Quantum Estimation of Parameters of Classical Spacetimes

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We describe a quantum limit to measurement of classical spacetimes. Specifically, we formulate a quantum Cramér-Rao lower bound for estimating the single parameter in any one-parameter family of spacetime metrics. We employ the locally covariant formulation of quantum field theory in curved spacetime, which allows for a manifestly background-independent derivation. The result is an uncertainty relation that applies to all globally hyperbolic spacetimes. Among other examples, we apply our method to detection of gravitational waves with the electromagnetic field as a probe, as in laser-interferometric gravitational-wave detectors. Other applications are discussed, from terrestrial gravimetry to cosmology.

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I. INTRODUCTION

The geometry of spacetime can be inferred from physical measurements made with clocks and rulers or, more generally, with quantum fields, sources, and detectors. We assume that the ultimate precision achievable is determined by quantum mechanics. In this paper we obtain parameter-based quantum uncertainty relations that bound the precision with which we can determine properties of spacetime in terms of stress-energy variances. Such uncertainty relations might become increasingly relevant to empirical observation as, for example, laser-interferometric gravitational-wave detectors are expected to approach quantum-limited sensitivity across a wide bandwidth in the near future.

An informative, high-level way to quantify the precision of a parameter measurement is by the inverse variance $\langle(\delta\tilde{\theta})^2\rangle$ of an estimator $\tilde{\theta}$. The best precision with which we can measure a parameter is determined by the quantum Fisher information [1, 2]. For pure states, the Fisher information reduces to a multiple of the variance $\langle(\Delta\hat{P})^2\rangle$ of an evolution operator $\hat{P}$ that describes how the quantum state changes with changes in the parameter. This determines a parameter-based uncertainty relation [3, 4],

$$\langle(\delta\tilde{\theta})^2\rangle\langle(\Delta\hat{P})^2\rangle \geq \frac{\hbar^2}{4}\ , \tag{1.1}$$

whose form is reminiscent of Heisenberg uncertainty relations.

Such parameter-based uncertainty relations can be applied to parameters associated with local changes of the spacetime metric. They are derived from a universal connection between local changes in the metric and relative changes in the states of quantum fields that live on the spacetime. These changes, used to sense the spacetime parameters, can be characterized in terms of an evolution of the state driven by a stress-energy integral with respect to the change in the metric, which gives the operator $\hat{P}$.

For the universal connection between changes in states of quantum fields and the stress-energy integrals, we rely on the locally covariant formalism for quantum fields on curved spacetime backgrounds [5]. For this purpose, we treat gravity classically as in general relativity, determined by a metric with signature $(-,+,+,+)$ on a spacetime manifold. This is treated as a fixed background on which the quantum fields used for measurements—we call these "probe fields"—propagate. In particular, we do not consider back action from the quantum fields on the metric. In any case, we expect that such back action would transfer uncertainty in the quantum field being measured to the metric and reduce the achievable precision, for otherwise, by an argument given in [6, 7], the uncertainty principle for matter would be violated (see also [8] for a study of the problem of back action when measuring the structure of spacetime).

We allow for the presence of classical fields that can propagate on the spacetime background. We only consider those fields that play a direct role as sources for the quantum fields used for measurement. These sources are determined by classical devices needed to implement the measurement. For the types of measurement considered here, the parametrized changes of the metric are independent of these equipment-related classical fields. In particular, these classical fields contain no information about the parameter of interest, and for this reason we do not model them explicitly.

Our work in this paper relies on the parameter-based uncertainty relation (1.1). To put such uncertainty relations in context, we consider now how the familiar Heisenberg uncertainty relations generalize to parameter-based uncertainty relations. The Heisenberg uncertainty

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relation for position and momentum states that the product of the uncertainties in position and momentum, for any quantum state, must be greater than a constant. In terms of the variances of position and momentum, the Heisenberg uncertainty relation is written as

$$\langle(\Delta\hat{x})^2\rangle\langle(\Delta\hat{p})^2\rangle \geq \frac{\hbar^2}{4} \ , \tag{1.2}$$

where $\hbar$ is the reduced Planck's constant. The Heisenberg uncertainty relation is derived in standard quantum mechanics, where position and momentum are both represented as Hermitian operators.

A similar relation, albeit with a different interpretation, exists between time and energy:

$$\langle(\delta t)^2\rangle\langle(\Delta\hat{H})^2\rangle \geq \frac{\hbar^2}{4} \ . \tag{1.3}$$

Unlike position, time in standard quantum mechanics is a classical parameter. The time-energy uncertainty relation is an example of a quantum limit on parameter estimation. One tries to estimate a classical parameter, in this case time, by making measurements on a quantum system, a "clock," whose evolution depends on time. In the time-energy uncertainty relation (1.3), $\langle(\delta t)^2\rangle$ is the classical variance of the estimate of t; this classical variance arises ultimately from quantum uncertainties in clock variables conjugate to the Hamiltonian H.

The most common way to make quantum mechanics compatible with classical relativity is to demote position to a parameter, just like time in the previous example. Physical systems can then be thought of as living on, and interacting with, the classical spacetime manifold. In this relativistic context, the position-momentum uncertainty relation naturally becomes a quantum limit on parameter-estimation [3].

This parameter-based approach is the natural, operational way to think of uncertainty relations. Nothing in the traditional Heisenberg uncertainty relation (1.2) refers directly to a measurement of position or momentum. In the parameter-based approach, one considers measurements of any sort whose results are used to estimate changes in a parameter; quantum mechanics then says, via the quantum Fisher information, that the uncertainty of the estimate is limited by the uncertainty in the operator that generates changes in the parameter, in a way that looks like, but is more powerful by being operational, a traditional Heisenberg uncertainty relation.

This approach was used by Braunstein, Caves, and Milburn [4] to develop optimal quantum estimation for spacetime displacements in flat Minkowski spacetime. In spacetime, not only can one move a fixed proper distance or time, one can also boost and rotate. Quantum parameter estimation was thus also developed for the parameters corresponding to these actions [4]. The results were developed with the quantized electromagnetic field as the probe. These results show that estimates of a spacetime translation can be made increasingly accurate as the uncertainty in the operator that generates the translation, boost, or rotation is made very large.

In this paper we are interested in limits on the precision of estimates of parameters of the classical gravitational field. In general relativity the gravitational field is a manifestation of the geometry of spacetime, which is described by a metric. The metric determines the length of the invariant (proper) interval between two spacetime events according to [9]

$$ds^2 = \mathrm{g}_{\mu\nu}(x)dx^\mu dx^\nu \ , \tag{1.4}$$

where $\mathrm{g}_{\mu\nu}(x)$ is the metric tensor, with indices $\mu,\nu = 0,1,2,3$ for time and the three spatial coordinates. The dx^μ are infinitesimal coordinate differences. We assume the Einstein summation convention, where repeated upper and lower indices are summed over.

Before we describe the relevant quantum parameter estimation, we ask the following: Can any insight be found by applying the Heisenberg uncertainty relation, in its parameter-based form, directly to a proper distance? It was along these lines that Unruh [10] derived an uncertainty relation for a component of the metric tensor. Once coordinates are chosen, there should only be quantum uncertainty in the proper time and the proper distance. As these are in turn related to the metric via Eq. (1.4), any uncertainty in the proper distance is equivalent to uncertainty in the metric. By applying the Heisenberg uncertainty relation to a proper distance, Unruh found a simple, yet insightful uncertainty relation for a single component of the metric. The Unruh uncertainty relation for the g_{11} component of the metric (assuming particular Cartesian-like coordinates), in terms of variances, is

$$\langle(\delta\mathrm{g}_{11})^2\rangle\langle(\Delta\hat{T}^{11})^2\rangle \geq \frac{\hbar^2}{V^2} \ , \tag{1.5}$$

where here, and henceforth, we use units such that $G = c = 1$. The conjugate variable to g_{11} is the corresponding component of the quantized stress-energy tensor, in this case the pressure $\hat{T}^{11}$ in the x^1 direction. The key feature of this uncertainty relation is the inverse proportionality to V^2, the square of the four-volume of the measurement.

We provide a general framework for deriving such an uncertainty relation, by formulating it as a problem in quantum estimation theory. The metric $\mathrm{g}_{\mu\nu}(x)$ is defined for each point x on the manifold. If the quantum probe (measurement device) occupies some four-volume V, then the probe's state depends on the metric at every point in that region. If we consider the metric to be an arbitrary function on the manifold, then we need to estimate an infinite number of parameters to define it completely. Instead, we consider regions of spacetime that can be described by metrics characterized by a single parameter θ. For example, the Schwarzschild metric, which describes the spacetime around a static nonrotating black hole, is defined by the single parameter M, the mass of the black hole. The task is to estimate this mass parameter by making measurements on physical systems living on the spacetime manifold.

There has recently been some promising work in this direction [11–13], focusing on quantum probes consisting of scalar fields in Gaussian states. Here we present a general formalism for relativistic quantum metrology, using arbitrary fields and states. In so doing, we address several related issues, which have, we believe, not previously received enough attention in this context. Ensuring that quantum observables in different spacetimes measure “the same” physical parameter is nontrivial. If the spacetimes differ by a global perturbation, the positions of measurement devices therein might correspondingly differ, further complicating the issue; more generally, spacetime points in two such spacetimes cannot unambiguously be identified with each other. Care must also be taken to ensure coordinate independence.

Fortunately, a framework exists for comparing quantum observables in perturbed, classical spacetimes in a coordinate-independent manner [5]. This locally covariant framework, which was developed in the context of algebraic quantum field theory, serves as our starting point. As the current work is geared more towards physical experiments than most literature invoking algebraic quantum field theory, it is worth a comment. The aim of algebraic quantum field theory is to put quantum field theory on rigorous mathematical footing, while the aim of what might be termed pragmatic quantum field theory is to make experimental predictions [14, 15]. Strides toward connecting the two have been made recently [5, 16–19], and this progress makes the current work possible.

The locally covariant framework directly connects the stress-energy to the change in state associated with a compactly supported change in the metric. The connection is via the concept of relative Cauchy evolution developed in [5] and leads directly to our bounds on measurement precision. An issue is that the bounds obtained are with respect to the best observable supported in a region that can be much bigger than the region containing the probes. While this means that the bounds are guaranteed to be optimistic in the sense that they suggest a higher-than-achievable precision, we generally wish to make the relationship between the measurement region, stress-energy variance, and measurement precision tighter. For this we show that the metric change can be localized, provided that the sensitivity of our measurement to the parameter of interest is not affected. Because the locally covariant framework requires compactly supported regions, the localization is necessary when the parameter is a global property of spacetime.

Another important issue—perhaps the most important issue for the interpretation and relevance of our results—is that computations of the relevant stress-energy variances can be difficult. In most situations, however, the probe devices introduce fields that have large mean values compared to a zero-mean reference state, which is typically the vacuum state. In these cases, the calculation can be simplified by recognizing that mean-field-independent contributions become negligible.

Once we have determined the general parameter-based uncertainty relations connecting measurement precision to stress-energy variances, we apply them to several situations of interest. The first involves estimating constant metric components and recovers the Unruh uncertainty relations. By considering specific metric components, we also obtain the parametrized form of the Heisenberg uncertainty relations. Next we study in detail the problem of interferometric gravitational-wave detection with light. This requires the full power of our approach. Here we take advantage of localization by means of a “bump” function supported in the region of the measurement devices and use the large-mean-field property to enable explicit calculation of a bound on precision that depends on the amplitude of the light fields used. This recovers the well-known shot-noise limit, but goes beyond this limit in two ways. First, for the case of a wideband mean field on top of vacuum fluctuations, we obtain a general shot-noise bound that does not make an assumption of narrow bandwidth detection of the probe field; in particular, we find a wideband shot-noise limit in terms of a frequency-weighted integration over mean photon numbers. Second, we find that wideband squeezing gives the optimal, sub-shot-noise sensitivity under the assumptions we are making. We briefly discuss other examples, including cosmological parameters and gravimetry.

Our work is organized as follows. In Sec. II we review quantum estimation theory including, in particular, the quantum Cramér-Rao bound, which is the expression of parameter-based uncertainty principles. In Sec. III we review the locally covariant approach to quantum field theory in perturbed, classical spacetime developed by Brunetti, Fredenhagen, and Verch [5]. In Sec. IV we show that application of the Cramér-Rao bound discussed in Sec. II to such a perturbed spacetime results in a coordinate-independent uncertainty relation between a local spacetime property and a quantum operator that depends on the probe field. In Sec. V we generalize this uncertainty relation to global spacetime properties. In Sec. VI we consider application of the formalism to estimation of metric components, proper time, and proper distance in a certain class of perturbed spacetimes. In Sec. VII we derive a quantum uncertainty bound on detection of gravitational waves using the electromagnetic field as a probe, as in laser-interferometric gravitational-wave detectors. Then, in Sec. VIII, we consider additional applications, and finally we conclude in Sec. IX.

