On Upper Bounds for D2D Group Size

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***Abstract*—In this paper, we derive upper bounds for the number of Device-to-Device (D2D)-capable out-of-coverage (OOC) User Equipments (UEs) that can share the Physical Sidelink Discovery Channel (PSDCH) while maintaining a minimum probability of discovery message decoding. We maximize these upper bounds with respect to the UEs' transmission probability threshold by exploiting the fact that the upper bound is nearly linear with respect to the number of resources in the discovery resource pool. The resulting simple approximate bound is accurate over a large range of parameter values. We validate our results using Monte Carlo simulations in MATLAB and the ns-3 simulation tool.**

I. Introduction

Device-to-device (D2D) communications offer a means for improving cellular network efficiency by reducing the load at the base station due to intra-cell traffic. The Proximity Services (ProSe) working group in the Third Generation Partnership Project (3GPP) has defined standards for D2D communications for UEs that are within a base station's coverage area and also for out-of-coverage (OOC) UEs, i.e., those that are outside any cellular coverage [1]. The latter case affects public safety users who may be deployed to remote areas or who may have to operate in regions where cellular service is offline due to a natural or other causes.

Each User Equipment (UE) in a group of UEs must discover the D2D applications hosted by other UEs in the group before it can establish D2D sessions with them. OOC UEs implement the discovery function by transmitting discovery messages over the Physical Sidelink Discovery Channel (PSDCH). The UEs randomly choose Physical Resource Block (PRB) pairs from a periodically recurring discovery resource pool whose dimensions are N_t, the number of subframe sets allocated for a transmission, by N_f, the number of PRB pairs in the frequency domain. There are $N_r = N_f N_t$ PRB pairs, i.e., PSDCH resources, in the discovery resource pool. Because they choose resources randomly, multiple UEs can select the same resource, which causes message collisions that reduce the PSDCH's throughput and delay UE discovery. The ProSe standard tries to fix this problem by incorporating a transmission probability threshold, θ, that UEs can use to throttle their transmissions; a UE will transmit on the PSDCH only if an internally randomly generated number in the unit interval $[0, 1]$ is less than θ [2, Clause 5.15.1.1]. The value of θ determines how many UEs can participate in the D2D discovery process while maintaining a given level of performance, i.e., at least a minimum discovery message decoding probability.

Our previous work found the optimal value of θ that maximizes the probability of a successful discovery message transmission from one UE to another [3]. In this paper, we extend this work by deriving an expression for the maximum number of OOC UEs whose members' probability of decoding discovery messages is above a given minimum threshold. In Section II, we obtain an expression for the group size, N_u, as a function of other PSDCH parameters and use it to derive the maximum group size that supports at least a minimum discovery message decoding probability, $P_{\min}$. In Section III, we maximize the group size upper bound with respect to θ. Because direct methods are not tractable, we obtain approximate closed-form expressions for the critical value of θ, and use them to get the maximum group size upper bound. In Section IV, we validate our model using simulations and discuss the implications of the results, and in Section V, we summarize our results.

II. The Maximum Group Size

Let N_u be the number of OOC UEs. We assume that every UE can communicate with every other UE in its neighborhood, and that the UEs are using Model A discovery, i.e., they are continuously sending discovery announcement messages. In each discovery period, each UE generates a discovery message, selects a pool resource, and transmits its message with probability θ. We randomly select two UEs from their group and designate them as UE X and UE Y. From [3], $P_{Y\to X}$, the probability that UE X successfully decodes UE Y's discovery message during a given period, is

$$P_{Y\to X} = \theta\left(1 - \frac{\theta}{N_t}\right)\left(1 - \frac{\theta}{N_r}\right)^{N_u-2}. \tag{1}$$

Solving Eq. (1) for N_u gives

$$N_u = \left\lfloor 2 + \frac{\log\left(P_{Y\to X}/\theta\right) - \log\left(1 - \theta/N_t\right)}{\log\left(1 - \theta/N_r\right)}\right\rfloor, \tag{2}$$

Since $P_{Y\to X} \leq 1$, $\theta \leq 1$, and both N_t and N_r are positive integers, it follows from Eq. (1) that $P_{Y\to X} < \theta$ and $\log\left(P_{Y\to X}/\theta\right) < 0$.

