High rank elliptic curves with torsion $\mathbb{Z}/4\mathbb{Z}$

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Abstract

Working over the field $\mathbb{Q}(t)$, Kihara constructed an elliptic curve with torsion group $\mathbb{Z}/4\mathbb{Z}$ and five independent rational points, showing the rank is at least five. Following his approach, we give a new infinite family of elliptic curves with torsion group $\mathbb{Z}/4\mathbb{Z}$ and rank at least five. This matches the current record for such curves. In addition, we give specific examples of these curves with high ranks 10 and 11.

Keywords: Elliptic curve, rank, torsion group

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1. Introduction

As is well-known, an elliptic curve E over a field $\mathbb{K}$ can be explicitly expressed by the generalized Weierstrass equation of the form

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

with $a_1, a_2, a_3, a_4, a_6 \in \mathbb{K}$. In this paper, we are interested in elliptic curves defined over the rationals, i.e., $\mathbb{K} = \mathbb{Q}$. The famous Mordell-Weil theorem says

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that every elliptic curve over $\mathbb{Q}$ has a commutative group $E(\mathbb{Q})$ which is finitely generated. That is, $E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus E(\mathbb{Q})_{tors}$, where r is a nonnegative integer and $E(\mathbb{Q})_{tors}$ is the subgroup of elements of finite order in $E(\mathbb{Q})$. This subgroup is called the torsion subgroup of $E(\mathbb{Q})$ and the integer r is known as the rank of E.

By Mazur's theorem [12], the torsion subgroup $E(\mathbb{Q})_{tors}$ can only be one of fifteen groups: $\mathbb{Z}/n\mathbb{Z}$ with $1 \le n \le 10$ or $n = 12$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2m\mathbb{Z}$ with $1 \le m \le 4$. While the possibilities for the torsion subgroup are finite, the situation is not as clear for the rank r. The folklore conjecture is that the rank can be arbitrarily large, but it seems to be very hard to find concrete examples of elliptic curves with large rank. The current record is an example of an elliptic curve over $\mathbb{Q}$ with rank at least 28, found by Elkies in May 2006 (see [5]). There is no known guaranteed algorithm to determine the rank and it is not known which integers can occur as ranks.

Let T be an admissible torsion group for an elliptic curve E over $\mathbb{Q}$. Define

$$B(T) = \sup\{\text{rank } E(\mathbb{Q}) : \text{torsion group of } E \text{ over } \mathbb{Q} \text{ is } T\},$$

$$G(T) = \sup\{\text{rank } E(\mathbb{Q}(t)) : \text{torsion group of } E \text{ over } \mathbb{Q}(t) \text{ is } T\},$$

$$C(T) = \limsup\{\text{rank } E(\mathbb{Q}) : \text{torsion group of } E \text{ over } \mathbb{Q} \text{ is } T\}.$$

There exists a conjecture in this setting that says $B(T)$ is unbounded for all T. Even though $B(T)$ is conjectured to be arbitrarily high, it appears difficult to find examples of curves with high rank. There has been much interest in finding high rank elliptic curves with specified torsion groups. See [2, 3] for tables with the best known lower bounds for $B(T), G(T)$, and $C(T)$, including references to the papers where each bound is found.

In this paper, we will consider elliptic curves with torsion group $\mathbb{Z}/4\mathbb{Z}$. The current record for the highest rank of an elliptic curve with this torsion group is 12, with a curve found by Elkies in 2006 [5], as well as another recently found by Dujella and Peral [2]. In 2004, Kihara [8] found an infinite one-parameter family of curves with torsion group $\mathbb{Z}/4\mathbb{Z}$ and having rank (at least) 4. He extended this to an infinite rank 5 family whose fifth point was parameterized by a positive rank curve. Later in [9], he improved his results to an unconditional family of rank (at least) 5. Dujella, et. al. [4], by using a suitable injective specialization, subsequently proved that the rank of Kihara's family over $\mathbb{Q}(t)$ is exactly equal to 5 and found explicit generators. In 2007, Elkies also found an infinite family with rank at least 5 and a rank 6 family dependent on a positive rank curve [6]. Thus, $B(\mathbb{Z}/4\mathbb{Z}) = 12$, $G(\mathbb{Z}/4\mathbb{Z}) \geq 5$, and $C(\mathbb{Z}/4\mathbb{Z}) \geq 6$.