II. QUANTUM ESTIMATION THEORY

In this paper we consider estimating an individual parameter of a spacetime metric, so we only need the theory of single-parameter quantum estimation. A general scheme for quantum parameter estimation is depicted in Fig. 1. A quantum state, represented by a density operator $\hat{\rho}_0$, undergoes a unitary transformation $\hat{U}(\theta)$ that depends on the parameter θ of interest, producing a one-parameter family of states, $\hat{\rho}(\theta) = \hat{U}(\theta)\hat{\rho}_0\hat{U}^\dagger(\theta)$. Mea-

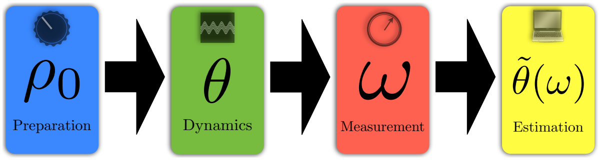

FIG. 1. Scheme for quantum parameter estimation.

surements are made on the system, with results ω, which are fed into an estimator $\tilde{\theta}(\omega)$ of the parameter.

We consider generalized measurements, described by positive-operator-valued measures (POVMs). For simplicity we consider such POVMs given by a positive-operator-valued density $\hat{E}(\omega)$ that satisfies the completeness property

$$\int d\omega\, \hat{E}(\omega) = \hat{1} \,, \tag{2.1}$$

where $\hat{1}$ is the identity operator. The outcomes of a particular measurement follow a probability distribution $p(\omega|\theta)$ conditional on the parameter θ. The probability distribution for the outcomes ω can be calculated as

$$p(\omega|\theta)\, d\omega = \mathrm{Tr}\Big(\hat{E}(\omega)\hat{\rho}(\theta)d\omega\Big) \,. \tag{2.2}$$

The problem of estimating the parameter θ is essentially that of choosing a value $\tilde{\theta}$ to make a good estimate of θ by considering the observed ω in relation to the known probability distributions $p(\omega|\theta)d\omega$. A common example is the maximum likelihood estimator, which is the choice of $\tilde{\theta}$ that retrospectively maximizes the probability of the observed measurement outcomes.

The variance of an unbiased estimate of the parameter θ, based on the distribution of measurement outcomes to be observed, is bounded by the classical Cramér-Rao lower bound [4],

$$\langle(\delta\tilde{\theta})^2\rangle \geq \frac{1}{F(\theta)} \,, \tag{2.3}$$

where $F(\theta)$ is the classical Fisher information for the measurement, given by

$$F(\theta) = \int d\omega\, \frac{1}{p(\omega|\theta)} \left(\frac{\partial p(\omega|\theta)}{\partial\theta}\right)^2 \,. \tag{2.4}$$

In this paper we consider the special case where $\hat{\rho}$ is differentiable at θ_0, with the differential generated by the self-adjoint operator $\hat{h}$, as expressed by

$$\frac{d\hat{\rho}}{d\theta} = -\frac{i}{\hbar}[\hat{h}, \hat{\rho}] \,, \tag{2.5}$$

Informally, we write

$$\hat{\rho}(\theta_0 + d\theta) = e^{-id\theta\hat{h}/\hbar}\hat{\rho}(\theta_0)e^{id\theta\hat{h}/\hbar} \,. \tag{2.6}$$

It can then be shown that for any POVM, the classical Fisher information satisfies $F(\theta) \leq 4\langle(\Delta\hat{h})^2/\hbar^2$ [1–3], where $\langle(\Delta\hat{h})^2\rangle$ is the quantum variance of the generator $\hat{h}$; moreover, there is a POVM that saturates this bound when $\hat{\rho}(\theta_0)$ is pure [3]. Applying this bound to Eq. (2.3) yields the quantum Cramér-Rao lower bound [4],

$$\langle(\delta\tilde{\theta})^2\rangle\langle(\Delta\hat{h})^2\rangle \geq \frac{\hbar^2}{4} \,. \tag{2.7}$$

One can now construct examples by identifying parameters and their corresponding generators. For example, since the Hamiltonian is the generator of time translations, this gives the time-energy uncertainty relation (1.3) presented in the Introduction. Since the momentum operator generates displacements, using it as the generator gives the parameter-based version of the Heisenberg uncertainty relation. Another important example is provided by the number operator and phase, which is the basis of Heisenberg-limited phase estimation [20–22].

III. RELATIVE CAUCHY EVOLUTION

We employ the locally covariant formulation of quantum field theory in curved spacetime as developed by Brunetti, Fredenhagen, and Verch [5]. The approach has been used to develop a notion of "identical physics" on different spacetimes [23]. A key result is a method for calculating how quantum observables respond to local changes in the background spacetime. Say we believe some particular region of the universe to be well described by a metric $\mathrm{g}^{(s)}_{\mu\nu}$ that depends on a parameter s. If s is assumed to parametrize a compactly supported perturbation, the locally covariant approach can be used to calculate how any observable $\hat{E}(s)$ responds to such a change. The response is evaluated as the rate of change of the observable with respect to the parameter. As we noted in the Introduction, this is just what we need to calculate the quantum Cramér-Rao lower bound.

We emphasize, however, that only compactly supported perturbations in spacetime are considered in [5]. The motivation for this restriction is similar to that for restricting the domain of distributions to test functions and ensures that relevant quantities are well defined. We consider how to approach more general perturbations in Sec. IV.

The locally covariant approach is formulated in a category-theory framework. It involves the category of globally hyperbolic spacetimes and the mapping of each to an algebra of observables. By this formalism, which is summarized in Appendix A, the evolution of an observable in response to a spacetime perturbation is made mathematically well defined.

Note that here a spacetime is a pair (M, g), where M is a 4-manifold admitting a Lorentzian metric and g is a Lorentzian metric. The additional property of global hyperbolicity is a restriction on the causal structure on the manifold. It removes the possibility of closed time-like curves and ensures the spacetime can be foliated into

Cauchy surfaces. This in turn ensures that any hyperbolic field equation (Klein-Gordon, Maxwell, etc.) has a well-posed initial-value formulation.

A one-parameter family of spacetimes $\{M, \mathrm{g}^{(s)}\}$ was considered in [5], all sharing "initial" and "final" Cauchy surfaces, as well as respective neighborhoods N_- and N_+ of those Cauchy surfaces. These spacetimes differ only in their geometry, and only within a compact region between N_- and N_+. We refer to the metric $\mathrm{g}^{(s)}$ as *perturbed* when $s \neq 0$ and *fiducial* when $s = 0$.

To be more precise, we make the following geometric assumptions:

1. $(M, \mathrm{g}^{(0)})$ is a globally hyperbolic spacetime.
2. We choose a Cauchy surface C in $(M, \mathrm{g}^{(0)})$ and two open subregions $(N_\pm, \mathrm{g}^{(0)}_{N_\pm})$ with the following properties:
   - N_+ is within the future and N_- is within the past causal regions of the Cauchy surface C.
   - $(N_\pm, \mathrm{g}^{(0)}_{N_\pm})$ are globally hyperbolic spacetimes.
   - $N_\pm$ contain Cauchy surfaces for the whole spacetime $(M, \mathrm{g}^{(0)})$.
3. $\{\mathrm{g}^{(s)}\}_{s\in[-1,1]}$ is a set of Lorentzian metrics on M with the following properties:
   - Each $\mathrm{g}^{(s)}$ deviates from $\mathrm{g}^{(0)}$ only on a compact subset of the region in the past of N_+ and the future of N_-.
   - Each $(M, \mathrm{g}^{(s)})$ is a globally hyperbolic spacetime.
   - C is also a Cauchy surface for each $(M, \mathrm{g}^{(s)})$.

The geometric assumptions listed here can be seen in greater mathematical detail in Sec. 4.1 of [5].

Consider now a Hilbert-space operator $\hat{A}(0)$ defined on $(M, g^{(0)})$; this is an operator acting on a representation of the algebra generated by the quantum fields on $(M, g^{(0)})$. This operator could, for example, be a POVM element for a particle detector and belong to the algebra of operators localized to the spatiotemporal extent where the detector is active.

It was shown in [5] that $\hat{A}(0)$ unitarily transforms under an s-parametrized metric perturbation into a new operator $\hat{A}(s)$. The functional derivative of the action of this unitary transformation with respect to the metric is defined as

$$\frac{d}{ds}\bigg|_0 \hat{A}(s) = \int_M d\mathring{\mu}(x)\, \frac{\delta \hat{A}(s)}{\delta \mathrm{g}_{\mu\nu}(x)}\, \frac{d}{ds}\bigg|_0 \mathrm{g}^{(s)}_{\mu\nu}(x)\ , \qquad (3.1)$$

where $d\mathring{\mu}(x) = \sqrt{|\det \mathrm{g}^{(0)}|}\, dx^0\, dx^1\, dx^2\, dx^3$ is the proper volume element for the metric $\mathrm{g}^{(0)}$. The interpretation of this is as follows: Fields are prepared in N_-, they then scatter off an intermediate region (the compact subset of geometric assumption 3) and are measured in N^+; s controls a localized perturbation within this region, and Eq. (3.1) gives the infinitesimal movement of the observables in the Hilbert-space representation of $\mathscr{A}(M, \mathrm{g}^{(0)})$ due to an infinitesimal perturbation ds around $s = 0$. Further properties and interpretations of this functional derivative and the relative Cauchy evolution can be found in [5, 23, 24].

For the case of the Klein-Gordon field, with its corresponding Weyl algebra of observables, it can be shown that both elements of this algebra and polynomials of field operators constructed from it obey the following relation (Theorem 4.3 from [5]):

$$\frac{\delta \hat{A}(s)}{\delta \mathrm{g}_{\mu\nu}(x)}\bigg|_{s=0} = \frac{i}{2\hbar}\left[\hat{A}(0), \hat{T}^{\mu\nu}(x)\right]\ . \qquad (3.2)$$

Here $\hat{T}^{\mu\nu}(x)$ is the renormalized stress-energy tensor on the relevant Hilbert space as discussed in [5] and satisfying Theorem 4.6.1 of [25]. We assume, more specifically, that it is normally ordered in accordance with the procedure advocated by Brown and Ottewill [26].

Inserting Eq. (3.2) into Eq. (3.1), we have

$$\frac{d}{ds}\bigg|_0 \hat{A}(s) = \int_M d\mathring{\mu}(x)\, \frac{i}{2\hbar}\left[\hat{A}(0), \hat{T}^{\mu\nu}(x)\right]\, \frac{d}{ds}\bigg|_0 \mathrm{g}^{(s)}_{\mu\nu}(x)\ , \qquad (3.3)$$

where we emphasize that the operator $\hat{A}$ does not depend on x. By defining the operator $\hat{P}$ as

$$\hat{P} = \frac{1}{2}\int_M d\mathring{\mu}(x)\, \hat{T}^{\mu\nu}(x)\, \frac{d}{ds}\bigg|_0 \mathrm{g}^{(s)}_{\mu\nu}(x)\ , \qquad (3.4)$$

we have

$$\frac{d}{ds}\hat{A}(s)\bigg|_{s=0} = \frac{i}{\hbar}\left[\hat{A}(0), \hat{P}\right]\ . \qquad (3.5)$$

The above relative Cauchy evolution equation has also been shown to hold for spin-$\frac{1}{2}$ and spin-1 fields [16–18], with an appropriately defined stress-energy tensor. For example, for the electromagnetic field, which we consider in Sec. VII, the Weyl algebra of the Klein-Gordon field is simply replaced by the Weyl algebra of gauge-equivalence classes of the vector potential. Then by Theorem 3.2.9 in [17], the functional derivative with respect to the perturbed metric of an operator $\hat{A}$ is again given by the commutator with the stress-energy tensor, as in Eq. (3.2).

IV. ESTIMATION OF SPACETIME PERTURBATION

Due to the dual nature of operators and states, a consequence of Eq. (3.5) is that we can write

$$\hat{\rho}(0 + ds) = e^{-ids\hat{P}/\hbar}\hat{\rho}(0)e^{ids\hat{P}/\hbar}\ , \qquad (4.1)$$

where $\hat{\rho}(s)$ is a density operator in the Gelfand-Neimark-Segal [27–29] representation of the algebra on $(M, \mathrm{g}^{(0)})$ after the action of $\hat{U}(s)$, the unitary in the Hilbert-space representation corresponding to the relative Cauchy evolution (more precisely to a unit-preserving automorphism on the algebra of observables, called $\beta_{\mathrm{g}(s)}$ in Appendix A). Then, noting the equivalence of Eq. (4.1) to Eq. (2.6), we obtain [30]

$$\langle(\delta\tilde{s})^2\rangle\langle(\Delta\hat{P})^2\rangle \geq \frac{\hbar^2}{4} \; . \tag{4.2}$$

where $\tilde{s}$ is the estimator for the perturbation parameter.

A limitation of the quantum Cramér-Rao bound (4.2) is that the spacetime perturbation is restricted to compact support, whereas physically interesting perturbations are typically not so restricted. For example, variation in the mass of the Earth would vary the metric at unbounded distances away. To apply our formalism to this situation, we approximate global perturbations compactly. As a first attempt, one might consider using compact perturbations that approach the global one in some limit, but this might lead to unbounded values of $(\Delta\hat{P})^2$, which would trivialize the bound (4.2). To avoid this we take advantage of the fact that the measurements that yield an estimate of $\tilde{\theta}$ are of observables that are accessible in a compact measurement region determined by the measurement device. Given this, the bounds above are necessarily conservative for perturbations with large extent: They apply to all observables in the region of the perturbation, even those causally separated from the measurement device. We can take advantage of a flexibility built into quantum estimation theory whereby we can obtain an uncertainty bound from any parameter that our estimator $\tilde{\theta}$ is sensitive to.