Let $P_{\min}$ be the minimum allowable value of $P_{Y\to X}$. We are interested in determining the largest number of UEs that the

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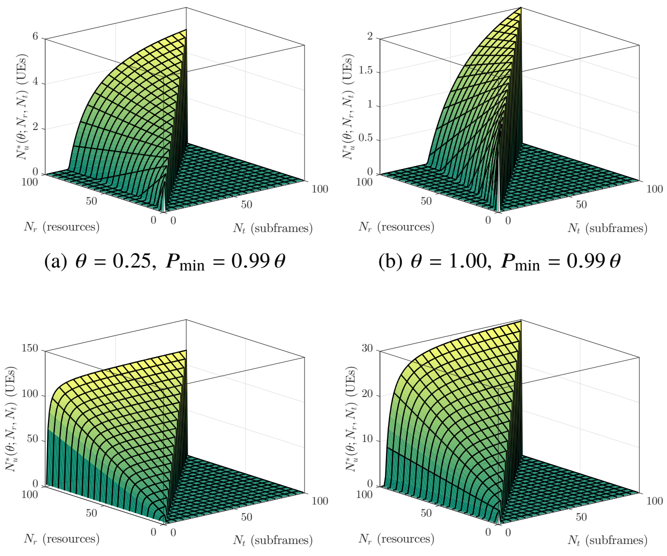


(a) $\theta = 0.25,\ P_{\min} = 0.99\,\theta$ (b) $\theta = 1.00,\ P_{\min} = 0.99\,\theta$

(c) $\theta = 0.25,\ P_{\min} = 0.75\,\theta$ (d) $\theta = 1.00,\ P_{\min} = 0.75\,\theta$

Fig. 1. Plots of $N_u^*(\theta; N_r, N_t)$ versus N_t and N_r, for various values of θ and $P_{\min}$.

discovery resource pool can support so that $P_{Y\to X} \geq P_{\min}$. The derivative of the argument of the floor function in Eq. (2) with respect to $P_{Y\to X}$ is $\big(P_{Y\to X}\log(1-\theta/N_r)\big)^{-1}$. Since $\log(1-\theta/N_r) < 0$, the argument of the floor function is monotonically decreasing with respect to $P_{Y\to X}$. Therefore the maximum value of N_u given $P_{Y\to X} \geq P_{\min}$ is

$$N_u^*(\theta; N_r, N_t) = \left\lfloor 2 + \frac{\log\big(P_{\min}/\theta\big) - \log\big(1-\theta/N_t\big)}{\log\big(1-\theta/N_r\big)} \right\rfloor. \quad (3)$$

Eq. (3) does not hold if $N_t > N_r$, since it is not possible for the number of subframes spanned by the resource pool to exceed the number of pool resources. Also, if $N_u^*(\theta; N_r, N_t) < 0$, then $P_{Y\to X} < P_{\min}$ for all values of N_u. This occurs when

$$N_t < \frac{\theta^2}{\theta - P_{\min}(1-\theta/N_r)^2} \leq \frac{\theta^2}{\theta - P_{\min}}, \quad (4)$$

where the higher upper bound is the limit of the lower upper bound as $N_r \to \infty$. Eq. (4) indicates that $P_{\min} > P_{Y\to X}$ for all values of N_u when N_t is sufficiently small, because of the half duplex effect.

Fig. 1 shows the effect of variations in θ and $P_{\min}$ on $N_u^*(\theta; N_r, N_t)$. In every subfigure in Fig. 1, we set $N_u^*(\theta; N_r, N_t) = 0$ when $N_t > N_r$, because the number of resources in the pool cannot be less than the number of subframes spanned by the pool. Also, when Eq. (4) holds, we set $N_u^*(\theta; N_r, N_t) = 0$, which produces the flat regions on the left side of each plot in Fig. 1a and Fig. 1b. Also, setting $P_{\min}$ close to θ reduces $N_u^*(\theta; N_r, N_t)$; for example, Fig. 1b shows that when $N_r = 100$ resources, at most two UEs can achieve $P_{Y\to X} \geq 0.99$ when $\theta = 1$.

Fig. 1 shows that the slope of the $N_u^*(\theta; N_r, N_t)$ surface is nearly linear in the N_r direction when N_r and N_t are large. The first and second order partial derivatives of $N_u^*(\theta; N_r, N_t)$ with respect to N_r are

$$\frac{\partial N_u^*(\theta; N_r, N_t)}{\partial N_r} = -\frac{\theta\big(\log(P_{\min}/\theta) - \log(1-\theta/N_t)\big)}{N_r(N_r-\theta)\log^2(1-\theta/N_r)} \quad (5)$$

and

$$\frac{\partial^2 N_u^*(\theta; N_r, N_t)}{\partial N_r^2} = \big(\log(P_{\min}/\theta) - \log(1-\theta/N_t)\big) \times \frac{2\theta^2 + (2\theta N_r - \theta^2)\log(1-\theta/N_r)}{N_r^2(N_r-\theta)^2\log^3(1-\theta/N_r)}. \quad (6)$$

From Eq. (5), when N_r is large the derivative approaches

$$\lim_{N_r\to\infty} \frac{\partial N_u^*(\theta; N_r, N_t)}{\partial N_r} = \frac{\log(1-\theta/N_t) - \log(P_{\min}/\theta)}{\theta}. \quad (7)$$

As N_t increases, the slope in Eq. (7) approaches a constant value: $\lim_{N_r\to\infty, N_t\to\infty} \partial N_u^*(\theta; N_r, N_t)/\partial N_r \approx -\log(P_{\min}/\theta)/\theta$. We confirm this by letting $N_r \to \infty$ in Eq. (6), which gives $\lim_{N_r\to\infty} \partial^2 N_u^*(\theta; N_r, N_t)/(\partial N_r^2) = 0$.