The main contribution of this work is a new family of elliptic curves with torsion group $\mathbb{Z}/4\mathbb{Z}$, and rank (at least) 5. In fact, we show the family has rank exactly 5 over $\mathbb{Q}(t)$. This family matches the best known results for high rank for an infinite family of elliptic curves with torsion group $\mathbb{Z}/4\mathbb{Z}$. We also find two elliptic curves with rank 11, and many with rank 10, all of which have not been previously published. According to [2], there are only two other known curves with rank 11 (and torsion group $\mathbb{Z}/4\mathbb{Z}$). This new family is thus a very good source for finding high rank curves.

Our starting point to find these families of curves is Kihara's original paper [8]. We review Kihara's method in Section 2, and in Section 3 find a new solution to some of Kihara's equations, leading to a different rank 4 family than Kihara found. In Section 4, we further specialize this family to create a fifth rational point. We show that the family has rank 5 over $\mathbb{Q}(t)$ in Section 5, and find the generators. We performed a computer search for specific curves in our families with high rank. The results are given in Section 6.

2. Kihara's Method

We briefly describe Kihara's construction [8]. Consider the projective curve

$$C : (x^2 - y^2)^2 + 2A(x^2 + y^2)z^2 + Bz^4 = 0.$$

C can be transformed into Weierstrass form by setting $X = (A^2 - B)y^2/x^2$ and $Y = (A^2 - B)y(Bz^2 + Ax^2 + Ay^2)/x^3$, resulting in the curve

$$E : Y^2 = X^3 + (2A^2 + 2B)X^2 + (A^2 - B)^2 X. \tag{2.1}$$

The point $P(A^2-B, 2A(A^2-B))$ is on E and it can be easily checked that $2P=(0,0)$ and $4P=\mathcal{O}$, the identity element of E. Now consider the affine model of C

$$H:(x^2-y^2)^2+2A(x^2+y^2)+B=0.$$

If we assume that the points $P_1(r,s)$ and $P_2(r,u)$ are on H, then it is required that $A=(2r^2-s^2-u^2)/2$ and $B=s^2u^2+s^2r^2+u^2r^2-3r^4$. We further assume that the points $P_3(s,p)$ and $P_4(u,q)$ are also on H, and so we must have

$$p^2=3s^2+u^2-3r^2, \tag{2.2}$$

$$q^2=s^2+3u^2-3r^2. \tag{2.3}$$

Kihara gave the following parametric solution to the Diophantine equations (2.2),(2.3):

$$\begin{aligned}
r &= t^2-33,\\
s &= t^2-2t-27,\\
u &= t^2-6t+33,\\
p &= t^2-12t+3,\\
q &= t^2-20t+27.
\end{aligned}$$

Thus, there are four $\mathbb{Q}(t)$-rational points on the affine curve H, and consequently four $\mathbb{Q}(t)$-rational points on the corresponding elliptic curve E.

3. A Family of Elliptic Curves with Rank at least 4

We solve the equations (2.2) and (2.3) in a different way. By subtracting (2.3) from (2.2), we have that

$$p^2+2u^2=q^2+2s^2. \tag{3.1}$$

Recall the well-known Brahmagupta identity

$$\begin{aligned}
(a^2+Nb^2)(c+Nd^2) &= (ac-Nbd)^2+N(ad+bc)^2\\
&= (ac+Nbd)^2+N(ad-bc)^2.
\end{aligned}$$

By setting $N = 2$, and letting

$$p = ac + 2bd,$$
$$q = ac - 2bd,$$
$$u = bc - ad,$$
$$s = bc + ad,$$

we see that we have a solution to (3.1).