To see this, let Θ denote the global parameter of interest for a spacetime with metric $g_{\mu\nu}(\Theta)$, with Θ_0 being the fiducial value of the parameter and $\tilde{\theta}$ denoting its estimator. We can choose a smooth, compactly supported "bump" function, $0 \leq \chi \leq 1$, with $\chi(x) = 1$ on the measurement region. We consider the localized perturbation $\mathrm{g}_{\mu\nu}(\theta) \equiv g_{\mu\nu}\big(\theta_0 + (\theta - \theta_0)\chi\big)$, parametrized in terms of θ, with $\theta_0 = \Theta_0$. If χ has sufficiently large extent and transitions to 0 sufficiently slowly (say, adiabatically), we can argue that the sensitivity of $\tilde{\theta}$ to θ, given by $(d\langle\tilde{\theta}_\theta\rangle/d\theta)|_{\theta=\theta_0}$ approaches that of $\tilde{\theta}$ to Θ, which is $(d\langle\tilde{\theta}\rangle_\Theta/d\Theta)|_{\Theta=\Theta_0} = 1$, where the latter identity follows from the assumption that $\tilde{\theta}$ is an unbiased estimator to first order in $\Theta - \Theta_0$. This means that $\tilde{\theta}$ is also an unbiased estimator of θ, to first order in $\theta - \theta_0$, so that the Cramér-Rao bound applies to $\tilde{\theta}$ with $\mathrm{g}_{\mu\nu}(\theta)$, giving

$$\langle(\delta\tilde{\theta})^2\rangle\langle(\Delta\hat{P})^2\rangle \geq \frac{\hbar^2}{4} \; , \tag{4.3}$$

where

$$\hat{P} = \frac{1}{2}\int_M d\mathring{\mu}\, \hat{T}^{\mu\nu}\, \frac{d}{d\theta}\bigg|_{\theta_0} \mathrm{g}_{\mu\nu}(\theta) \; . \tag{4.4}$$

How the bump function transitions from 1 on the measurement region to 0 is arbitrary. The choice affects the bound, however, through excess contributions to the variance $\langle(\Delta\hat{P})^2\rangle$ in the bump function's support outside the measurement region. To get the best bounds on precision, we choose bump functions that minimize this excess variance while achieving the desired sensitivity. In the examples to be considered, this excess variance can be attributed to contributions from a reference state such as the Minkowski vacuum. This is because the state associated with the measurement and from which the bound is computed is a localized deviation from the reference state and the bump function necessarily extends beyond the region of localization. We observe that the shape of the transition of the bump function from 1 to 0 affects the excess variance [31–33]. In particular, the excess can be reduced by ensuring that the transition from 1 to 0 is slow. This is analogous to the adiabatic limit. For the case of the Minkowski vacuum, the reference-state contribution can be made arbitrarily small by this method, as demonstrated for example in [31].

The use of slowly varying bump functions is expected to reduce excess variance from the reference state, but does not necessarily lead to readily computable bounds. For this, we observe that informative measurements rely on deviations from the reference state with relatively large localized mean fields. This is both out of necessity and to maximize the signal to noise. Typical measurements are designed not to detect the reference-state contributions to the variance, but rather to detect an effect in the presence of a strong mean field, which greatly enhances the signal we are looking for. Indeed, for arbitrary curved spacetimes, it is not known how to calculate the reference-state contributions, nor is it known how to design a measurement on the probe field that detects the corresponding mean-field-independent effects. Neglect of reference-state contributions can then be regarded as a way of finding quantum limits on the kinds of measurements we know how to do, which involve large mean fields.

For the case of free fields, the large-mean-field scenario is formalized by considering the measurement state as a displacement by a local Weyl unitary of a reference state with mean field zero. In Minkowski space, these displacements are enacted by conventional modal displacement operators, where the displacement is by an amount determined by the mean field. We show in Sec. VII that the variance $\langle(\Delta\hat{P})^2\rangle$ has terms that grow with the mean field as well as mean-field independent terms. We identify the mean-field-independent terms as the reference-state contribution to the variance. For a fixed bump function, but large mean field, the reference-state contribution becomes negligible. This is the main strategy used for the analysis of gravitational-wave detection in Sec. VII.

For the above discussion, we assumed that the measurement device is contained in a finite measurement region, where an incoming reference state such as the vacuum state is temporarily modified for the measurement. This modification is usually necessary to enhance the signal that we are looking for. In the case of an interferometric measurement using light, the modification is accomplished by introducing a large amplitude light field confined between mirrors. To accommodate these modifications in the generally covariant formalism, the background includes externally introduced classical sources with fixed relationships to the manifold, meaning that these relationships are unchanged by the perturbation of the metric under investigation. The formalism and relative Cauchy evolution still applies, as suggested in [5]. For the example of gravitational-wave detection, the fixed relationship can be justified by the observation that the classical sources follow geodesics for the original as well as the perturbed metric and are thus manifestly independent of the perturbation in Gaussian normal coordinates.

While our examples involving large mean-field deviations are on flat backgrounds, we expect that the justification for neglecting reference-state contributions based on large mean-field deviations also applies to general curved backgrounds. For any free field in a (globally hyperbolic) spacetime, there exist zero-mean-field reference states, called Hadamard states, characterized by well-defined two-point correlators [16, 19, 25]. Such a reference state allows us to perform normal ordering via the point-splitting approach. A corresponding stress-energy tensor can then be constructed, with a well-defined expectation value [25], which is unique up to terms which cancel in the commutator (3.5)) and in the variance (and is thus sufficient for our purposes). Moreover, while Hadamard reference states are generally not unique, their contribution to the relevant variance is mean field independent.

V. COORDINATE INDEPENDENCE AND COMPACT PERTURBATIONS

The formulation given so far is generally covariant. Consider two compactly supported perturbations of the metric where one is obtained from the other by a local isometry, that is one acting as the identity except on a compact region in the past of N_+ and the future N_-. Then both perturbations induce the same relative Cauchy evolution, as shown in [5] (see also Appendix B). A complication to this obvious conclusion of general covariance arises, however, when the parameter of interest is expressed in a coordinate-dependent way and we require the use of a bump function to localize the associated metric perturbation. This is the situation when estimating global parameters. An example is the invariant mass of a black hole, where there are a number of different standard coordinate systems to choose from. When the bump function is expressed in the first coordinate system so as to be independent of the invariant mass, in the second it can depend on the invariant mass. This means that in the second coordinate system, the bump-function-modified metric perturbation includes a term coming from the derivative of the mass-dependent bump-function with respect to mass, and this leads to discrepancies in the values of the variances and Cramér-Rao bounds depending on which coordinate system is used to define the bump function. If our sensitivity argument for the choice of bump function is valid, we should obtain valid bounds regardless of coordinate system. It is desirable, however, to choose bump functions for which the dependence on coordinate system is negligible. In this section, we show that such is the case for the variance of $\hat{P}$ in an arbitrary spacetime, assuming a sufficiently large mean field. To demonstrate the basic mechanism by which coordinate independence is achieved, we first consider as an illustrative example the expectation of $\hat{P}$ in Schwarzschild spacetime.

The fiducial metric in Schwarzschild coordinates is

$$\mathrm{g}^S_{\mu\nu} dx^\mu_S dx^\nu_S = -\left(1 - \frac{2m_0}{r}\right) dt^2 + \left(1 - \frac{2m_0}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \;, \tag{5.1}$$

and in isotropic coordinates it is

$$\mathrm{g}^I_{\mu\nu} dx^\mu_I dx^\nu_I = -\left(\frac{1 - m_0/2\rho}{1 + m_0/2\rho}\right)^2 dt^2 + \left(1 + \frac{m_0}{2\rho}\right)^4 (d\rho^2 + \rho^2 d\Omega^2) \;. \tag{5.2}$$

By our prescription for approximating a global perturbation by a compact perturbation, we add to every instance of the fiducial mass m_0 a bump function of the form $(m - m_0)\chi(t, r, \theta, \phi)$ or $(m - m_0)\chi(t, \rho, \theta, \phi)$:

$$\mathrm{g}^S_{\mu\nu} dx^\mu_S dx^\nu_S = -\left(1 - \frac{2(m_0 + (m - m_0)\chi)}{r}\right) dt^2 + \left(1 - \frac{2(m_0 + (m - m_0)\chi)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \;, \tag{5.3}$$

$$\mathrm{g}^I_{\mu\nu} dx^\mu_I dx^\nu_I = -\left(\frac{1 - [m_0 + (m - m_0)\chi]/2\rho}{1 + [m_0 + (m - m_0)\chi/2\rho]}\right)^2 dt^2 + \left(1 + \frac{m_0 + (m - m_0)\chi}{2\rho}\right)^4 (d\rho^2 + \rho^2 d\Omega^2) \;. \tag{5.4}$$

The resulting metrics $\mathrm{g}^S_{\mu\nu}(m)$ and $\mathrm{g}^I_{\mu\nu}(m)$ are not related by a coordinate transformation and thus are no longer physically equivalent. This reflects the fact that there is no unique way to approximate a global perturbation with a compact perturbation. Our particular choice depends on our initial coordinates, out of convenience. These metrics are, however, locally related by a coordinate transformation on a patch restricted to the region where $\chi = 1$. Therein, both metrics are locally indistinguishable from that of a black hole with mass m and are related by the m-dependent coordinate transformation that relates Schwarzschild and isotropic coordinates.

Now letting $m_0 = 0$ for simplicity, consider the (normally ordered) expectation value of $\hat{P}$, where the nonzero expectation value of the stress-energy is assumed to be confined to a region K in which $\chi = 1$, i.e., $K = \mathrm{supp}(\langle{:}\hat{T}^{\mu\nu}{:}\rangle)$ and $\chi(x) = 1$ for $x \in K$. Note that K is strictly contained in the interior of $\mathrm{supp}(\chi)$, since as discussed in the previous section we assume the transition of the bump χ from 1 to 0 is both smooth and gradual. We have

$$\langle{:}\hat{P}{:}\rangle = \frac{1}{2}\int_K d\mathring{\mu}\,\langle{:}\hat{T}^{\mu\nu}{:}\rangle\,\frac{d}{dm}\bigg|_0 \mathrm{g}_{\mu\nu}(m)\ . \tag{5.5}$$

Notice that $\mathrm{g}^S_{\mu\nu}(0) = \mathrm{g}^I_{\mu\nu}(0)$, since both Schwarzschild and isotropic coordinates reduce to standard spherical coordinates in this limit. Yet

$$\frac{d}{dm}\bigg|_0 \mathrm{g}^S_{\mu\nu}(m) \neq \frac{d}{dm}\bigg|_0 \mathrm{g}^I_{\mu\nu}(m)\ , \tag{5.6}$$

nor are these two tensor fields related by any coordinate transformation. To understand this, observe that $\mathrm{g}^S_{\mu\nu}(m) - \mathrm{g}^S_{\mu\nu}(0)$ and $\mathrm{g}^I_{\mu\nu}(m) - \mathrm{g}^I_{\mu\nu}(0)$ also represent distinct tensor fields; the first terms in the two tensor fields can be made equal by an m-dependent coordinate transformation, but not without making the second terms unequal. It should come as no surprise, then, that $\langle{:}\hat{P}_S{:}\rangle$ and $\langle{:}\hat{P}_I{:}\rangle$ appear to be unequal:

$$\langle{:}\hat{P}_S{:}\rangle = \int_K \frac{1}{r}\Big(\langle{:}\hat{T}^{tt}{:}\rangle + \langle{:}\hat{T}^{rr}{:}\rangle\Big) r^2 \sin\vartheta\, dt\, dr\, d\vartheta\, d\varphi\ , \tag{5.7}$$

$$\langle{:}\hat{P}_I{:}\rangle = \int_K \frac{1}{r}\Big(\langle{:}\hat{T}^{tt}{:}\rangle + \langle{:}\hat{T}^{rr}{:}\rangle + r^2\langle{:}\hat{T}^{\vartheta\vartheta}{:}\rangle + r^2\sin^2\!\vartheta\langle{:}\hat{T}^{\varphi\varphi}{:}\rangle\Big) r^2 \sin\vartheta\, dt\, dr\, d\vartheta\, d\varphi\ . \tag{5.8}$$

This appearance is deceptive, however, as we see from

$$\int_K \langle{:}\nabla_\alpha \hat{T}^{\alpha r}{:}\rangle\, r^2 \sin\vartheta\, dt\, dr\, d\vartheta\, d\varphi = \int_{\partial K} d\lambda\, n_\alpha \langle{:}\hat{T}^{\alpha r}{:}\rangle - \int_K \Big(r\langle{:}\hat{T}^{\vartheta\vartheta}{:}\rangle + r\sin^2\!\vartheta\langle{:}\hat{T}^{\varphi\varphi}{:}\rangle\Big) r^2 \sin\vartheta\, dt\, dr\, d\vartheta\, d\varphi\ , \tag{5.9}$$

where $d\lambda$ is the surface element induced on the boundary ∂K of K and n_μ is the corresponding surface normal. Since $\langle{:}\hat{T}^{\mu\nu}{:}\rangle$ vanishes on ∂K and assuming $\nabla_\mu \hat{T}^{\mu\nu} = 0$ (as required for a properly defined stress-energy tensor [25]), which implies that the left-hand side of Eq. (5.9) vanishes identically, we conclude that

$$\langle{:}\hat{P}_I{:}\rangle - \langle{:}\hat{P}_S{:}\rangle = \int_K \Big(r\langle{:}\hat{T}^{\vartheta\vartheta}{:}\rangle + r\sin^2\!\vartheta\langle{:}\hat{T}^{\varphi\varphi}{:}\rangle\Big) r^2 \sin\vartheta\, dt\, dr\, d\vartheta\, d\varphi = 0\ . \tag{5.10}$$

Note the critical role played by the vanishing divergence of $\hat{T}^{\mu\nu}$ in the above demonstration of coordinate independence. This is no coincidence; for more discussion of the relevance of the stress-energy tensor to diffeomorphism invariance in the context of relative Cauchy evolution, see [5].