Thus, for any value of N_t, we can approximate $N_u^*(\theta; N_r, N_t)$ as a linear function of N_r. We use a Taylor series expansion about the point $N_r = N_t$. If we retain only the constant and linear terms, we get

$$N_u^*(\theta; N_r, N_t) \approx \left(1 + \frac{\log(P_{\min}/\theta)}{\log(1-\theta/N_t)}\right) - \frac{\theta\left(\log(P_{\min}/\theta) - \log(1-\theta/N_t)\right)}{N_t(N_t-\theta)\log^2(1-\theta/N_t)}(N_r - N_t). \quad (8)$$

We want to find the maximum group size, given a pool of N_r resources, over the range of values for N_t. We define this upper bound as follows:

$$N_u^{\max}(\theta; N_r) = \max_{N_t} N_u^*(\theta; N_r, N_t). \quad (9)$$

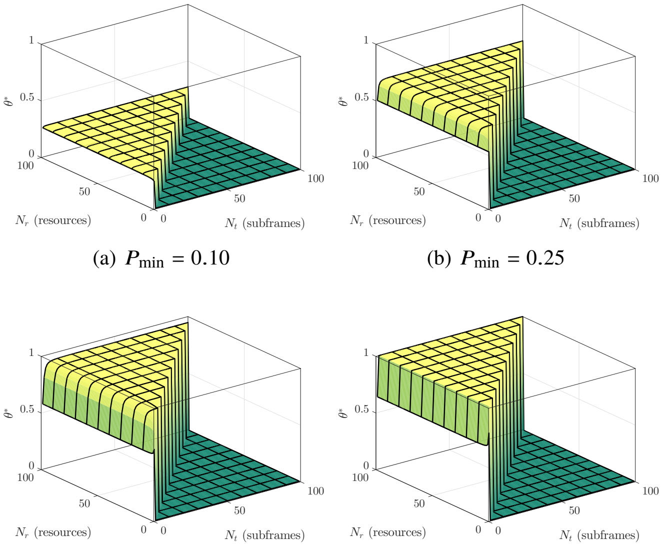
We can use Eq. (3) to derive $N_u^{\max}(\theta; N_r)$. The derivative of the argument of the floor function in Eq. (3) with respect to N_t is $-\theta/[N_t(N_t-\theta)\log(1-\theta/N_r)]$. Since $\log(1-\theta/N_r) < 0$ and $\theta < N_t \leq N_r$, the derivative is positive over the interval of interest, so that $N_u^*(\theta; N_r, N_t)$ is a non-decreasing function of N_t. Setting $N_t = N_r$ gives

$$N_u^{\max}(\theta; N_r) = \left\lfloor 1 + \frac{\log\big(P_{\min}/\theta\big)}{\log\big(1-\theta/N_r\big)} \right\rfloor \geq N_u^*(\theta; N_r, N_t). \quad (10)$$

III. Maximum Group Size Bounds with Respect to θ

In this section, we maximize N_u^* in Eq. (3) and $N_u^{\max}$ in Eq. (10) with respect to θ. We assume that all N_u UEs use the same value for θ. We let θ^* be the value of θ that maximizes $N_u^*(\theta; N_r, N_t)$, and we let $\theta^{\max}$ be the value of θ that maximizes $N_u^{\max}(\theta; N_r)$.

We can simplify the problem by showing that θ^* is nearly constant with respect to N_r and N_t, and by showing that $\theta^* \approx \theta^{\max}$ unless N_t is small. Consider a set of pool dimensions where $N_r \leq 100$ resources and $N_t \leq N_r$ subframes. For each ordered pair (N_r, N_t), we use numerical methods to find θ^*,



(a) $P_{\min} = 0.10$ (b) $P_{\min} = 0.25$

(c) $P_{\min} = 0.35$ (d) $P_{\min} = 0.50$

Fig. 2. Plots of θ^*, which is the value of θ that minimizes Eq. (3), versus N_t and N_r, for various values of $P_{\min}$.

and we use the following values of $P_{\min}$: 0.10, 0.25, 0.35, and 0.50; the resulting plots of θ^* are shown in Fig. 2.

Fig. 2 shows that θ^* is insensitive to N_r and N_t, except when N_t is small (i.e., $N_t < 10$ subframes). Fig. 2a shows that when $P_{\min}$ is small, θ^* is nearly constant over all values of N_r and N_t, and Fig. 2d shows that θ^* varies with respect to N_t only when $N_t = 1$ subframe. Fig. 2b and Fig. 2c show more variation of θ^* with respect to N_t, but this occurs when $N_t < 10$ subframes.