From (2.2), we require $r^2 = (3s^2 + u^2 - p^2)/3$. Substituting in, this translates to

$$r^2 = (4/3)b^2c^2 + (4/3)a^2d^2 - (1/3)a^2c^2 - (4/3)b^2d^2. \tag{3.2}$$

In order to find a parametric solution to (3.2) we fix c and d. Now we rewrite (3.2) in the form

$$4b^2(c^2 - d^2) + 4a^2(d^2 - c^2/4) = 3r^2. \tag{3.3}$$

If we consider $4(c^2 - d^2) = \alpha$ and $4(d^2 - c^2/4) = \beta$, then (3.3) can be written $\alpha b^2 + \beta a^2 = 3r^2$, with parametric solution given by

$$a = (d^2 - c^2)m^2 + 3n^2,$$
$$b = (d^2 - c^2)m^2 - 3cmn - 3n^2,$$
$$r = c(d^2 - c^2)m^2 + (4(d^2 - c^2))mn - 3cn^2,$$

for any c, d. Therefore

$$r = 4c^2mn - cm^2d^2 + 3cn^2 - 4mnd^2 + c^3m^2,$$
$$s = cm^2d^2 - c^3m^2 - 3c^2mn - 3cn^2 + m^2d^3 - dm^2c^2 + 3dn^2,$$
$$u = cm^2d^2 - c^3m^2 - 3c^2mn - 3cn^2 - m^2d^3 + dm^2c^2 - 3dn^2,$$
$$p = cm^2d^2 - c^3m^2 + 3cn^2 + 2m^2d^3 - 2dm^2c^2 - 6dcmn - 6dn^2,$$
$$q = cm^2d^2 - c^3m^2 + 3cn^2 - 2m^2d^3 + 2dm^2c^2 + 6dcmn + 6dn^2.$$

If we write the elliptic curve E from (2.1) in the form

$$Y^2 = X^3 + A_4X^2 + B_4X,$$

then a simple calculation yields

$$\begin{aligned}
A_4 = &- 240c^7m^7d^4n - 16c^{11}m^7n - 432c^5mn^7 - 16c^2m^8d^{10} + 48m^6d^{10}n^2 \\
&+ 432m^2d^6n^6 - 1512c^6m^2n^6 - 808m^4d^8n^4 - 16c^6m^8d^6 - 736c^9m^5n^3 \\
&+ 4c^8m^8d^4 - 1708c^8m^4n^4 + 24c^4m^8d^8 - 168c^{10}m^6n^2 - 2208c^7m^3n^5 \\
&+ 4m^8d^{12} + 324d^4n^8 + 112c^9m^7d^2n - 2136c^6m^6d^4n^2 + 4528c^7m^5d^2n^3 \\
&+ 1680c^4m^6d^6n^2 + 1104c^8m^6d^2n^2 + 9712c^6m^4d^2n^4 - 12840c^4m^4d^4n^4 \\
&- 7440c^5m^5n^3d^4 + 208c^5m^7nd^6 + 11376c^5m^3n^5d^2 - 528c^2m^6d^8n^2 \\
&+ 5968c^2m^4d^6n^4 - 10944c^3n^5d^4m^3 + 1728c^3n^7d^2m - 64c^3m^7d^8n \\
&+ 4672c^3m^5d^6n^3 + 6912c^4m^2d^2n^6 - 3888c^2m^2d^4n^6 + 3072m^3d^6n^5c \\
&- 1024m^5d^8n^3c,
\end{aligned}$$

$$\begin{aligned}
B_4 = &\ 16m^2n^2(2n+mc)^2(-d+c)^2(2d+c)^2(-2d+c)^2(d+c)^2(n+dm+mc)^2 \\
&\times (3n-dm+mc)^2(3n+dm+mc)^2(3cn+2c^2m-2d^2m)^2(n-dm+mc)^2.
\end{aligned}$$

With the values given above, the curve $E(c,d,m,n)$ has a point of order 4, as well as four rational points. Using specialization, the four rational points can easily be shown to be independent. For instance, when $(c,d,m,n) = (3,2,1,1)$, the height pairing matrix has determinant 357.065396133752 as computed by SAGE [13]. Thus P_1, P_2, P_3, and P_4 are independent.