In the above example, we only considered the first moment of $\hat{P}$, whereas the Cramér-Rao bound involves the variance of $\hat{P}$. We deal with this question now by considering an arbitrary s-dependent coordinate transformation from unprimed coordinates to primed coordinates, on a coordinate patch that is assumed to cover the support K of the mean stress energy (for a more general treatment see Appendix B). In classic index notation, the metric components in the two systems are related by

$$\mathrm{g}^{(s)}_{\alpha'\beta'}(x') = L^\mu{}_{\alpha'}(s) L^\nu{}_{\beta'}(s)\mathrm{g}^{(s)}_{\mu\nu}(x)\ , \tag{5.11}$$

where $L^\mu{}_{\alpha'}(s) = \partial x^\mu(x', s)/\partial x^{\alpha'}$. We are interested in the tensor fields

$$\frac{\partial}{\partial s}\bigg|_0 \mathrm{g}^{(s)}_{\alpha\beta}(x) \qquad \text{and} \qquad \frac{\partial}{\partial s}\bigg|_0 \mathrm{g}^{(s)}_{\alpha'\beta'}(x')\ , \tag{5.12}$$

where we now write the s-derivatives as partial derivatives to emphasize that the respective coordinates are held fixed while taking the s-derivative. Because the coordinate transformation is s dependent, these two tensors are not the same, but are related by

$$\begin{aligned}
\frac{\partial}{\partial s}\bigg|_0 \mathrm{g}^{(s)}_{\alpha'\beta'}(x') &= L^\mu{}_{\alpha'}(0)L^\nu{}_{\beta'}(0)\left(\frac{\partial}{\partial s}\bigg|_0 \mathrm{g}^{(s)}_{\mu\nu}(x) + \frac{\partial X^\gamma}{\partial x^\mu}\mathrm{g}^{(0)}_{\gamma\nu} + \frac{\partial X^\gamma}{\partial x^\nu}\mathrm{g}^{(0)}_{\mu\gamma} + X^\gamma \frac{\partial \mathrm{g}^{(0)}_{\mu\nu}}{\partial x^\gamma}\right) \\
&= L^\mu{}_{\alpha'}(0)L^\nu{}_{\beta'}(0)\left(\frac{\partial}{\partial s}\bigg|_0 \mathrm{g}^{(s)}_{\mu\nu}(x) + \nabla_\nu X_\mu + \nabla_\mu X_\nu\right) \\
&= L^\mu{}_{\alpha'}(0)L^\nu{}_{\beta'}(0)\,\frac{\partial}{\partial s}\bigg|_0 \mathrm{g}^{(s)}_{\mu\nu}(x) + \nabla_{\beta'} X_{\alpha'} + \nabla_{\alpha'} X_{\beta'}\ ,
\end{aligned} \tag{5.13}$$

where

$$X^\gamma = \frac{\partial}{\partial s}\bigg|_0 x^\gamma(x',s)\ . \tag{5.14}$$

Now we find

$$\begin{aligned}
\int_K d\mathring{\mu}'\,\hat{T}^{\alpha'\beta'}\,\frac{\partial}{\partial s}\bigg|_0 \mathrm{g}^{(s)}_{\alpha'\beta'} &= \int_K d\mathring{\mu}\,\hat{T}^{\mu\nu}\left(\frac{\partial}{\partial s}\bigg|_0 \mathrm{g}^{(s)}_{\mu\nu} + \nabla_\nu X_\mu + \nabla_\mu X_\nu\right) \\
&= \int_K d\mathring{\mu}\left(\hat{T}^{\mu\nu}\,\frac{\partial}{\partial s}\bigg|_0 \mathrm{g}^{(s)}_{\mu\nu} + 2\nabla_\mu(\hat{T}^{\mu\nu}X_\nu) - 2(\nabla_\mu \hat{T}^{\mu\nu})X_\nu\right) \\
&= \int_K d\mathring{\mu}\,\hat{T}^{\mu\nu}\,\frac{\partial}{\partial s}\bigg|_0 \mathrm{g}^{(s)}_{\mu\nu} + 2\int_{\partial K} d\lambda\, n_\mu \hat{T}^{\mu\nu} X_\nu\ ,
\end{aligned} \tag{5.15}$$

where in the last line we assume that $\nabla_\mu \hat{T}^{\mu\nu} = 0$ and where we convert a volume integral over K to a surface integral over the boundary ∂K.

As before, let $K = \mathrm{supp}(\langle{:}\hat{T}^{\mu\nu}{:}\rangle)$, so that $\langle{:}\hat{T}^{\mu\nu}{:}\rangle|_{\partial K} = 0$, and we assume that $\chi|_K = 1$ (and thus K is strictly contained within $\mathrm{supp}(\chi)$). In addition, consider a θ_0-dependent coordinate transformation of $\mathrm{g}_{\mu\nu}(\theta_0)$, denoted $\mathrm{g}'_{\mu\nu}(\theta_0)$, where we switch back from classic index notation to denoting a coordinate change with a prime on the tensor itself. As previously prescribed, to each instance of θ_0 in the coordinate components of this new metric, add $(\theta-\theta_0)\chi$, denoting the result as $\mathrm{g}'_{\mu\nu}(\theta) = \mathrm{g}'_{\mu\nu}\big(\theta_0 + (\theta-\theta_0)\chi\big)$. It proves convenient to divide $\hat{P}$ and $\hat{P}'$ into two parts,

$$\hat{P} = \frac{1}{2}\int_M d\mathring{\mu}\,\hat{T}^{\mu\nu}\,\frac{d}{d\theta}\bigg|_{\theta_0} \mathrm{g}_{\mu\nu}(\theta) = \frac{1}{2}\int_K d\mathring{\mu}\,\hat{T}^{\mu\nu}\,\frac{d}{d\theta}\bigg|_{\theta_0} \mathrm{g}_{\mu\nu}(\theta) + \frac{1}{2}\int_{\bar{K}} d\mathring{\mu}\,\hat{T}^{\mu\nu}\,\frac{d}{d\theta}\bigg|_{\theta_0} \mathrm{g}_{\mu\nu}(\theta) = \hat{P}_K + \hat{P}_{\bar{K}}\ , \tag{5.16}$$

$$\hat{P}' = \frac{1}{2}\int_M d\mathring{\mu}'\,\hat{T}'^{\mu\nu}\,\frac{d}{d\theta}\bigg|_{\theta_0} \mathrm{g}'_{\mu\nu}(\theta) = \frac{1}{2}\int_K d\mathring{\mu}'\,\hat{T}'^{\mu\nu}\,\frac{d}{d\theta}\bigg|_{\theta_0} \mathrm{g}'_{\mu\nu}(\theta) + \frac{1}{2}\int_{\bar{K}} d\mathring{\mu}'\,\hat{T}'^{\mu\nu}\,\frac{d}{d\theta}\bigg|_{\theta_0} \mathrm{g}'_{\mu\nu}(\theta) = \hat{P}'_K + \hat{P}'_{\bar{K}}\ , \tag{5.17}$$

where $\bar{K} = M\backslash K$. The integrals over $\bar{K}$ are restricted to the neighborhood of K where the bump function is nonzero, but in which $\langle{:}\hat{T}^{\mu\nu}{:}\rangle = 0$. The content of Eq. (5.15) is that

$$\hat{P}'_K = \hat{P}_K + \int_{\partial K} d\lambda\, n_\mu \hat{T}^{\mu\nu} X_\nu = \hat{P}_K + \hat{B}\ , \tag{5.18}$$

where $\hat{B}$ is the boundary term. By construction, we have $\langle{:}\hat{P}'_{\bar{K}}{:}\rangle = \langle{:}\hat{P}_{\bar{K}}{:}\rangle = \langle{:}\hat{B}{:}\rangle = 0$, so

$$\langle{:}\hat{P}'{:}\rangle = \langle{:}\hat{P}'_K{:}\rangle = \langle{:}\hat{P}_K{:}\rangle = \langle{:}\hat{P}{:}\rangle\ ; \tag{5.19}$$

this is the general version of what we showed for the particular case of Schwarzschild and isotropic coordinates.

What we need for the Cramér-Rao bound (4.3) are variances, not mean values, and it is in the variances that the problem with reference-state contributions arises. Equation (5.16) gives us

$$\begin{aligned}
\langle(\Delta\hat{P})^2\rangle &= \langle{:}\hat{P}{:}{:}\hat{P}{:}\rangle - \langle{:}\hat{P}{:}\rangle^2 \\
&= \big\langle{:}(\hat{P}_K + \hat{P}_{\bar{K}}){:}{:}(\hat{P}_K + \hat{P}_{\bar{K}}){:}\big\rangle - \langle{:}\hat{P}_K{:}\rangle^2 \\
&= \langle(\Delta\hat{P}_K)^2\rangle + \langle{:}\hat{P}_K{:}{:}\hat{P}_{\bar{K}}{:}\rangle + \langle{:}\hat{P}_{\bar{K}}{:}{:}\hat{P}_K{:}\rangle + \langle{:}\hat{P}_{\bar{K}}{:}{:}\hat{P}_{\bar{K}}{:}\rangle\ ,
\end{aligned} \tag{5.20}$$

where $\langle(\Delta\hat{P}_K)^2\rangle = \langle{:}\hat{P}_K{:}{:}\hat{P}_K{:}\rangle - \langle{:}\hat{P}_K{:}\rangle^2$. We now decompose ${:}\hat{P}_K{:}$ into its expectation value and a correction, ${:}\hat{P}_K{:} = \langle{:}\hat{P}_K{:}\rangle + {:}\Delta\hat{P}_K{:}$, and use the fact that $\langle{:}\hat{P}_K{:}{:}\hat{P}_{\bar{K}}{:}\rangle = \langle{:}\hat{P}_K{:}\rangle\langle{:}\hat{P}_{\bar{K}}{:}\rangle + \langle{:}\Delta\hat{P}_K{:}{:}\hat{P}_{\bar{K}}{:}\rangle = \langle{:}\Delta\hat{P}_K{:}{:}\hat{P}_{\bar{K}}{:}\rangle$ to write the variance in the form

$$\langle(\Delta\hat{P})^2\rangle = \langle(\Delta\hat{P}_K)^2\rangle + \langle{:}\Delta\hat{P}_K{:}{:}\hat{P}_{\bar{K}}{:}\rangle + \langle{:}\hat{P}_{\bar{K}}{:}{:}\Delta\hat{P}_K{:}\rangle + \langle{:}\hat{P}_{\bar{K}}{:}{:}\hat{P}_{\bar{K}}{:}\rangle\ . \tag{5.21}$$

The final three terms are all reference-state contributions: the middle two terms express correlations between the reference state inside and outside of K; the third term is a reference-state contribution from outside the support of the probe's mean stress-energy.

Now, using Eqs. (5.17) and (5.18) to write $\hat{P}' = \hat{P}_K + \hat{Q}$, with $\hat{Q} = \hat{B} + \hat{P}'_{\bar{K}}$, the same considerations give us

$$\begin{aligned} \langle(\Delta\hat{P}')^2\rangle &= \langle{:}\hat{P}'{:}{:}\hat{P}'{:}\rangle - \langle{:}\hat{P}'{:}\rangle^2 \\ &= \big\langle{:}(\hat{P}_K+\hat{Q}){:}{:}(\hat{P}_K+\hat{Q}){:}\big\rangle - \langle{:}\hat{P}_K{:}\rangle^2 \\ &= \langle(\Delta\hat{P}_K)^2\rangle + \langle{:}\hat{P}_K{:}{:}\hat{Q}{:}\rangle + \langle{:}\hat{Q}{:}{:}\hat{P}_K{:}\rangle + \langle{:}\hat{Q}{:}{:}\hat{Q}{:}\rangle \\ &= \langle(\Delta\hat{P}_K)^2\rangle + \langle{:}\Delta\hat{P}_K{:}{:}\hat{Q}{:}\rangle + \langle{:}\hat{Q}{:}{:}\Delta\hat{P}_K{:}\rangle + \langle{:}\hat{Q}{:}{:}\hat{Q}{:}\rangle\ . \end{aligned} \tag{5.22}$$

Again, the final three terms are reference-state contributions with the same sort of interpretation as that given above for $\langle(\Delta\hat{P})^2\rangle$, except that now there is a contribution from the boundary ∂K.

