Economics of Networked Infrastructures at the Edge of Undesirable Contagion: A Case of SIS Infection

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Economic and convenience benefits of interconnectivity drive current explosive emergence and growth of networked systems. However, these benefits of interconnectivity are inherently associated with various risks, including risk of undesirable contagion [1]. Due to reliance on networked infrastructures, understanding and ability to manage the fundamental risk/benefit tradeoffs of interconnectivity is one of the most important challenges faced by modern society. Generally, this is a difficult problem especially for large-scale networked infrastructures when centralized management may not be feasible. Often, different parts of infrastructure have different owners, which further complicates the problem.

This situation can be naturally modelled as a non-cooperative game of selfish system components, which are either under different ownership or under control by different entities due to lack of centralized control. Inefficiency of the selfish vs. socially optimal management can be quantified by the corresponding Price of Anarchy (PoA). This inefficiency is due to the positive externalities since an investment in contagion risk mitigation by a network component reduces likelihood of the contagion and thus benefits other system components [2]. Since quantitative comparison of various inefficiency mitigation strategies is typically computationally prohibitive, we propose to explore a possibility of developing practical mitigation strategies under practically plausible assumption of large losses due to the contagion.

Our results indicate that under natural assumption on the cost structure, a socially optimal strategy keeps the system on the boundary of contagion free region in the space of system parameters. However, a selfish strategy keeps the system "close" to boundary of the contagion region within this region with finite contagion losses. Thus, inefficiency mitigation includes (a) elimination of the contagion losses, and (b) moving the system operational point towards socially optimal point by rebalancing individual investments through a combination of regulations and incentives.

We consider a Susceptible-Infected- Susceptible (SIS) model on an undirected connected graph with N nodes and irreducible symmetric incidence matrix $A=(A_{ij})_{i,j=1}^{N}$, where $A_{ij}=A_{ji}=1$ if nodes i and j are connected by a link and $A_{ij}=A_{ji}=0$ otherwise. Once node i becomes infected, it spreads infection to each of its neighboring nodes j at fixed rate $\lambda>0$. Node i recovery time is distributed exponentially with average τ_i. It is known [3] that SIS model is infection free if $\lambda\le 1/\gamma$, and has finite portion of nodes persistently infected otherwise, where γ is the Perron-Frobenius (P-F) eigenvalue of matrix $B=(\tau_i A_{ij})_{i,j=1}^{N}$.

Following [3] we assume that nodes can invest in the reduction of their expected recovery time: average node i recovery time $\tau_i=\tau_i(c_i)$ is a decreasing and strongly convex function of the node i investment $c_i\ge 0$. "Large" investment makes recovery "very fast" and "small" investment makes recovery time "very slow," i.e., $\tau_i(c)\uparrow\infty$ as $c\downarrow 0$, and $\tau_i(c)\downarrow 0$ as $c\uparrow\infty$, $i=1,..,N$. We also assume that infected node i suffers a "large" loss $H_i=h_i/\varepsilon$, where $h_i=O(1)$ as $\varepsilon\to 0$. Thus the expected node i loss is

$$L_i(c)=(h_i/\varepsilon)p_i(c)+c_i\,, \tag{1}$$

where steady-state probability of node i being infected, p_i depends on the entire vector of investments $c=(c_1,..,c_N)$.

Socially optimal investments $c^{opt}=(c_1^{opt},..,c_N^{opt})$ minimize the aggregate loss:

$$c^{opt}=\arg\min_{c_i\ge 0}\sum\nolimits_i[(h_i/\varepsilon)p_i(c)+c_i]\,. \tag{2}$$

We show that the optimal infection probabilities are of the order of ε^2 as $\varepsilon\to 0$:

$$p_i[c^{opt}(\varepsilon)]=p_{i0}^{opt}\varepsilon^2+O(\varepsilon^3)\,, \tag{3}$$

where $p_{i0}^{opt}>0$. Since socially optimal investments result in asymptotically zero contagion losses: $\lim_{\varepsilon\to 0}(h_i/\varepsilon)p_i[c^{opt}(\varepsilon)]=0$, socially optimal investments $c^{opt}=\lim_{\varepsilon\to 0}c^{opt}(\varepsilon)$ asymptotically minimize the aggregate system investment subject to system being infection free:

$$c^{opt}=\arg\min\nolimits_{\gamma(c)\le 1/\lambda}\sum\nolimits_i c_i\,, \tag{4}$$

where P-F eigenvalue $\gamma(c)$ depends on the vector of node investments $c=(c_1,..,c_N)$, as shown in Figure 1.