4. An Infinite Family with Rank 5

Following the approach of Kihara's second paper [9], we seek to force a fifth point $P_5(p, M)$ on H. The point P_5 will only be rational if we have a rational solution to the equation

$$M^2 = 6s^2 + 3u^2 - 8r^2.$$

Substituting in the expresssions for r, s, and u in terms of c, d, m, and n, we note that the expression $6s^2 + 3u^2 - 8r^2$ is a quartic in m. In fact, this is expression is

$$((c+3d)(c^2-d^2))^2m^4 + \ldots + (3n^2(c-3d))^2.$$

If we set this equal to $(t_2m^2 + t_1m + t_0)^2$, where $t_2 = (c + 3d)(c^2 - d^2)$ and $t_0 = 3n^2(c - 3d)$ then a little bit of algebra finds that setting

$$t_1 = -cn\frac{5c^2 - 9cd - 32d^2}{c + 3d}$$

leads to $6s^2 + 3u^2 - 8r^2 = (t_2m^2 + t_1m + t_0)^2$, if

$$m = -12\frac{cdn(c + 3d)}{(c^2 - d^2)(3c^2 + 8cd + 12d^2)}.$$

This leads to an infinite family with five rational points, in terms of c, d, and n. To simplify the coefficients, we perform an isomorphism $(x, y) \to (k^2x, k^3y)$ where

$$k = \frac{(c^2 - d^2)^2(3c^2 + 8dc + 12d^2)^4}{36n^4(c^2 - 4d^2)^2}.$$

The resulting family is the curve $E : y^2 = x^3 + A_5x^2 + B_5x$, where A_5 and B_5 are homogenous polynomials in c and d. We can thus set $d = 1$, obtaining

$$\begin{aligned} A_5 = {} & 8748c^{19} + 67068c^{18} + 140940c^{17} - (668655/4)c^{16} - 986796c^{15} \\ & + 481455c^{14} + 11101764c^{13} + (68553243/2)c^{12} + 58405056c^{11} \\ & + 57810663c^{10} + 12960480c^9 - (219842399/4)c^8 - 89552688c^7 \\ & - 59540580c^6 - 2331072c^5 + 31437720c^4 + 27682560c^3 + 9844416c^2 \\ & + 2239488c + 419904, \end{aligned}$$

$$\begin{aligned} B_5 = {} & 36(c-1)^2(c+1)^2c^4(c-2)^2(c+3)^2(c+2)^2(3c^2+c+6)^2(3c^2+7c+6)^2 \\ & (3c^2+8c+12)^2(3c^2-13c-6)^2(3c^2+5c-6)^2(3c^2+2c+3)^2. \end{aligned}$$

We denote this curve by E_c, since the parametrization is only dependent on c (and not n or d). Thus, we have an infinite number of curves in this family with rank at least 5, which can be proved by specialization at $c = -6/5$, where the height pairing matrix has determinant 5062.58320537396.

To verify that this family is different than Kihara's family, let $j(t)$ be the j-invariant of the elliptic curve E_t given in Kihara's paper [9]. Let $j(c)$ be the j-invariant of the curve E_c given above. We checked that there are no solutions to the equation $j(t) - j(c)$, for any value of $c = a/b$, with $0 < |a|, b \leq 100$. If the two families were isomorphic, then there would exist solutions.