If we take advantage of the improved signal-to-noise that comes from a large mean field, we expect that our probe fields have a sufficiently substantial mean component that $\langle(\Delta\hat{P}_K)^2\rangle$ makes the dominant contribution to the variances and thus determines the quantum Cramér-Rao bound. Under these assumptions we conclude that $\langle(\Delta\hat{P}')^2\rangle$ and $\langle(\Delta\hat{P})^2\rangle$ are both equal to $\langle(\Delta\hat{P}_K)^2\rangle$ up to subleading, mean-field-independent terms.

A complementary point of view acknowledges that for a real perturbation, not one that has been modified by a bump function, the changes in the reference state do contribute to the quantum Cramér-Rao bound. These changes in the reference state (in some cases, the reference state could be vacuum, when that can be properly defined) could presumably be used to detect the perturbation in the absence of a mean field. Yet we do not know how to calculate the reference-state contributions in general curved spacetimes, nor do we know how to measure the corresponding modifications of the reference state. It would be desirable to determine how to do the necessary measurements, which might involve particle emission or Casimir-type effects. What we can say in the present context is that when we assume that the mean-field terms predominate, we are finding quantum Cramér-Rao bounds on measurements that we do know how to perform, which involve large mean fields.

VI. SIMPLE EXAMPLES

A. Estimation of Constant Metric Components

Consider now making measurements in a local inertial frame where the fiducial metric g(0) is flat or sufficiently flat for differences to be negligible in our calculations. In this case we define a local inertial coordinate system where the fiducial metric is $\eta_{\mu\nu}$. Now further suppose that the perturbed metric has variation $g_{\mu\nu}(\theta) = \eta_{\mu\nu} + \theta\delta_\mu^{\mu_0}\delta_\nu^{\nu_0}$, for some fixed μ_0 and ν_0. In other words, assume that the parameter of interest is θ, and the fixed local coordinates are such that $\delta g_{\mu_0\nu_0} = \theta$. Then the uncertainty relation (4.3) becomes

$$\langle(\delta g_{\mu_0\nu_0})^2\rangle\left\langle\left(\Delta\int_K d\mathring{\mu}\,\hat{T}^{\mu_0\nu_0}\right)^2\right\rangle \geq \hbar^2\ , \tag{6.1}$$

where in this case there is no sum over the repeated indices as we are dealing with a particular metric component specified by the fixed values μ_0 and ν_0. This inequality is reminiscent of the Unruh uncertainty relation (1.5). Indeed, the same restrictions were needed to derive Eq. (1.5) as were used to produce Eq. (6.1).

It should be emphasized that the stress-energy tensor $\hat{T}^{\mu\nu}$ in Eq. (6.1) is for the probe field. It is not the stress-energy tensor for the matter distribution which gives rise to $g_{\mu\nu}$ via Einstein's field equations. This is to be expected, as Eq. (6.1) is essentially an uncertainty relation for the probe field: we are estimating the metric with field measurements, so the uncertainty in the field in Eq. (6.1) has been replaced by uncertainty in our estimate of the metric, just as uncertainty in some "clock" variable is replaced by uncertainty in a time estimate in the time-energy uncertainty relation (1.3). The uncertainty relation (6.1) can be thought of as the minimum uncertainty achievable when attempting to verify with measurements that the metric takes the Minkowski form in the local coordinate system one has defined.

B. Estimation of Proper Time and Proper Distance

Suppose we are interested in the proper time as measured by a stationary observer in a perturbed Minkowski spacetime. Assuming the metric perturbation is compactly supported in space, we might consider the passage of time as measured by an atomic clock at rest within the perturbed region, relative to the passage of time as measured by an atomic clock at rest in flat spacetime outside the perturbed region. Both can be considered proper time, but the latter is also equivalent to our Minkowskian coordinate time. The proper time in our locally defined inertial frame is related to the coordinate time by

$$\tau = \int dt\,\sqrt{-g_{00}}\ . \tag{6.2}$$

In these coordinates, then, we are interested in a metric perturbation of the form $g^{(s)}_{00} = \eta_{00} + sa(x)$, where a is a smooth function of the coordinate 4-position x. Using the approximation

$$\langle[\delta f(X)]^2\rangle \simeq \left(\left.\frac{df}{dX}\right|_{X=\langle X\rangle}\right)^2 \langle(\delta X)^2\rangle \,, \tag{6.3}$$

the uncertainty in the metric is related to the uncertainty in the proper time by

$$\langle(\delta s)^2\rangle = 4\left(\int dt\, a(x)\right)^{-2} \langle(\delta\tau)^2\rangle \,. \tag{6.4}$$

Then the relation (4.2) becomes

$$\left(\int dt\, a(x)\right)^{-2} \langle(\delta\tau)^2\rangle \left\langle\left(\Delta\int d^4x\, \hat{T}^{00}(x)a(x)\right)^2\right\rangle \geq \frac{\hbar^2}{4} \,. \tag{6.5}$$

Now we assume that the effective spatial volume of the confined probe field is small enough that the spacetime perturbation can be considered spatially uniform throughout. Then, recognizing that the integral of the energy density over the spatial component of the four-volume is the Hamiltonian, we have

$$\int d^4x\, \hat{T}^{00}(x)a(t) = \int dt\, \hat{H}(t)a(t) \,. \tag{6.6}$$

Further assuming a time-independent Hamiltonian, the time-integrals cancel and the uncertainty relation reduces to

$$\langle(\delta\tau)^2\rangle\langle(\Delta\hat{H})^2\rangle \geq \frac{\hbar^2}{4} \,. \tag{6.7}$$

This demonstrates that the standard time-energy uncertainty relation is a special case of the metric uncertainty relation (4.2).

Using a similar argument, one can derive a corresponding uncertainty relation for proper distance X:

$$\langle(\delta X)^2\rangle\langle(\Delta\hat{P}_X)^2\rangle \geq \frac{\hbar^2}{4} \,. \tag{6.8}$$

This is the parametric version of the Heisenberg uncertainty relation, where $\hat{P}_X$ is the momentum in the direction of the displacement. It demonstrates the consistency between the metric uncertainty relation (4.2) and the earlier work on parameter-based uncertainty relations for the Lorentz group in flat spacetime [4].

VII. QUANTUM-LIMITED GRAVITATIONAL-WAVE DETECTION

We now consider estimating the amplitude of a gravitational wave. To a good approximation, a gravitational wave can be modeled by a small perturbation of Minkowski spacetime satisfying the linearized Einstein equations. Thus we write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \tag{7.1}$$

where for a plane-fronted, parallel-propagating wave, linearly polarized along the x and y axes and cast in the transverse-traceless gauge [9], the nonvanishing components of the metric perturbation are

$$h_{xx} = -h_{yy} = A_+ f(z-t) \,. \tag{7.2}$$

Physical solutions have a suitably localized envelope along the propagation direction, which can be approximated by a compactly supported function of $z - t$. Physical solutions also are not exactly plane-fronted, but rather are confined in the directions transverse to the propagation direction or, as in the case of astrophysical sources, have spherical wave fronts. We assume for our analysis that any physical deviations from a plane-fronted wave are negligible on the spatial scales of our probe field. Assuming that the gravitational-wave detector, which might be the electromagnetic field confined within a laser-powered interferometer, is compactly supported in space, the intersection with the detector's support is compactly supported in spacetime. The volume integrals of concern to us are therefore well defined.

For simplicity, however, we analyze broadband detection of a gravitational wave, i.e., detection that is essentially instantaneous compared to the scale of variation (period or wavelength) of the gravitational wave. What this means is that the probe field's support is sufficiently confined spatially and temporally relative to the gravitational wave's envelope and wavelength that within the probe's window of observation, the gravitational wave is well approximated by a constant:

$$h_{xx} = -h_{yy} \simeq A \,. \tag{7.3}$$

For consistency with this assumption and to ensure that the relevant volume integrals remain well defined, we assume a finite duration of detection. Since our perturbed metric happens to be in Gaussian normal coordinates (i.e., $g_{00} = -1$ and $g_{0i} = 0$), the resulting coordinate bounds of integration are independent of the perturbation. Our assumptions amount to saying that the probe field is to be turned on and off, i.e., emitted and absorbed, within a compact spatial region in such a way that it senses an essentially instantaneous amplitude of the gravitational wave over this compact spatial region.

The generator (4.4) of changes in the probe field is

$$\begin{aligned}
\hat{P} &= \frac{1}{2}\int d^4x\, \hat{T}^{\mu\nu} \left.\frac{d}{dA}\right|_0 g_{\mu\nu}(A) \\
&= \frac{1}{2}\int d^4x \left(\hat{T}^{xx} \left.\frac{d}{dA}\right|_0 g_{xx}(A) + \hat{T}^{yy} \left.\frac{d}{dA}\right|_0 g_{yy}(A)\right) \\
&= \frac{1}{2}\int d^4x\, (\hat{T}^{xx} - \hat{T}^{yy}) \,,
\end{aligned} \tag{7.4}$$

where we have assumed the fiducial amplitude is zero (i.e., perturbation about flat spacetime). The domain of integration encloses the finite extent of the probe mean field.

We now take the probe field to be the electromagnetic field, having a large mean field that is turned on and off, as we have discussed. We assume that the domain of integration in Eq. (7.4) is large, both temporally and spatially, compared to the scales of variation (periods and wavelengths) of the mean electromagnetic field. We could regard the probe electromagnetic field as being confined within a laser-powered interferometer, as in the LIGO detectors [34–36], but there is no need to specialize to this particular field configuration. Instead, we let the probe be a free electromagnetic field: the field is turned on, receives an imprint from the gravitational wave as it propagates freely through the gravitational wave, and is then turned off. Recall that the Cramér-Rao bound optimizes over all measurements we could make on the probe field, so we do not have to specify what measurement is used to read out the imprint of the gravitational wave on the electromagnetic field, although we will have something to say about this as we proceed. This approach allows us to use the free electromagnetic field and the free-field commutators. In this approach, it is clear that we do not find a "standard quantum limit" that is enforced by back-action forces that act on masses that confine the field, because there are no such masses.

Notice that if we did regard the field as being in an interferometric configuration, there would need to be beam splitters and mirrors to split, confine, and recombine the field. To neglect back-action and thus to be consistent with the present calculation, we could make these optical elements sufficiently massive that they are unaffected by the field's back-action radiation-pressure noise and thus move on geodesics. All of this is consistent with the now well-established result that there is no back-action-enforced "standard quantum limit" that fundamentally limits interferometric gravitational-wave detectors. The absence of a back-action-enforced fundamental limit for interferometric detectors follows from a substantial body of work on specialized, back-action-evading designs for laser-interferometer gravitational-wave detectors [37–39] and from general analyses of quantum limits on the detection of waveforms [40, 41].

For the electromagnetic field, the diagonal components of the stress tensor are

$$\hat{T}^{jj} = \frac{1}{8\pi}\big(\hat{E}_x^2+\hat{E}_y^2+\hat{E}_z^2+\hat{B}_x^2+\hat{B}_y^2+\hat{B}_z^2\big) - \frac{1}{4\pi}\big(\hat{E}_j^2+\hat{B}_j^2\big)\ , \tag{7.5}$$

where $\hat{E}_j^2$ and $\hat{B}_j^2$ are normally ordered and we use cgs Gaussian units with $c = 1$. The local generator of changes in the field due to the gravitational wave is

$$\begin{aligned}\frac{1}{2}(\hat{T}^{xx} - \hat{T}^{yy}) &= \frac{1}{8\pi}(\hat{E}_y^2 - \hat{E}_x^2 + \hat{B}_y^2 - \hat{B}_x^2)\\ &= \frac{1}{8\pi}\sum_\sigma r_\sigma{:}\hat{f}_\sigma^2{:}\ .\end{aligned} \tag{7.6}$$

Here we let $\hat{f}_1 = \hat{E}_y$, $\hat{f}_2 = \hat{E}_x$, $\hat{f}_3 = \hat{B}_y$, $\hat{f}_4 = \hat{B}_x$ and $r_1 = r_3 = 1$, $r_2 = r_4 = -1$. Beginning with the last form, we indicate normal ordering explicitly where it is needed. The generator (7.4) becomes

$$:\!\hat{P}\!: = \frac{1}{8\pi}\sum_\sigma r_\sigma \int d^4x\, {:}\hat{f}_\sigma^2{:}\ . \tag{7.7}$$

Now we express the electric and magnetic fields as a sum of a mean field and field fluctuations, defined as the deviation from the mean:

$$\hat{f}_\sigma(\mathbf{x}, t) = \langle \hat{f}_\sigma(\mathbf{x}, t)\rangle + \Delta\hat{f}_\sigma(\mathbf{x}, t)\ . \tag{7.8}$$

This puts the generator in the form

$$:\!\hat{P}\!: = P + \Delta\hat{\mathcal{X}}_1 + {:}\hat{F}{:}\ , \tag{7.9}$$

where

$$P = \frac{1}{8\pi}\sum_\sigma r_\sigma \int d^4x\, \langle \hat{f}_\sigma\rangle^2\ , \tag{7.10}$$

$$\hat{\mathcal{X}}_1 = \frac{1}{4\pi}\sum_\sigma r_\sigma \int d^4x\, \langle \hat{f}_\sigma\rangle \hat{f}_\sigma\ , \tag{7.11}$$

$$:\!\hat{F}\!: = \frac{1}{8\pi}\sum_\sigma r_\sigma \int d^4x\, {:}(\Delta\hat{f}_\sigma)^2{:}\ . \tag{7.12}$$

Notice that we do not need to normal order $\hat{\mathcal{X}}_1$ because it is linear in field operators.

