)^{i+1}} - (-1)^i i! \sum_{j=1}^{n-1} \frac{j}{(l+j)^{i+1}} \\ &= n\psi_i(l) + n(\psi_i(n+l) - \psi_i(l)) - (-1)^i i! \sum_{j=1}^{n-1} \frac{l+j-l}{(l+j)^{i+1}} \\ &= n\psi_i(n+l) - (-1)^i i! \sum_{j=1}^{n-1} \frac{1}{(l+j)^i} + (-1)^i i! l \sum_{j=1}^{n-1} \frac{1}{(l+j)^{i+1}} \\ &= n\psi_i(n+l) + i \left(\psi_{i-1}(n+l) - \psi_{i-1}(l) - \frac{(-1)^{i-1} (i-1)!}{l^i} \right) \\ & \quad + l \left(\psi_i(n+l) - \psi_i(l) - \frac{(-1)^i i!}{l^{i+1}} \right) \\ &= (n+l)\psi_i(n+l) - l\psi_i(l) + i(\psi_{i-1}(n+l) - \psi_{i-1}(l)). \end{aligned}$$

Note that the equality (17a) and the special case $i = 1$ of the equality (17b) appeared in [12, Eq. (A1)] and [12, Eq. (A7)], respectively, where no proofs were provided. ■

With the results in Lemma 1, we now turn to the proof of Proposition 1.

Proof: As previously mentioned, the key ingredient of our results is to make use of the relevant matrix integral instead of the integral over the eigenvalue density (5). Mathematically, the Jacobi MIMO channel (4) is a specific truncation of an unitary matrix, where the corresponding density of the channel matrix $\mathbf{H}\mathbf{H}^\dagger$ is given by [8, p. 357]

$$\frac{\Gamma_m(\alpha+l-n)}{\Gamma_m(\alpha)\Gamma_m(l-n)} \det(\mathbf{H}\mathbf{H}^\dagger)^{\alpha-m} \det(\mathbf{I}_m - \mathbf{H}\mathbf{H}^\dagger)^{l-m-n}. \quad (20)$$

Here, $\Gamma_m(\alpha)$ denotes the multivariate gamma function [8]

$$\Gamma_m(\alpha) = \pi^{\frac{1}{2}m(m-1)} \prod_{k=0}^{m-1} \Gamma(\alpha - k) \quad (21)$$

and the parameter α in the density (20) equals n .

Now the cumulant generating function (7) of the random variable (6) over the matrix density (20) is calculated as

$$\begin{aligned} K(s) &= \ln \mathbb{E} \left[e^{s \ln \det(\mathbf{H}\mathbf{H}^\dagger)} \right] \\ &= \ln \mathbb{E} \left[\det(\mathbf{H}\mathbf{H}^\dagger)^s \right] \\ &= \ln \frac{\Gamma_m(\alpha+l-n)}{\Gamma_m(\alpha)\Gamma_m(l-n)} + \ln \int \det(\mathbf{H}\mathbf{H}^\dagger)^{s+\alpha-m} \times \\ & \quad \det(\mathbf{I}_m - \mathbf{H}\mathbf{H}^\dagger)^{l-m-n} d\mathbf{H}\mathbf{H}^\dagger \\ &= \ln \frac{\Gamma_m(\alpha+l-n)}{\Gamma_m(\alpha)\Gamma_m(l-n)} + \ln \frac{\Gamma_m(s+\alpha)\Gamma_m(l-n)}{\Gamma_m(s+\alpha+l-n)}. \end{aligned}$$

The integral in the above can be considered as a trivial deformation of the density (20), which is directly obtained by replacing in the normalization constant the appearance of α by $s+\alpha$. Setting $\alpha = n$, according to the definition (8) the i -th cumulant $\tilde{\kappa}_i$ of (6) can now be computed as

$$\begin{aligned} \tilde{\kappa}_i &= \frac{d^i}{ds^i} K(s) \Big|_{s=0} \\ &= \frac{d^i}{ds^i} \ln \frac{\Gamma_m(s+n)}{\Gamma_m(s+l)} \Big|_{s=0} \\ &= \frac{d^i}{ds^i} \left(\sum_{k=0}^{m-1} \ln \Gamma(s+n-k) - \sum_{k=0}^{m-1} \ln \Gamma(s+l-k) \right) \Big|_{s=0} \\ &= \sum_{k=0}^{m-1} \psi_{i-1}(n-k) - \sum_{k=0}^{m-1} \psi_{i-1}(l-k) \\ &= \sum_{k=1}^m \psi_{i-1}(n-m+k) - \sum_{k=1}^m \psi_{i-1}(l-m+k), \end{aligned}$$

where we have used the definitions (21) and (13). Finally, by using Lemma 1 and the relation between the cumulants (9), we prove Proposition 1. ■

It is seen that instead of directly studying the moments, the use of cumulant generating function turns out to be more convenient, where all the cumulants are obtained at once via the polygamma functions.