5. The Generators of the Rank 5 Family

Similarly as done in [4], we find the generators of the family E_c and prove the rank is 5 over $\mathbb{Q}(c)$. The key result needed is a theorem of Gusíc and Tadíc [7], for elliptic curves E given by $y^2 = x^3 + A(t)x^2 + B(t)x$, where $A, B \in \mathbb{Z}[t]$, with exactly one nontrivial 2-torsion point over $\mathbb{Q}(t)$. If $t \in \mathbb{Q}$ satisfies the condition that for every nonconstant square-free divisor h of $B(t)$ or $A(t)^2 - 4B(t)$ in $\mathbb{Z}[t]$ the rational number $h(t_0)$ is not a square in $\mathbb{Q}$, then the specialized curve E_{t_0} is elliptic and the specialization homomorphism at t_0 is injective.

If additionally there exist $P_1, \cdots, P_r \in E(\mathbb{Q}(t))$ such that $P_1(t_0), \cdots, P_r(t_0)$ are the free generators of $E(t_0)(\mathbb{Q})$, then $E(\mathbb{Q}(t))$and $E(t_0)(\mathbb{Q})$ have the same rank r, and $P_1, \cdots, P_r$ are the free generators of $E(\mathbb{Q}(t))$.

Just as in [4], the points P_i, for $i = 2, 3, 4, 5$, all satisfy $P_1 + P_i = 2Q_i$ for some point Q_i on $E(c)$. Concretely,

$$\begin{aligned}
Q_2 = {} & ((1/4(c-1))(c+1)(3c^2+c+6)(3c^2+7c+6)(3c^2-13c-6)(3c^2+5c-6) \\
& \times (9c^4+48c^3+115c^2+48c+36)^2, (1/8(9c^4+48c^3+115c^2+48c+36)) \\
& \times (3c^2+c+6)(c+1)(3c^2+5c-6)(3c^2+7c+6)(c-1)(9c^4-61c^2-96c-108) \\
& \times (3c^2-13c-6)c(216c^9+1449c^8+3624c^7+4446c^6+1728c^5-1103c^4-2784c^3 \\
& + 216c^2+3456c+1296)), \\
Q_3 = {} & (12c^3(c-1)(c+2)(c+1)(3c^2+c+6)(3c^2+8c+12)(3c^2-13c-6)(c+3)^2 \\
& \times (3c^2+5c-6)^2, 6(c+3)^2(c+2)(162c^{1}0+324c^9-459c^8-3840c^7-8880c^6 \\
& - 9924c^5-4175c^4+11040c^3+18360c^2+8640c+1296)(3c^2+8c+12) \\
& \times (3c^2+5c-6)^2(c+1)(c-1)c^3(3c^2+c+6)(3c^2-13c-6)), \\
Q_4 = {} & (12c^3(c-2)(3c^2+c+6)(3c^2+8c+12)(3c^2-13c-6)(c+3)^2(c+1)^2(6c^2-5c+6)^2 \\
& \times (3c^2+7c+6)^2/(7c+6)^2, 6(c+3)^2(c+2)(162c^{1}0+324c^9-459c^8-3840c^7-8880c^6 \\
& - 9924c^5-4175c^4+11040c^3+18360c^2+8640c+1296)(3c^2+8c+12)(3c^2+5c-6)^2 \\
& \times (c+1)(c-1)c^3(3c^2+c+6)(3c^2-13c-6)),
\end{aligned}$$

$$
\begin{aligned}
Q_5 = {} & (16c^3(c-1)(c-2)(c+3)(3c^2+2c+3)(3c^2+8c+12)(3c^2-13c-6)(3c^2+5c-6) \\
& \times (3c^2+7c+6)^2, 6(c+3)^2(c+2)(162c^10+324c^9-459c^8-3840c^7-8880c^6-9924c^5 \\
& -4175c^4+11040c^3+18360c^2+8640c+1296)(3c^2+8c+12)(3c^2+5c-6)^2(c+1) \\
& \times (c-1)c^3(3c^2+c+6)(3c^2-13c-6)).
\end{aligned}
$$