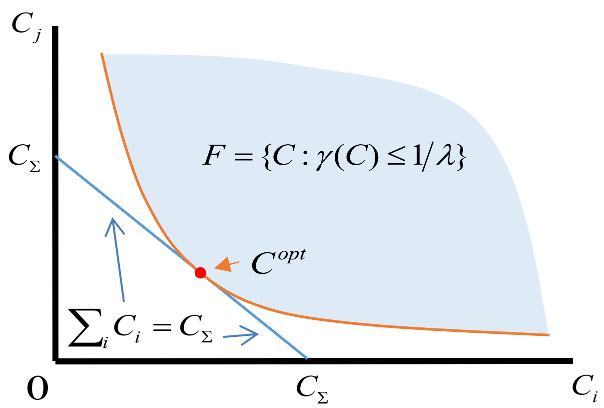

Fig. 1. Social optimization: large infection losses.

Following [3], we model selfish node investment in the recovery capability as a non-cooperative game G, where each node $i = 1,..,N$ attempts to minimize its expected individual expected loss (1) over this node investment c_i, given investments by other nodes $c_{-i} := (c_j, j \ne i)$. We show that under our and some additional technical assumptions, game G is strictly concave, and thus has unique pure Nash equilibrium $c^* = (c_1^*,..,c_N^*)$, which solves the following optimization problem:

$$c_i^* = \arg\min_{c_i \ge 0} [(h_i/\varepsilon) p_i(c_i, c_{-i}^*) + c_i]. \quad (5)$$

Our analysis indicates that investments (5) result in the infection probabilities of the order of ε as $\varepsilon \to 0$:

$$p_i(c^*) = p_{i0}^* \varepsilon + O(\varepsilon^2), \quad (6)$$

where $p_{i0}^* > 0$, and thus selfish investments result in non-diminishing infection losses $h_i p_{i0}^* > 0$ as $\varepsilon \to 0$.

Inefficiency of selfish investment can be quantified by the corresponding Price of Anarchy

$$PoA(\varepsilon) = \frac{\sum_i \{(h_i/\varepsilon) p_i[c^*(\varepsilon)] + c_i^*(\varepsilon)\}}{\sum_i \{[(h_i/\varepsilon) p_i[c^{opt}(\varepsilon) + c_i^{opt}(\varepsilon)]\}}. \quad (7)$$

We have from (3), (6):

$$\lim_{\varepsilon \to 0} PoA(\varepsilon) = PoA_1 + PoA_2, \quad (8)$$

where the aggregate optimal and selfish investments are $c_\Sigma^{opt} = \Sigma_i c_i^{opt}$ and $c_\Sigma^* = \Sigma_i c_i^*$ respectively. Component $PoA_1 = (1/c_\Sigma^{opt}) \sum_i h_i p_{i0}^* \ge 0$ quantifies inefficiency due to contagion losses, and component $PoA_2 = c_\Sigma^* / c_\Sigma^{opt}$ quantifies inefficiency due to imbalance of investments by selfish nodes as compared to the socially optimal investments as $\varepsilon \to 0$. It follows from (6) that "small" additional investments of the order of ε as $\varepsilon \to 0$ can eliminate inefficiency due to contagion losses thus making $PoA_1 = 0$. Since $PoA \ge 1$ implies $PoA_2 \ge 1$, we conjecture that the leading contributor to inefficiency of selfish investments is the investment imbalance as compared to the socially optimal investments, which challenges a conventional view that selfish nodes underinvest in the contagion avoidance.

We demonstrate how selfish investments (5) and corresponding infection probabilities (6) can be derived from known results on P-F perturbation theory [4]. The corresponding expansions allow us not only to evaluate PoA (8), but also to develop and evaluate practical inefficiency mitigation techniques through combination of regulations and incentives. Broadly speaking, regulations/incentives should force/incentivize "systemically important" nodes to invest more in the recovery capability. "Central planner," being capable of measuring the aggregate loss due to contagion $h_{ave}^{agg} := N h_{ave} p_{ave}$, where the portion of infected nodes is $p_{ave} = N^{-1} \Sigma_i p_i$ and the overall average node loss due to infection is $h_{ave} := (N p_{ave})^{-1} \Sigma_i h_i p_i(c)$, mandates penalty/payment to each node $i = 1,..,N$ according to this node centrality measure π_i where $\Sigma_i \pi_i = 1$. Specifically, the central planner imposes penalty on node i equal to $\pi_i h_{ave} - p_i h_i$ if $\pi_i h_{ave} > p_i h_i$ or provides payment $p_i h_i - \pi_i h_{ave}$ to node i if $\pi_i h_{ave} < p_i h_i$.

We show that for a general network, the properly defined centrality measure π_i can be expressed in terms of eigenvectors of matrix $B = (\tau_i A_{ij})_{i,j=1}^N$. For a random uncorrelated network, where for each node $i = 1,..,N$ infection loss h_i and expected recovery time as a function of this node investment c, $\tau_i(c)$ depend only in this node degree $d_i = \sum_{j \ne i} A_{ij}$, we derive explicit expressions for the selfish equilibrium investments (5), infection probabilities (6), and the optimal centrality measure π_i, such that penalties/payments of $\pi_i h_{ave} - p_i h_i$ eliminate inefficiency by transforming competitive equilibrium in the socially optimal equilibrium.

References

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