Note that by our formalism, the quantum fields in Eqs. (7.5)–(7.12) need only be evaluated in the fiducial spacetime, which in the present case is flat. Therefore, the vacuum state is unambiguous, and the splitting of the field operators into positive- and negative-frequency parts and the use of normal ordering are appropriate and well defined. (For a discussion of the issues arising in curved spacetime, see Sec. 1 of [5].) Thus we have

$$\begin{aligned}{:}\Delta\hat{f}_\sigma^2{:} &= {:}[\Delta\hat{f}_\sigma^{(+)} + \Delta\hat{f}_\sigma^{(-)}]^2{:}\\ &= 2\Delta\hat{f}_\sigma^{(-)}\Delta\hat{f}_\sigma^{(+)} + \Delta\hat{f}_\sigma^{(+)2} + \Delta\hat{f}_\sigma^{(-)2}\ .\end{aligned} \tag{7.13}$$

The free-field commutators and vacuum correlators that we need are summarized in Appendix C.

Our separation of the field operators into a mean field plus field fluctuations is different from our treatment in Sec. V, where we separated the stress-energy, which is generally quadratic in field operators, into its mean and its fluctuation about the mean. To identify the mean-field-independent contributions to the variance, we view the state as being obtained from a zero-mean-field state by a displacement operator D, which is generated by a linear function of the fields. This operator is determined by requiring that $D^\dagger f_\sigma D = f_\sigma + \langle f_\sigma\rangle$. The displacement parameter is the mean field. For the present purposes, the zero-mean-field state is the reference state and is considered fixed. One example is where this reference state is the vacuum state. Our initial arguments apply to all zero-mean-field reference states, except where noted otherwise, and we eventually get to the case of a squeezed-vacuum state as the reference state that provides optimal sensitivity under the assumptions we make. Our main conclusions are aimed at the case where the displacement is large, in which case we only keep the terms that are leading order in the displacement.

The expectation value of the generator (7.9) is

$$\langle{:}\hat{P}{:}\rangle = P + \langle{:}\hat{F}{:}\rangle \;. \tag{7.14}$$

Using

$$\begin{aligned}{:}\hat{P}{:}{:}\hat{P}{:} = P^2 + 2P{:}\hat{F}{:} + (\Delta\hat{\mathcal{X}}_1)^2 + {:}\hat{F}{:}{:}\hat{F}{:}\\ + 2P\Delta\hat{\mathcal{X}}_1 + \Delta\hat{\mathcal{X}}_1{:}\hat{F}{:} + {:}\hat{F}{:}\Delta\hat{\mathcal{X}}_1 \;,\end{aligned} \tag{7.15}$$

we have

$$\begin{aligned}\langle{:}\hat{P}{:}{:}\hat{P}{:}\rangle &= P^2 + 2P\langle{:}\hat{F}{:}\rangle + \langle(\Delta\hat{\mathcal{X}}_1)^2\rangle + \langle{:}\hat{F}{:}{:}\hat{F}{:}\rangle\\ &= \langle{:}\hat{P}{:}\rangle^2 + \langle(\Delta\hat{\mathcal{X}}_1)^2\rangle + \langle{:}\hat{F}{:}{:}\hat{F}{:}\rangle - \langle{:}\hat{F}{:}\rangle^2 \;.\end{aligned} \tag{7.16}$$

Here we assume that the odd moments of the reference (zero-mean-field) state are zero, which is the case if the reference state is Gaussian or is invariant under parity and time reversal.

Rewriting Eq. (7.16) in terms of the variance, we get

$$\langle(\Delta\hat{P})^2\rangle = \langle{:}\hat{P}{:}{:}\hat{P}{:}\rangle - \langle{:}\hat{P}{:}\rangle^2 = \langle(\Delta\hat{\mathcal{X}}_1)^2\rangle + \langle(\Delta\hat{F})^2\rangle \;. \tag{7.17}$$

We subsume the normal ordering into the definition of the definition of the variance $(\Delta\hat{F})^2$ when using this notation. The term $\langle(\Delta\hat{F})^2\rangle$ is mean field independent, so for large mean field, we can drop it. Before doing so, however, it is worth taking a closer look at the mean-field-independent contributions. When we put the right-hand side of Eq. (7.13) into the spacetime integral (7.12) to get ${:}\hat{F}{:}$, we can expand the field operators in the last two terms of Eq. (7.13) into integrals over the wave vectors of free-field plane-wave modes, as in Appendix C. Performing the spacetime integral first, the amplitudes for a pair of wave vectors, $\mathbf{k}$ and $\mathbf{k}'$, average to nearly zero, except for field modes whose period and wavelength are as large or larger than the temporal and spatial extent of the region of integration. Realistic measurement devices such as laser interferometers are neither designed for nor capable of detecting such low-frequency photons. If we neglect these essentially DC contributions, we are left with

$${:}\hat{F}{:} = \frac{1}{4\pi}\sum_\sigma r_\sigma \int d^4x\,\Delta\hat{f}_\sigma^{(-)}\Delta\hat{f}_\sigma^{(+)} \;, \tag{7.18}$$

where the equals sign now assumes that we have omitted the DC contributions. The corresponding variance is

$$\begin{aligned}\langle(\Delta\hat{F})^2\rangle = \langle{:}\hat{F}{:}{:}\hat{F}{:}\rangle - \langle{:}\hat{F}{:}\rangle^2 = \frac{1}{16\pi^2}\sum_{\sigma,\sigma'} r_\sigma r_{\sigma'}\int d^4x\,d^4x'\Big(&\big\langle\Delta\hat{f}_\sigma^{(-)}(\mathbf{x},t)\Delta\hat{f}_\sigma^{(+)}(\mathbf{x},t)\Delta\hat{f}_{\sigma'}^{(-)}(\mathbf{x}',t')\Delta\hat{f}_{\sigma'}^{(+)}(\mathbf{x}',t')\big\rangle\\ &- \big\langle\Delta\hat{f}_\sigma^{(-)}(\mathbf{x},t)\Delta\hat{f}_\sigma^{(+)}(\mathbf{x},t)\big\rangle\big\langle\Delta\hat{f}_{\sigma'}^{(-)}(\mathbf{x}',t')\Delta\hat{f}_{\sigma'}^{(+)}(\mathbf{x}',t')\big\rangle\Big) \;.\end{aligned} \tag{7.19}$$

If the electromagnetic field is excited into a coherent state, where the field fluctuations are those of vacuum, both $\langle{:}\hat{F}{:}\rangle$ and $\langle(\Delta\hat{F})^2\rangle$, as calculated from Eqs. (7.18) and (7.19) vanish. For coherent states, the only mean-field-independent contributions to the total variance of $\hat{F}$ come from the DC terms discarded in going from Eq. (7.13) to Eq. (7.18).

If the field fluctuations are redistributed relative to vacuum, as in the squeezed state discussed below, the terms of $\langle(\Delta\hat{F})^2\rangle$ given in Eq. (7.19) make the dominant mean-field-independent contribution, expressing the fact that these nonvacuum field fluctuations are affected by the presence of a gravitational wave and can be used to detect the wave. For sufficiently large mean field, these mean-field-independent contributions are small compared to $\langle(\Delta\hat{\mathcal{X}})_1^2\rangle$, leaving us with

$$\langle(\Delta\hat{P})^2\rangle = \langle(\Delta\hat{\mathcal{X}}_1)^2\rangle \;, \tag{7.20}$$

as we assume henceforth.

We can now summarize our results by saying that the Cramér-Rao bound (4.2) on the estimate of the gravitational-wave amplitude A is

$$\langle(\delta\tilde{A})^2\rangle\langle(\Delta\hat{\mathcal{X}}_1)^2\rangle \geq \frac{\hbar^2}{4} \;. \tag{7.21}$$

We could stop here, having confirmed the valuable lesson, generic to Cramér-Rao bounds, that precise determination of the gravitational-wave amplitude requires that the observable $\hat{\mathcal{X}}_1$, which in the presence of a large mean field generates the change in the probe-field state, be as uncertain as possible. In this case, however, we can say considerably more.

Since $\hat{\mathcal{X}}_1$ is linear in the fields, one can find an observable $\hat{\mathcal{X}}_2$, conjugate to $\hat{\mathcal{X}}_1$ and also linear in the fields, which is the observable one should measure to effect the precise determination of A. The commutator of $\hat{\mathcal{X}}_1$ and $\hat{\mathcal{X}}_2$ is

$$[\hat{\mathcal{X}}_1, \hat{\mathcal{X}}_2] = i\hbar C \ , \tag{7.22}$$

where the real constant C is to be determined (we can make C positive by, say, changing the sign of $\hat{\mathcal{X}}_2$). The commutator implies a Heisenberg uncertainty relation, precisely analogous to the position-momentum uncertainty relation (1.2),

$$\langle(\Delta\hat{\mathcal{X}}_1)^2\rangle\langle(\Delta\hat{\mathcal{X}}_2)^2\rangle \ge \frac{\hbar^2}{4}C^2 \ . \tag{7.23}$$

To put these two observables on the same footing relative to vacuum, we require that

$$\langle 0|\hat{\mathcal{X}}_1^2|0\rangle = \langle 0|\hat{\mathcal{X}}_2^2|0\rangle = \frac{\hbar}{2}C \ . \tag{7.24}$$

The observables $\hat{\mathcal{X}}_1$ and $\hat{\mathcal{X}}_2$ are generalized quadrature components [42–44] for the single field mode that is determined with respect to the Minkowski vacuum by the mean field according to the definition of $\hat{\mathcal{X}}_1$, with their vacuum level of noise given by $\hbar C/2$. We calculate $\hat{\mathcal{X}}_2$ explicitly below after restricting to the case of a plane wave.

Equation (3.5) specifies the response of $\hat{\mathcal{X}}_2$ to the gravitational wave,

$$\frac{d\hat{\mathcal{X}}_2}{dA} = \frac{i}{\hbar}[\hat{\mathcal{X}}_2, \hat{\mathcal{X}}_1] = C \ . \tag{7.25}$$

Linear-response analysis gives the variance of an estimate of A based on a measurement of $\hat{\mathcal{X}}_2$,

$$\langle(\delta\tilde{A})^2\rangle = \frac{\langle(\Delta\hat{\mathcal{X}}_2)^2\rangle}{|d\langle\hat{\mathcal{X}}_2\rangle/dA|^2} = \frac{\langle(\Delta\hat{\mathcal{X}}_2)^2\rangle}{C^2} \ge \frac{\hbar^2}{4}\frac{1}{\langle(\Delta\hat{\mathcal{X}}_1)^2\rangle} \ , \tag{7.26}$$

matching the Cramér-Rao bound (7.21). If the probe field is placed in a minimum-uncertainty state relative to the uncertainty relation (7.23), the bound (7.26) is saturated, and it is particularly useful to write the variance of the estimate as

$$\langle(\delta\tilde{A})^2\rangle = \frac{\langle(\Delta\hat{\mathcal{X}}_2)^2\rangle}{C^2} = \frac{\hbar}{2C}\sqrt{\frac{\langle(\Delta\hat{\mathcal{X}}_2)^2\rangle}{\langle(\Delta\hat{\mathcal{X}}_1)^2\rangle}} \ . \tag{7.27}$$

We stress that Eq. (7.27) is not a general expression for the Cramér-Rao bound, but rather is the form the bound assumes for minimum-uncertainty states relative to the uncertainty relation (7.23).

The physical content here is that if the field is excited into a coherent state, the uncertainties in the quadrature components are equal, and the variance of the estimate of A, equal to $\hbar/2C$, is set by the vacuum-level noise in $\hat{\mathcal{X}}_1$ and $\hat{\mathcal{X}}_2$. To achieve a sensitivity better than $\hbar/2C$, one should squeeze the vacuum so that the uncertainty in $\hat{\mathcal{X}}_2$ decreases and the uncertainty in $\hat{\mathcal{X}}_1$ increases, as in the original proposal for decreasing shot noise in a laser-interferometer gravitational-wave detector by using squeezed light [45], a proposal that has been implemented in large-scale laser-interferometer detectors [46, 47] and might be incorporated into Advanced LIGO [48].