Checking the conditions of Gusíc and Tadíc's specialization theorem, a calculation shows $c = -6/5$ satisfies the squarefree requirements. Furthermore $E_{-6/5}$ has rank 5, with generators

$$
\begin{aligned}
W_1 = {} & (-942065755318848806144/95367431640625, -493277904978566951687763787 \\
& 776/931322574615478515625), \\
W_2 = {} & (-2647983756027101184/3814697265625, -720916856841461761306106 2656/3 \\
& 7252902984619140625), \\
W_3 = {} & (955663310445871104/19073486328125, 951313387939019346584744 7552/3725 \\
& 2902984619140625), \\
W_4 = {} & (9863389228799361024/95367431640625, 3595584771925801844733609 24672/93 \\
& 1322574615478515625), \\
W_5 = {} & (5527805029051604832092 16/11539459228515625, -42153339642395572521 4166 \\
& 054407766016/12395903468 13201904296875).
\end{aligned}
$$

It can be checked that (disregarding torsion), $Q_1 = -2W_1 + W_5$, $Q_2 = -2W_1 + W_2 + W_5$, $Q_3 = W_1$, $Q_4 = 2W_1 - W_4 - W_5$, $Q_5 = W_1 - W_3 - W_5$. The matrix of conversion has determinant 1.

6. Examples of curves with high rank

The highest known rank of an elliptic curve over $\mathbb{Q}$ with torsion subgroup $\mathbb{Z}/4\mathbb{Z}$ is rank 12 (see [2, 5]). From Dujella's table [2], there are also two known examples of curves with rank 11, and some elliptic curves with rank 10. Doing a computer search, we found two new curves with rank 11 and many curves with rank 10. We actually only list a few of the many rank 10 curves we found (over 40 rank 10 curves). Note that the curves listed below are all new, meaning they have never appeared in the literature (to the best of our knowledge). We refer to [2] for the details of the other high rank curves with torsion group $\mathbb{Z}/4\mathbb{Z}$.

A common strategy for finding high rank elliptic curves over $\mathbb{Q}$ is the construction of families of elliptic curves with high generic rank, and then searching for adequate specialization with efficient sieving tools. One popular tool is the Mestre-Nagao sum, see for example [10, 11]. These sums are of the form

$$S(n, E) = \sum_{p \leq n,\ p \text{ prime}} \left(1 - \frac{p-1}{\#E(\mathbb{F}_p)}\right) \log p. \tag{6.1}$$

For our search, we used the family of elliptic curves with rank at least 4 given in Section 3. We attempted to search the rank 5 family, but the large coefficients proved too much of an impediment in the calculations. Since the curve in Section 3 with parameters $[c, d, m, n]$ is isomorphic to the curve with parameters $[cm/n, dm/n, 1, 1]$, we can take $m = n = 1$. Using the Mestre-Nagao sums (6.1), we looked for those curves E with $S(523, E) > 20$ and $S(1979, E) > 32$. We ranged over the values $c = p/q$ and $d = r/s$, with $-100 \leq p \leq 120, 1 \leq q \leq 100, -100 \leq r \leq 100$ and $1 \leq s \leq 100$.

After this initial sieving, we calculated the rank of the remaining curves with `mwrank` [1], though we note we were not always able to determine the rank exactly. Table 1 summarises the results.