The insensitivity of θ^* to N_r and N_t, and the fact that $\theta^* = \theta^{\max}$ when $N_r = N_t$, implies that $\theta^* \approx \theta^{\max}$ unless N_t is very small. Additional analysis shows that if $P_{\min} = 1/\mathsf{e}$, $\theta^* = 1$ over almost the entire set of values for N_t and N_r. Thus, we can obtain a good approximation for θ^* by finding $\theta^{\max}$.

As we will show, $N_u^{\max}(\theta; N_r)$ is linear with respect to the pool size, N_r. We expand $N_u^{\max}(\theta; N_r)$ in a Taylor series about $N_r = a$:

$$\begin{aligned} N_u^{\max}(\theta; N_r) = \Bigg\lfloor & \left(1 + \tfrac{\log(P_{\min}/\theta)}{\log(1-\theta/a))}\right) - \tfrac{\theta \log(P_{\min}/\theta)}{a(a-\theta)\log^2(1-\theta/a))}(N_r - a) \\ & + \tfrac{1}{2}\tfrac{\log(P_{\min}/\theta)\left[2\theta^2 - (\theta^2 - 2a\theta)\log(1-\theta/a)\right]}{a^2(a-\theta)^2\log^3(1-\theta/a)}(N_r - a)^2 \\ & + O\big((N_r - a)^3\big)\Bigg\rfloor. \end{aligned} \tag{11}$$

Derivatives of order two and higher in Eq. (11) vanish as a becomes very large, so the limit as $a \to \infty$ is

$$\begin{aligned} \lim_{a\to\infty} N_u^{\max}(\theta; N_r) &= \Bigg\lfloor \lim_{a\to\infty}\left(1 + \tfrac{[\theta + (a-\theta)\log(1-\theta/a)]\log(P_{\min}/\theta)}{(a-\theta)\log^2(1-\theta/a)}\right) \\ &\qquad - N_r \lim_{a\to\infty} \tfrac{\theta\log(P_{\min}/\theta)}{a(a-\theta)\log^2(1-\theta/a)}\Bigg\rfloor \\ &= \left\lfloor 1 + \left(\tfrac{1}{2} - \tfrac{N_r}{\theta}\right)\log(P_{\min}/\theta)\right\rfloor. \end{aligned} \tag{12}$$

To find our approximate value for $\theta^{\max}$, we take the derivative of the argument of the floor function with respect to θ:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[1 + \left(\tfrac{1}{2} - \tfrac{N_r}{\theta}\right)\log\left(\tfrac{P_{\min}}{\theta}\right)\right] = \frac{2N_r(1 + \log(P_{\min}/\theta)) - \theta}{2\theta^2}. \tag{13}$$

Setting the numerator in Eq. (13) equal to zero and rearranging the resulting expression gives:

$$\frac{\mathsf{e}P_{\min}}{2N_r} = \frac{\theta^{\max}}{2N_r}\mathsf{e}^{\theta^{\max}/(2N_r)}. \tag{14}$$

To solve Eq. (14) for $\theta^{\max}$, we use Lambert's W function, $\mathsf{W}(z)$, which is defined $\forall z \in \mathbb{C}$ as $z = \mathsf{W}(z)\exp\big(\mathsf{W}(z)\big)$ [4, Eq. (1.5)]. We match Eq. (14) to the W function's definition by letting $z = \mathsf{e}P_{\min}/(2N_r)$ and $\mathsf{W}(z) = \theta^{\max}/(2N_r)$, giving

$$\theta^{\max} = 2N_r\,\mathsf{W}(z) = 2N_r\,\mathsf{W}\big(\mathsf{e}P_{\min}/(2N_r)\big). \tag{15}$$

We also obtain a simple approximate value for $\theta^{\max}$ as follows. The Taylor series expansion of the W function about the point $x = 0$ has the form [4, Eq. (3.1)]

$$\mathsf{W}(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!}\,x^n, \tag{16}$$

and from this and Eq. (15), we get the series expansion for $\theta^{\max}$:

$$\theta^{\max} = \mathsf{e}P_{\min} - \frac{(\mathsf{e}P_{\min})^2}{2N_r} + O\left(\frac{(P_{\min})^3}{N_r^2}\right). \tag{17}$$

Since $P_{\min} > 0$ and $\mathsf{W}(0) = 0$, letting $N_r \to \infty$ gives

$$\lim_{N_r\to\infty} \theta^{\max} = \mathsf{e}P_{\min}\exp\big(-\mathsf{W}(0)\big) = \mathsf{e}P_{\min}, \tag{18}$$

Thus we have the following approximation for $\theta^{\max}$:

$$\theta^{\max} \approx \begin{cases} \mathsf{e}P_{\min}, & 0 < P_{\min} \le 1/\mathsf{e} \\ 1, & 1/\mathsf{e} < P_{\min} \le 1 \end{cases} \tag{19}$$