There is one task remaining, quite an important one, and that is to evaluate the constant C. To do that, we specialize a bit, to the case where the mean probe field is that of a nearly plane wave propagating in the x direction and linearly polarized along the y axis. We do not need to assume that this wave is close to monochromatic, but we do assume that the transverse extent of the wave is much larger than the wave's typical wavelengths. We neglect the small corrections to a plane wave due to the finite transverse extent. With these assumptions, we have $\langle\hat{E}_x\rangle = \langle\hat{E}_z\rangle = \langle\hat{B}_x\rangle = \langle\hat{B}_y\rangle = 0$ and

$$\langle\hat{E}_y\rangle = \langle\hat{B}_z\rangle = E_1(\mathbf{x}, t) \ , \tag{7.28}$$

where the (real) waveform $E_1(\mathbf{x}, t)$ is mainly a function of $x - t$ and only a weak function of y and z. With these assumptions, we have

$$\hat{\mathcal{X}}_1 = \frac{1}{4\pi}\int d^4x\, E_1(\mathbf{x}, t)\hat{E}_y(\mathbf{x}, t) \ . \tag{7.29}$$

It is useful to divide E_1 into positive- and negative-frequency parts and to write these in terms of the Fourier transform,

$$E_1(\mathbf{x}, t) = E_1^{(+)}(\mathbf{x}, t) + E_1^{(-)}(\mathbf{x}, t) \ , \tag{7.30}$$

$$E_1^{(+)}(\mathbf{x}, t) = E_1^{(-)*}(\mathbf{x}, t) = i\sum_\sigma \int \frac{d^3k}{(2\pi)^3}\sqrt{2\pi\hbar\omega}\,\alpha_{1;\mathbf{k}\sigma}\mathbf{e}_{\mathbf{k}\sigma}\cdot\mathbf{e}_y e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = i\int \frac{d^3k}{(2\pi)^3}\sqrt{2\pi\hbar\omega}\,\alpha_{1,\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \ , \tag{7.31}$$

where $\alpha_{1;\mathbf{k}\sigma} = \langle a_{\mathbf{k}\sigma}\rangle$. The assumption of a nearly plane wave propagating in the x direction is that $\alpha_{1;\mathbf{k}\sigma}$ has substantial support only for $\mathbf{k}$ pointing nearly along the x direction, with linear polarization nearly along the y direction, in which case we drop the polarization index and write it as $\alpha_{1,\mathbf{k}}$ (formally, we might write $\alpha_{1;\mathbf{k}\sigma} = \delta_{\sigma y}\alpha_{1,\mathbf{k}}$); this leads to the final form in Eq. (7.31).

We assume that $\hat{\mathcal{X}}_2$ looks the same as $\hat{\mathcal{X}}_1$,

$$\hat{\mathcal{X}}_2 = \frac{1}{4\pi}\int d^4x\, E_2(\mathbf{x},t)\hat{E}_y(\mathbf{x},t)\;, \tag{7.32}$$

but with a different (real) waveform E_2, which is also a nearly plane wave propagating in the x direction, with linear polarization nearly along the y direction,

$$E_2(\mathbf{x},t) = E_2^{(+)}(\mathbf{x},t) + E_2^{(-)}(\mathbf{x},t)\;, \tag{7.33}$$

$$E_2^{(+)}(\mathbf{x},t) = E_2^{(-)*}(\mathbf{x},t) = i\sum_\sigma\int\frac{d^3k}{(2\pi)^3}\sqrt{2\pi\hbar\omega}\,\alpha_{2;\mathbf{k}\sigma}\mathbf{e}_{\mathbf{k}\sigma}\cdot\mathbf{e}_y e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = i\int\frac{d^3k}{(2\pi)^3}\sqrt{2\pi\hbar\omega}\,\alpha_{2,\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\;. \tag{7.34}$$

Again we understand that $\alpha_{2,\mathbf{k}}$ has support only for $\mathbf{k}$ close to the x direction and corresponds to linear polarization nearly along the y direction ($\alpha_{2;\mathbf{k}\sigma} = \delta_{\sigma y}\alpha_{2,\mathbf{k}}$). The following calculations show that the above assumption is warranted. Specifically, we find in Eq. (7.43) that E_2 can be obtained from E_1 by a 90° phase shift of every monochromatic mode that contributes to E_1, as one might expect for a broadband version of conjugate quadrature components.

Notice that in the expressions (7.29) and (7.32) for $\hat{\mathcal{X}}_1$ and $\hat{\mathcal{X}}_2$, we can extend the spatial integrals over all of space because the waveforms $E_1(\mathbf{x},t)$ and $E_2(\mathbf{x},t)$ are zero outside the original domain of spatial integration.

To determine C, we use the field commutators and vacuum correlators of Appendix C [see Eqs. (C12) and (C14)] to find the commutator (7.22) and the second moments (7.24):

$$\begin{aligned}\big[\hat{\mathcal{X}}_1,\hat{\mathcal{X}}_2\big] &= \frac{1}{16\pi^2}\int d^4x\,d^4x'\,E_1(\mathbf{x},t)E_2(\mathbf{x}',t')\big[\hat{E}_y(\mathbf{x},t),\hat{E}_y(\mathbf{x}',t')\big]\\ &= \frac{i\hbar}{16\pi^2}\int d^4x\,d^4x'\,E_1(\mathbf{x},t)E_2(\mathbf{x}',t')\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}\right)G(\mathbf{x}-\mathbf{x}',t-t')\;,\end{aligned} \tag{7.35}$$

$$\begin{aligned}\langle 0|\hat{\mathcal{X}}_a^2|0\rangle &= \frac{1}{16\pi^2}\int d^4x\,d^4x'\,E_a(\mathbf{x},t)E_a(\mathbf{x}',t')\frac{1}{2}\langle 0\big|[\hat{E}_y(\mathbf{x},t)\hat{E}_y(\mathbf{x}',t')+\hat{E}_y(\mathbf{x}',t')\hat{E}_y(\mathbf{x},t)]\big|0\rangle\\ &= \frac{\hbar}{16\pi^3}\int d^4x\,d^4x'\,E_a(\mathbf{x},t)E_a(\mathbf{x}',t')\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial y^2}\right)D(\mathbf{x}-\mathbf{x}',t-t')\;,\quad a=1,2.\end{aligned} \tag{7.36}$$

Here $G(\mathbf{x},t)$, the difference between retarded and advanced Green functions, is defined in Eq. (C10), and $D(\mathbf{x},t)$, the principal value of the inverse of the invariant interval, is defined in Eq. (C11).

In Eqs. (7.35) and (7.36), we can integrate by parts twice on the y derivatives. The boundary terms vanish because we can take the boundary of the region of integration to be outside the spatial extent of the waveforms E_1 and E_2, and we can neglect the resulting integrals because E_1 and E_2 are weak functions of y. The upshot is that we can omit the y derivatives in Eqs. (7.35) and (7.36). Using Eqs. (C10) and (C11) to start getting back into the Fourier domain, we have

$$\big[\hat{\mathcal{X}}_1,\hat{\mathcal{X}}_2\big] = \frac{i\hbar}{4\pi}\,\mathrm{Im}\int\frac{d^3k}{(2\pi)^3}\,\omega\left(\int dt\,e^{-i\omega t}\int d^3x\,E_1(\mathbf{x},t)e^{i\mathbf{k}\cdot\mathbf{x}}\right)\left(\int dt'\,e^{i\omega t'}\int d^3x'\,E_2(\mathbf{x}',t')e^{-i\mathbf{k}\cdot\mathbf{x}'}\right)\;, \tag{7.37}$$

$$\langle 0|\hat{\mathcal{X}}_a^2|0\rangle = \frac{\hbar}{8\pi}\,\mathrm{Re}\int\frac{d^3k}{(2\pi)^3}\,\omega\left(\int dt\,e^{-i\omega t}\int d^3x\,E_a(\mathbf{x},t)e^{i\mathbf{k}\cdot\mathbf{x}}\right)\left(\int dt'\,e^{i\omega t'}\int d^3x'\,E_a(\mathbf{x}',t')e^{-i\mathbf{k}\cdot\mathbf{x}'}\right)\;,\quad a=1,2. \tag{7.38}$$

The spatial Fourier transforms are

$$e^{-i\omega t}\int d^3x\,E_a(\mathbf{x},t)e^{i\mathbf{k}\cdot\mathbf{x}} = \sqrt{2\pi\hbar\omega}\big(-i\alpha^*_{a,\mathbf{k}} + i\alpha_{a,-\mathbf{k}}e^{-2i\omega t}\big)\;. \tag{7.39}$$

The counter-rotating terms average to nearly zero in the temporal integrals, so we discard them and obtain

$$\int dt\,e^{-i\omega t}\int d^3x\,E_a(\mathbf{x},t)e^{i\mathbf{k}\cdot\mathbf{x}} = -i\tau\sqrt{2\pi\hbar\omega}\,\alpha^*_{a,\mathbf{k}}\;, \tag{7.40}$$

where τ is the time interval over which the mean field is turned on. Our final results for the commutator and vacuum second moment are

$$[\hat{\mathcal{X}}_1,\hat{\mathcal{X}}_2]=i\hbar\bigg(\frac{1}{2}\hbar\int\frac{d^3k}{(2\pi)^3}\,(\omega\tau)^2\mathrm{Im}\big(\alpha^*_{1,\mathbf{k}}\alpha_{2,\mathbf{k}}\big)\bigg)\;, \tag{7.41}$$

$$\langle 0|\hat{\mathcal{X}}_a^2|0\rangle=\frac{\hbar}{2}\bigg(\frac{1}{2}\hbar\int\frac{d^3k}{(2\pi)^3}\,(\omega\tau)^2|\alpha_{a,\mathbf{k}}|^2\bigg)\;,\quad a=1,2. \tag{7.42}$$

A glance at Eqs. (7.22) and (7.24) shows that the quantities in large parentheses are all equal to C. Since we want this to be true whatever the probe waveform is, we must have

$$\alpha_{2,\mathbf{k}}=i\alpha_{1,\mathbf{k}}\;; \tag{7.43}$$

i.e., as promised, E_2 is obtained from E_1 by a 90° phase shift of every monochromatic mode that contributes to E_1. Finally, we obtain

$$C=\frac{1}{2}\hbar\int\frac{d^3k}{(2\pi)^3}\,(\omega\tau)^2|\alpha_{1,\mathbf{k}}|^2\;. \tag{7.44}$$

Consider now a nearly monochromatic mean field with wave vector $\mathbf{k}=\omega\mathbf{e}_x$. The vacuum level of noise in the estimate of the gravitational-wave amplitude is $\hbar/2C=1/(\omega\tau)^2\bar{n}$, where $\bar{n}$ is the number of photons carried by the mean field. We have $E_1^{(+)}\propto\alpha_2 e^{i\omega(x-t)}$ and $E_2^{(+)}\propto i\alpha_2 e^{i\omega(x-t)}$; writing $\alpha_2=|\alpha_2|e^{i\phi}$, we have $E_1\propto\alpha_2\cos[\omega(t-x)-\phi]$ and $E_2\propto\alpha_2\sin[\omega(t-x)-\phi]$. Thus $\hat{\mathcal{X}}_1$ and $\hat{\mathcal{X}}_2$ are proportional to the standard quadrature components for a monochromatic field mode.

In the case of a nearly monochromatic mean field, the quantum-limited sensitivity (7.27) for detecting a gravitational-wave amplitude becomes

$$\langle(\delta\tilde{A})^2\rangle=\frac{1}{(\omega\tau)^2\bar{n}}\sqrt{\frac{\langle(\Delta\hat{\mathcal{X}}_2)^2\rangle}{\langle(\Delta\hat{\mathcal{X}}_1)^2\rangle}}\;. \tag{7.45}$$

If the field is excited into a nearly monochromatic coherent state, the quadrature components have equal, vacuum-level uncertainties, $\bar{n}=\langle\hat{n}\rangle=\langle(\Delta\hat{n})^2\rangle$ is the expectation value and the variance in the number of photons, and the sensitivity is shot-noise-limited, i.e., $\langle(\delta\tilde{A})^2\rangle^{1/2}=1/\omega\tau\langle\hat{n}\rangle^{1/2}$. This result has a physically intuitive interpretation. The gravitational wave changes the coordinate speed of light in the x direction by $-A/2$, leading to a phase shift $\delta\phi=(\omega\tau)A/2$; the shot-noise limit on detecting the gravitational-wave amplitude translates to $\langle(\delta\phi)^2\rangle\langle(\Delta\hat{n})^2\rangle=\frac{1}{4}$, which is the conventional uncertainty-principle bound on phase and photon number. To do better than shot noise, one can squeeze the $\hat{\mathcal{X}}_2$ quadrature, reducing its uncertainty while increasing the uncertainty in the $\hat{\mathcal{X}}_2$ quadrature.

A bonus of our approach is that Eq. (7.27) gives us the quantum limit on detecting a gravitational wave using a nearly plane-wave, but broadband probe field:

$$\langle(\delta\tilde{A})^2\rangle=\bigg(\int\frac{d^3k}{(2\pi)^3}\,(\omega\tau)^2|\alpha_{1,\mathbf{k}}|^2\bigg)^{-1}\sqrt{\frac{\langle(\Delta\hat{\mathcal{X}}_2)^2\rangle}{\langle(\Delta\hat{\mathcal{X}}_1)^2\rangle}}\;. \tag{7.46}$$

Comparison to the monochromatic sensitivity (7.45) shows that the way to generalize $(\omega\tau)^2\bar{n}$ to a broadband mean field is to integrate over contributions from all the monochromatic modes. If the field is excited into a broadband coherent state, the quantum-limited sensitivity is given by a sort of generalized shot noise quantified by this frequency-weighted integration over mean photon numbers in the monochromatic modes.