We give some details on the rank 11 curves. For the parameters $(c, d, m, n) = (99/2, 99/10, 1, 1)$ in the family from Section 3 we may write the first curve with rank equal to 11 in the form

$$\begin{aligned} y^2 + xy = x^3 &- 8359895892458790946485434683076630177 0x \\ &+ 2945584756350286890221962366255202390319646506418231089 00, \end{aligned}$$

Table 1: High rank elliptic curves with torsion subgroup $\mathbb{Z}/4\mathbb{Z}$

c	d	rank
99/2	99/10	11
108/71	-74/71	11
1	3/34	10
41/22	71/66	10
67/13	24/13	10
74/83	61/83	10
82/3	45	10
88/75	47/30	10
82/63	11/9	10
115/79	23/79	10
139/16	97/16	10
-89/55	13/22	10
-96/7	81	10
-98/39	35/13	10
-99/32	51/8	10
-1/28	41/70	10
-9/7	66/91	10

with the 11 rational points

$P_1 = [-376658071791198860, 1805528347444782339748789303 0],$

$P_2 = [-10533246148223735060, 254379054304084801844464903 0],$

$P_3 = [15646393449996084277898349826 0/33904961689, 165704052959916119272410328299126 17594508110/6243022310680637],$

$P_4 = [-885711103196628014898367036982440460/418008125605759441, 581009956292043808 342474664202960105645965798704050817 0/270257083714411845345707239],$

$P_5 = [231062783343861856620727144066 0/1518703775449, 244662015713267131101350527183 07507812214184590/1871585228601003293],$

$P_6 = [637984027011974650607003560/91718929, 6186779273059247399067481789000045078 39 0/878392183033],$

$P_7 = [-2787164183682069030974 0/2809, 1806199110727433199159170522308910/148877],$

$P_8 = [883414680732734546357429746 0/2255965009, 559250800133575059743294150493626041 731190/107151570032473],$

$P_9 = [135830457044382821003687057651862514 0/168942036456899281, 827898920114464004 9 483997882863838635543982428959411 30/694395003794405730005000 71],$

$P_{10} = [1694354547857797725807403203760/7739952529, 220360325100630052208992113623208 3170878234490/680937803643833],$

$P_{11} = [154143538630738814816283133954788005083334459928492868125827227071427138252 2 3124/54305095374461809030392666163215731920626834813916017776704 9, 577030171952$

The second curve with rank 11 has the parameters $(c, d, m, n) = (108/71, -74/71, 1, 1)$, and can be put in the form

$$
\begin{aligned}
y^2 + xy + y = x^3 - x^2 &- 67334058510805774154754542219325854993040 47x \\
&+ 48003254557985483908241440909204492663637885904804498365016001 19,
\end{aligned}
$$

with the 11 rational points

$$
\begin{aligned}
P_1 = [&-85904048046668413539940994 6637/357701569, -570851739448307375446298\\
&525772903969793199258/6765209774497],\\
P_2 = [&-866662884337837827654200301821689 46493/403989240106987441, -2027547\\
&3440123939103498192448687826487891735533319052199446/256776158512087\\
&335734025239],\\
P_3 = [&-74094409735977494521721607243178 53/20224187231161, -772709352834881\\
&228831369088396219202079393912664 7174/909508193470582990 91],\\
P_4 = [&-35731599195624446874160819005 33/2363029321, -1233150435614496041957\\
&0336229779531484558668734/114869218323131],\\
P_5 = [&50457757689930470323988196055818601/239245222129, 113333723486760862\\
&452556255013362749204856022951844 76/117021297764291383],\\
P_6 = [&75332028075844614055259981404446 7/192925628289, 165280395951455987 24\\
&6360474125117366436913008724 86/84739302490262337],\\
P_7 = [&1139959040731317261449545984982 7/3813186001, 2512861449644935888439 8\\
&093023284000254237912586/235468048747751],\\
P_8 = [&48033786325942023283804892481881 7/751461863161, -179284591717340572 4\\
&586418811703022872432009565232 6/6514189938565129 09],\\
P_9 = [&59551531952372825064410936761 87/220789881, -1446684474104748724311369\\
&0100138781156335681006/3280716841779],\\
P_{10} = [&1375362162769333581779723763/543169, -25271915734573400160308817 68\\
&3399948364442/400315553],\\
P_{11} = [&21920079230111700414086050318699/4818025, 102627348546574245945365\\
&125069231207443139803042/10575564875].
\end{aligned}
$$

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