Using Eq. (19) with Eq. (10), we get the following approximate expression for the maximum group size:

$$N_u^{\max}(\theta; N_r) \approx \begin{cases} \left\lfloor 1 - \frac{1}{\log\left(1 - \frac{\mathsf{e}P_{\min}}{N_r}\right)}\right\rfloor, & 0 < P_{\min} \le 1/\mathsf{e} \\ \left\lfloor 1 + \frac{\log(P_{\min})}{\log\left(1 - \frac{1}{N_r}\right)}\right\rfloor, & 1/\mathsf{e} < P_{\min} \le 1 \end{cases} \tag{20}$$

If we expand the two terms in Eq. (20) as a pair of Taylor series about $N_r = a$ and then let $a \to \infty$, we get

$$1 - \frac{1}{\log\left(1 - \frac{\mathsf{e}P_{\min}}{N_r}\right)} = \frac{N_r}{\mathsf{e}P_{\min}} + \frac{1}{2} - \frac{\mathsf{e}P_{\min}}{12N_r} - \frac{\mathsf{e}^2P_{\min}^2}{24N_r^2} + O\left(\left(\frac{1}{N_r}\right)^3\right) \tag{21a}$$

$$\begin{aligned} 1 + \frac{\log(P_{\min})}{\log\left(1 - \frac{1}{N_r}\right)} &= -N_r\log(P_{\min}) + \left(1 + \frac{\log(P_{\min})}{2}\right) \\ &\quad + \frac{\log(P_{\min})}{12N_r} + \frac{\log(P_{\min})}{24N_r^2} + O\left(\left(\frac{1}{N_r}\right)^3\right). \end{aligned} \tag{21b}$$

If N_r is large, we can use Eq. (21) to get the following approximate expression for $N_u^{\max}(\theta; N_r)$:

$$N_u^{\max}(\theta; N_r) \approx \begin{cases} \left\lfloor \frac{1}{2} + N_r/(\mathrm{e}P_{\min}) \right\rfloor, & 0 < P_{\min} \le 1/\mathrm{e} \\ \left\lfloor 1 - \left(N_r - \frac{1}{2}\right)\log(P_{\min}) \right\rfloor, & 1/\mathrm{e} < P_{\min} \le 1 \end{cases} \tag{22}$$

The approximations are very close to the actual values. In Fig. 3 we plot the approximation error for $\theta^{\max}$ and $N_u^{\max}(\theta; N_r)$ versus N_r. First, in Fig. 3a, we plot the absolute error between the exact value of $\theta^{\max}$ and its approximate value from Eq. (19). The error decreases as N_r increases and $P_{\min}$ decreases. By increasing $P_{\min}$, the error increases to an asymptotic value associated with $P_{\min} = 1/\mathrm{e}$, for any N_r. Beyond $P_{\min} = 1/\mathrm{e}$, $\theta^{\max} = 1$, and the error remains at the level associated with $P_{\min} = 1/\mathrm{e}$. The error is significant for high values of $P_{\min}$ and small pool sizes; for example, it is greater than 0.1 for $N_r < 5$ resources when $P_{\min} \ge 1/\mathrm{e}$. However, discovery resource pools that are so small do not seem likely to be implemented in practical systems. For $N_r > 50$ resources, the greatest error in the value of $\theta^{\max}$ is less than 0.01, and the error is less than 0.001 for $N_r > 500$ resources.

We plot the absolute error in $N_u^{\max}(\theta; N_r)$ versus N_r in Fig. 3b, comparing the exact value from Eq. (10) with the approximation from Eq. (22). We use a different set of values for $P_{\min}$ to show the effect when $P_{\min} > 1/\mathrm{e}$. In this case, the error is less than 1/2 over the full range of values of N_r. Thus, the approximation in Eq. (22) is accurate enough that it can be used in all cases.

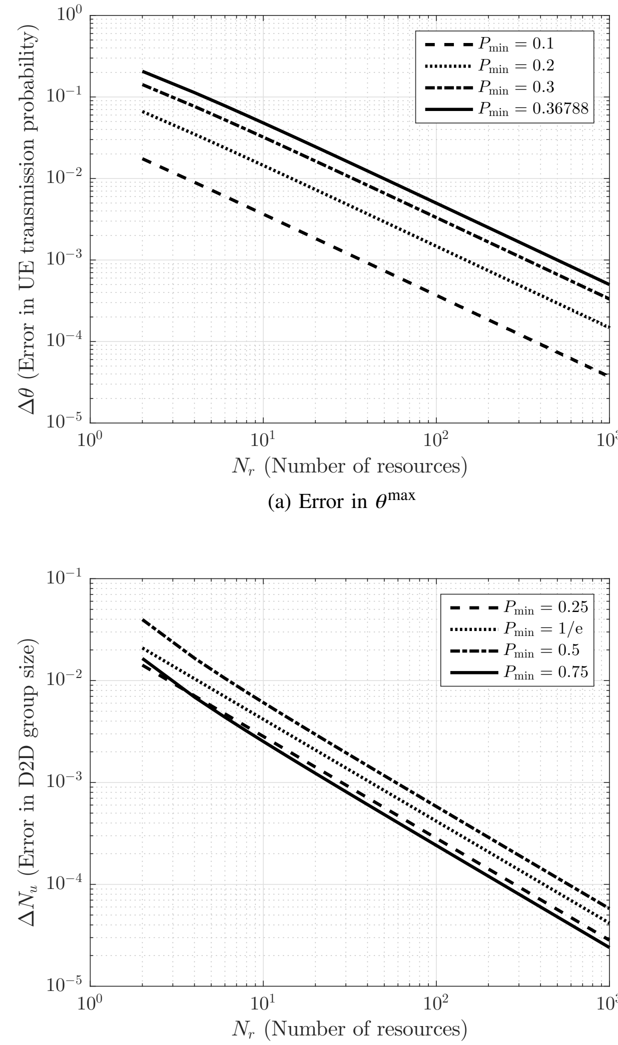
IV. Validation and Numerical Results

We validated our results from Section II and Section III by performing Monte Carlo simulations of discovery pool resource selection in MATLAB, and we compared the resulting maximum group size with what was predicted by Eqs. (3) and (10). We also examined the impact of the discovery pool dimensions on the group size upper bound by performing simulations in ns-3 of a group of OOC UEs exchanging discovery messages over the PSDCH, in which we obtained the maximum group size $N_u^*(\theta; N_r, N_t)$ versus $P_{\min}$, using the following four values for θ: 1/4, 1/2, 3/4, and 1.

A. Monte Carlo Simulations

We simulated the discovery resource selection process for a discovery pool size of $N_r = 40$ resources, using the parameters shown in Table I. As the extent of the resource pool in the time domain decreases, the half-duplex effect has a greater impact on the ability of UEs to decode discovery messages. This will reduce the maximum OOC group size that can achieve $P_{Y \to X} \ge P_{\min}$.

For each set of pool parameters, we used the following three values for $P_{\min}$: 0.2, 0.3, and 0.4. We used Eq. (19) to generate the following respective values of θ: 0.5437, 0.8155, and 1. We assigned the generated value of θ to each UE in the OOC group, and we computed $N_u^*(\theta; N_r, N_t)$ and $N_u^{\max}(\theta; N_r)$, which are listed in Table I.



(a) Error in $\theta^{\max}$

(b) Error in $N_u^{\max}(\theta; N_r)$

Fig. 3. Errors between approximate and exact values of $\theta^{\max}$ and $N_u^{\max}(\theta; N_r)$

TABLE I
Monte Carlo Simulation Parameters

N_f	N_t	$P_{\min}$	θ	N_u^*	$N_u^{\max}$
1	40	0.2	0.5437	74	74
		0.3	0.8155	49	49
		0.4	1.0000	37	37
4	10	0.2	0.5437	70	74
		0.3	0.8155	46	49
		0.4	1.0000	34	37
10	4	0.2	0.5437	64	74
		0.3	0.8155	39	49
		0.4	1.0000	26	37

For each set of pool parameters and value of $P_{\min}$, we simulated N_u UEs, where N_u varied from $N_u^* - 2$ to $N_u^{\max} + 2$, because we are interested in examining the behavior of $P_{Y \to X}$ in the vicinity of the theoretical upper bounds on N_u. For each value of N_u we performed $N_{\text{runs}} = 100$ runs, with $N_{\text{trials}} = 1000$ trials per run. For each trial, we designated a random UE as UE_0, the transmitter of interest, and we chose a second random UE to be the receiver. Each UE independently generated a random transmission decision variable, $p1$. If $p1 \leq \theta$, the UE transmitted; otherwise, it did not. UEs that decided to transmit discovery messages chose one of the N_r resources in the pool with uniform probability. Each UE's resource choice was independent of the choices of all other UEs. UEs kept track of which subframe set they were using to send their discovery messages. The trial was a success if UE_0 decided to transmit, no other UE chose UE_0's resource, and the receiver did not transmit in the same subframe set as UE_0. Any other outcome was a failure.

The estimated value of $P_{Y\to X}$ for each run is the ratio of the number of successful trials to the number of trials in the run. The output of each simulation was a set of N_{runs} estimates for $P_{Y\to X}$: $\{\widehat{P}_{Y\to X}(n)\}_{n=1}^{N_{\text{runs}}}$. The estimate of $P_{Y\to X}$ is

$$\widehat{P}_{Y\to X} = \frac{1}{N_{\text{runs}}} \sum_{n=1}^{N_{\text{runs}}} \widehat{P}_{Y\to X}(n) \tag{23}$$

The standard deviation of the set of estimates of $P_{Y\to X}$ is:

$$\widehat{\sigma}_{\widehat{P}_{Y\to X}} = \sqrt{\frac{1}{N_{\text{runs}}-1} \sum_{n=1}^{N_{\text{runs}}} \left(\widehat{P}_{Y\to X}(n) - \widehat{P}_{Y\to X}\right)^2} \tag{24}$$

Using Eqs. (23) and (24), we plot the 95 % confidence interval associated with the simulation result, whose limits are

$$\widehat{P}_{Y\to X} \pm 1.96\,\widehat{\sigma}_{\widehat{P}_{Y\to X}} / \sqrt{N_{\text{runs}}}. \tag{25}$$

Fig. 4 shows the simulation results that we obtained using the parameters listed in Table I. In each subfigure, we plot $P_{Y\to X}$ versus N_u for a given set of pool parameters, with different markers for each of the three values of $P_{\min}$ that we used. We show the theoretical values of N_u^*, listed in Table I, in each subfigure with additional ticks on the N_u-axis. In each subfigure, $P_{Y\to X}$ decreases with respect to N_u, so that N_u^* is the largest value of N_u such that $P_{Y\to X} \geq P_{\min}$, which validates the model. The slope of $P_{Y\to X}$ decreases as $P_{\min}$ decreases, so that $P_{Y\to X}$ becomes less sensitive to N_u. Note that in Fig. 4a, $N_u^* = N_u^{\max}$. Also, the sensitivity of N_u^* to $P_{\min}$ increases as $P_{\min}$ becomes small.

Figs. 4b and 4c show the effect of modifying the discovery resource pool parameters while keeping N_r constant. For a pool with $N_f = 4$ PRB pairs and $N_t = 10$ subframe sets, N_u^* is below $N_u^{\max}$ by 5.71 %, 6.52 %, and 8.82 % for $P_{\min}$ values of 0.2, 0.3, and 0.4, respectively. The effect is greater when $N_f = 10$ PRBs and $N_t = 4$ subframe sets, where N_u^* is below $N_u^{\max}$ by 15.63 %, 25.64 %, and 42.31 % for $P_{\min}$ values of 0.2, 0.3, and 0.4, respectively.

B. Simulations using ns-3

Our ns-3 simulations examine the impact of the pool parameters on the group size upper bound. We examined four values of θ: 0.25, 0.50, 0.75, and 1.00. For each value of θ, we created groups of UEs whose sizes were multiples of 15, up to a maximum of 150 UEs, and we examined groups consisting of 2 UEs. All UEs used the same value of θ. For each combination of group size and θ, we considered the same three discovery resource pool configurations that we examined in Section IV-A, each of which has $N_r = 40$ resources.

For each set of parameters, we conducted 10 runs, with 10 trials per run. We obtained estimates of $P_{Y\to X}$ for each trial, and obtained estimates of the mean and standard deviation using Eqs. (23) and (24), and used Eq. (25) to obtain 95 % confidence intervals. Fig. 5 shows the results; we use connecting lines to make the graphs legible and to illustrate trends in the data.

We compared our results to the theoretical model and found excellent agreement; we do not show the theoretical results in Fig. 5 for the sake of clarity in the subfigures. The figures collectively show that the group size upper bounds are most sensitive to $P_{\min}$ when θ is large, since increasing each UE's transmission rate increases the offered load, which increases the collision rate. Also, the effect of the pool parameters on the upper bound is more pronounced when $P_{\min}$ is small and when θ is large. The plots show that if we want $P_{\min}$ to be close to θ (e.g., within 90 %), N_u must be small. For example, when $\theta = 0.25$, we can satisfy $P_{\min} \approx 0.9\,\theta$ with a group that comprises at most about 15 UEs. Larger upper bounds for the group size are achievable only with a larger resource pool, preferably one where $N_t > N_f$.

V. Summary

In this paper, we derived an upper bound for the size of a group of OOC UEs whose members' probability of decoding discovery messages is at least $P_{\min}$. We showed that the value of θ that maximizes the group size upper bound is insensitive to the discovery pool dimensions. By using a linear approximation for the upper bound, we developed a closed-form expression for the optimal value of θ. We then showed that the optimal value of θ is proportional to $P_{\min}$, which yields a simple expression for the maximum group size. We validated our results using both Monte Carlo simulations in MATLAB and simulations of the PSDCH used by OOC UEs in the ns-3 network simulation tool.

References

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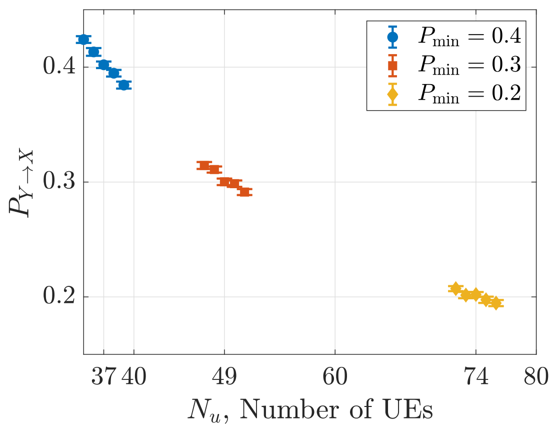
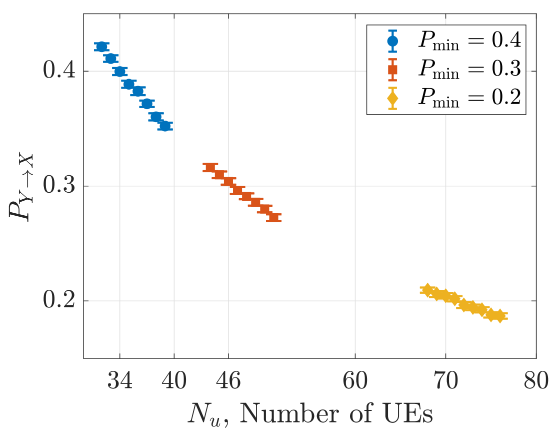
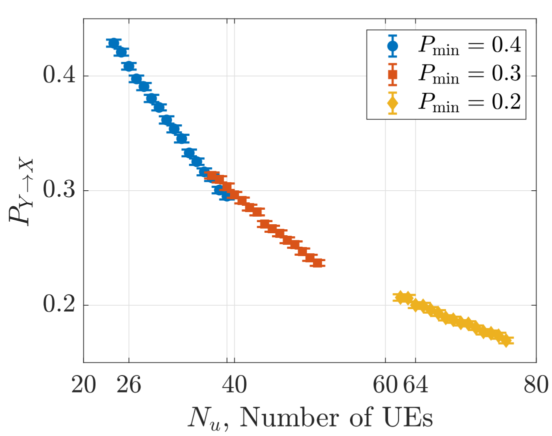


(a) N_f = 1 PRB and N_t = 40 subframe sets

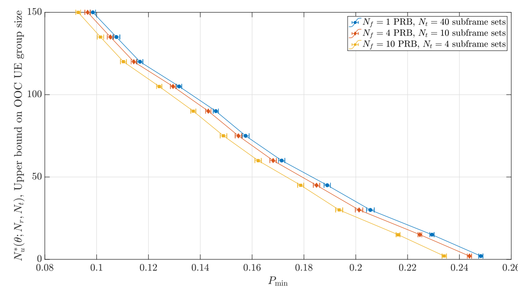


(b) N_f = 4 PRBs and N_t = 10 subframe sets

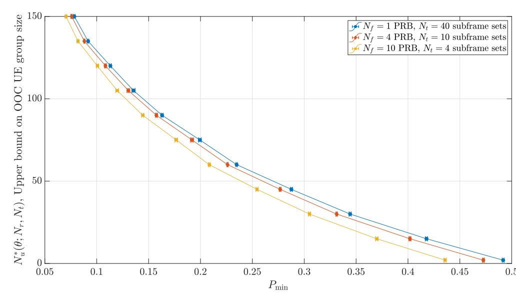


(c) N_f = 10 PRBs and N_t = 4 subframe sets

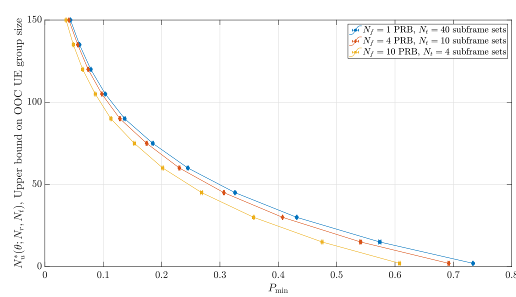
Fig. 4. Monte Carlo simulation results showing estimates of $P_{Y \to X}$ versus OOC UE group size N_u, for various pool size parameters and values of $P_{\min}$, with 95 % confidence intervals shown.



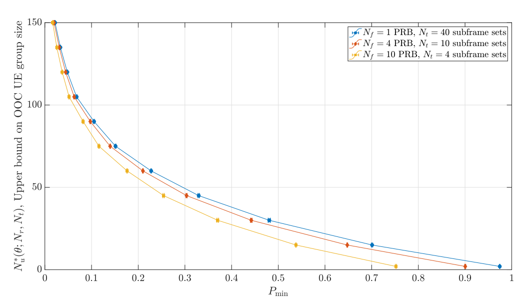
(a) θ = 0.25



(b) θ = 0.50



(c) θ = 0.75



(d) θ = 1.00

Fig. 5. Simulation results from ns-3 showing estimates of $N_u * (\theta; N_r, N_t)$ versus $P_{\min}$, for various values of θ, with 95 % confidence intervals shown.