To do better than shot-noise-limited sensitivity, one should put the appropriate field mode into a squeezed state. We can write down the required squeezed state by noting that for a nearly plane wave, the field quadratures take the form $\hat{\mathcal{X}}_1=\sqrt{\hbar C/2}\,(\hat{b}+\hat{b}^\dagger)$ and $\hat{\mathcal{X}}_2=\sqrt{\hbar C/2}\,(-i\hat{b}+i\hat{b}^\dagger)$ (see Appendix C), where

$$\hat{b}=\frac{\tau}{\sqrt{2\hbar C}}\int\frac{d^3k}{(2\pi)^3}\,\hbar\omega\,\alpha^*_{1,\mathbf{k}}\hat{a}_{\mathbf{k}y} \tag{7.47}$$

and $\hat{b}^\dagger$ satisfy the canonical bosonic commutation relation, $[\hat{b},\hat{b}^\dagger]=1$. This means that the desired minimum-uncertainty state is the squeezed state

$$e^{\mu\hat{b}^\dagger-\mu^*\hat{b}}\exp\bigg(\frac{1}{2}r\big[(\hat{b}^\dagger)^2-\hat{b}^2\big]\bigg)|0\rangle\;, \tag{7.48}$$

where μ and r are real, with μ chosen to give the assumed mean field $E_1(\mathbf{x},t)$. Indeed, one can see that this state has

$$\langle a_{\mathbf{k}y}\rangle=\frac{\mu\tau}{\sqrt{2\hbar C}}\,\hbar\omega\,\alpha_{1,\mathbf{k}}\;, \tag{7.49}$$

so consistency requires that $\mu=\sqrt{2\hbar C}/\hbar\omega\tau$. The squeezed state (7.48) has $\langle\hat{b}\rangle=\mu$, $\langle\hat{\mathcal{X}}_1\rangle=\sqrt{2\hbar C}\mu=2C/\omega\tau$, $\langle\hat{\mathcal{X}}_2\rangle=0$, and

$$\begin{aligned}\langle(\Delta\hat{\mathcal{X}}_1)^2\rangle&=\frac{\hbar C}{2}e^{2r}\;,\\ \langle(\Delta\hat{\mathcal{X}}_2)^2\rangle&=\frac{\hbar C}{2}e^{-2r}\;.\end{aligned} \tag{7.50}$$

It thus beats shot-noise-limited sensitivity by a factor of e^{-2r}.

VIII. OTHER APPLICATIONS

We now consider briefly a few of the many other applications of our formalism.

Cosmology is one field where accurate measurement of gravitational parameters is of obvious interest. For a simple example, consider the spatially closed Friedmann-Lemaitre-Robertson-Walker spacetime. This is a universe filled with a uniform density of matter, e.g., "galaxies," and radiation. At any instant in time, in the comoving frame of the galaxies, the universe looks the same everywhere (homogeneous) and in all directions (isotropic). The metric for this universe is given by [9]

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \ , \tag{8.1}$$

where t is the proper time of an observer comoving with any of the galaxies. The spatial coordinates χ, θ, ϕ describe homogeneous and isotropic three-spheres of constant proper time t. The function $a(t)$, known as the expansion parameter, is the ratio of the proper distance between any two galaxies at the initial time $t = 0$ and the time t.

During an infinitesimal duration of proper time dt a photon travels the distance $d\eta = dt/a(t)$. It is convenient to use η, known as the conformal time coordinate, as the time parameter. Transforming to conformal time has the effect of shunting the time dependence into a conformal factor. We furthermore consider a universe dominated by matter, in which case $a(\eta) = a_{\max}(1 - \cos\eta)$ and the metric becomes

$$ds^2 = \frac{a_{\max}^2}{4}(1-\cos\eta)^2\left[-d\eta^2 + d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \ , \tag{8.2}$$

where η runs between 0 at the beginning of expansion to 2π at the end of recontraction. We wish to estimate the parameter $a_{\max}$ which controls the maximum size the universe reaches before contraction commences. Since $dg_{\mu\nu}/da_{\max} = (2/a_{\max})g_{\mu\nu}$, the operator (4.4) becomes

$$\begin{aligned}\hat{P}(a_{\max}) &= \frac{a_{\max}^3}{16}\int_M d\eta\, d\chi\, d\theta\, d\phi\, (1-\cos\eta)^4 \sin^2\chi \sin\theta\, g_{\mu\nu}\hat{T}^{\mu\nu} \\ &= \frac{a_{\max}^5}{64}\int_M d\eta\, d\chi\, d\theta\, d\phi\, (1-\cos\eta)^6 \sin^2\chi \sin\theta\Big[-\hat{T}^{\eta\eta} + \hat{T}^{\chi\chi} + \sin^2\chi\big(\hat{T}^{\theta\theta} + \hat{T}^{\phi\phi}\sin^2\theta\big)\Big] \ .\end{aligned} \tag{8.3}$$

It is interesting to note that since we are estimating a scale factor, the above integrand is proportional to the trace of the stress-energy tensor. Thus, for a any field with a traceless stress-energy tensor, such as the free electromagnetic field, $\langle(\Delta\hat{P})^2\rangle$ vanishes, and we get no information about the scale factor. This is an expression of the well-known scale invariance of the electromagnetic field.

To give another cosmological example, suppose we are interested in measuring the cosmological constant Λ in a de Sitter universe. Again using the conformal time coordinate, the line element can be written as

$$ds^2 = \frac{3}{\Lambda}\sec^2\eta\left(-d\eta^2 + d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right) \ , \tag{8.4}$$

which gives us

$$\hat{P}(\Lambda) = -\frac{9}{2\Lambda^3}\int_M d\eta\, d\chi\, d\theta\, d\phi\, \sec^4\eta \sin^2\chi \sin\theta\, g_{\mu\nu}\hat{T}^{\mu\nu} \ . \tag{8.5}$$

Again, because we are estimating a scale factor, the integrand is proportional to the trace of the stress-energy tensor, which vanishes for the free electromagnetic field.

These and other cosmological parameters represent overall scale factors of the universe. As such, the conformally invariant electromagnetic field alone is not an adequate probe. For example, it is only possible to measure the cosmological redshift of light if the atomic emission spectrum of its source is also known. Indeed, Penrose has essentially argued that should all matter in the universe decay into photons, the scale of the universe would become unobservable and thus physically irrelevant [49]. So a more useful calculation should include fermionic fields, which break this scale invariance. Alternatively, a massive scalar or boson field would also yield a finite Cramér-Rao bound, since its stress-energy has nonvanishing trace.

A more down-to-earth application is that of a gravimeter. One way to deal with this case is to approximate the near-earth spacetime by a Schwarzschild metric, and assume the floor of the laboratory has constant Schwarzschild coordinate radius; then the problem of gravimetry becomes one of estimating the Schwarzschild mass of the Earth. This problem certainly lends itself to our method, provided the appropriate stress-energy correlations in a Schwarzshild background are calculated. Of course, the same calculation would also be applicable to a black hole.

Our method is also applicable to estimation of dynamical quantities of a probe in flat spacetime, such as acceleration, rotation, etc. One approach is to assume such dynamics are due to coupling with a nongravitational, classical field. The locally covariant approach then "applies, *mutatis mutandis*, also to this case" [5], and so also does our Cramér-Rao bound.

IX. CONCLUSION

In this paper we presented the quantum Cramér-Rao lower bound for the uncertainty in estimating parameters describing a spacetime metric. Our specific derivation applies for any quantum state on an arbitrary globally hyperbolic manifold. To demonstrate the utility of our formalism, we applied it to estimation of metric components and found uncertainty principles akin to those found by Unruh [10] using heuristic arguments. We also considered quantum estimation of gravitational-wave amplitude and obtained generalizations of known quantum limits for laser interferometers such as LIGO.

Appendix A: Category theory framework

In this appendix we briefly review category theory as employed by Brunetti, Fredenhagen, and Verch in [5]. A primary motivation is that it provides a convenient way to rigorously define the change in a quantum observable due to a spacetime perturbation, which we outline here. To begin with, a *category* (or more precisely, a *concrete category*) consists of objects and functions between objects called *morphisms*. A category of particular relevance for our purposes, given in [5], is the following:

Definition. $\mathfrak{Man}$ *is the category whose objects are globally hyperbolic spacetimes and whose morphisms are isometric embeddings (or in other words inclusion maps).*

We next consider quantum fields in those spacetimes, formulated in terms of C^*-algebras. The relevant category of C^*-algebras is given in [5] as follows:

Definition. $\mathfrak{Alg}$ *is the category whose objects are C^*-algebras and whose morphisms are injective *-homomorphisms.*

To associate C^*-algebras to our spacetimes, we use a covariant functor from the spacetimes to C^*-algebras. A *functor* is a function between categories which maps objects to objects and morphisms to morphisms, such that the identity maps to the identity and compositions map to compositions. A *covariant functor* is a functor that maps domains to domains and images to images (pictorially, it preserves the directions of morphism arrows). Thus we arrive at the following [5]:

Definition. *A locally covariant quantum field theory is a covariant functor,*

$$\mathscr{A} : \mathfrak{Man} \to \mathfrak{Alg} \ .$$

Note that a locally covariant quantum field theory is local in the sense that it maps submanifolds of manifolds to subalgebras of the corresponding algebras.

We further require that any causal, locally covariant quantum field theory obey the following axiom [5]:

Axiom. (Time-Slice Axiom) *If $\mathscr{A}$ is a locally covariant quantum field theory, $(N, g), (M, g) \in \mathfrak{Man}$, and $\psi \in \hom\big((N, g), (M, g)\big)$ such that $\psi(N, g)$ contains a Cauchy surface of (M, g), then $\mathscr{A}(\psi)(\mathscr{A}(N, g)) = \mathscr{A}(M, g)$.*

In other words, the algebra associated with a Cauchy surface of a manifold determines the algebra associated with the entire manifold.

The perturbed spacetime discussed above, along with the embedded subregions $N_\pm$, the locally covariant quantum field theory thereon, and the relevant morphisms, can all be represented by the diagram in Fig. 2, where $\mathring{\mathrm{g}} = \mathrm{g}^{(0)}$ and $\mathrm{g} = \mathrm{g}^{(s)}$. Note that the assumption of the time-slice axiom implies the morphisms $\mathscr{A}(N_\pm, \mathrm{g}_{N_\pm}) \to \mathscr{A}(M, g)$ are bijective. Therefore, the arrows in the right-hand side of the diagram are invertible. This allows these morphisms to be composed in such a way as to construct an automorphism on $\mathscr{A}(M, \mathrm{g}^{(0)})$:

$$\beta_\mathrm{g} = \alpha_{\psi_\circ^-} \circ \alpha^{-1}_{\psi_\mathrm{g}^-} \circ \alpha_{\psi_\mathrm{g}^+} \circ \alpha^{-1}_{\psi_\circ^+} \ . \tag{A1}$$

By the Gelfand-Neimark-Segal construction [27–29], every C^*-algebra admits a linear *-representation π by bounded operators on a Hilbert space. Thus β_g induces an automorphism on Hilbert-space operators. Thus any operator $\hat{A} = \pi(A)$, where $A \in \mathscr{A}(M, \mathrm{g}^{(0)})$, i.e., any operator associated with our fiducial spacetime, can be said to evolve under our s-parametrized spacetime perturbation into $\hat{A}(s) = \pi(\beta_{\mathrm{g}(s)} A)$. This process is termed *relative Cauchy evolution* [5].

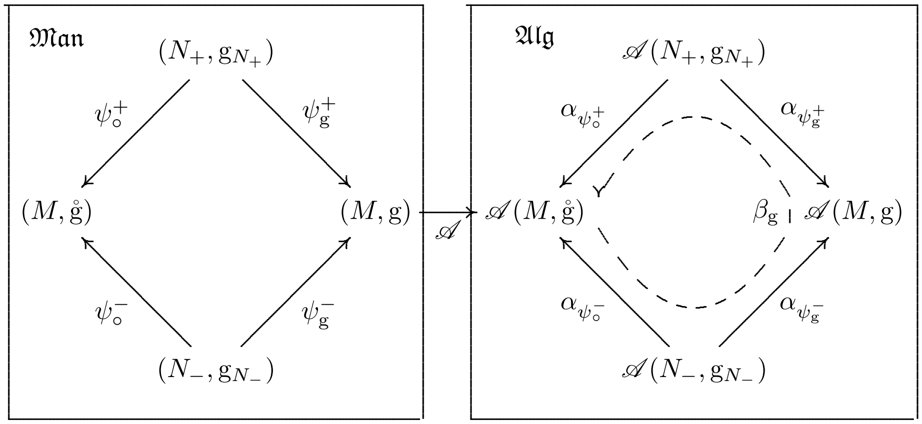

FIG. 2. Diagram of the perturbed spacetime, along with the embedded subregions $N_\pm$, the locally covariant quantum field theory thereon, and the relevant morphisms, where $\mathring{\mathrm{g}} = \mathrm{g}^{(0)}$ and $\mathrm{g} = \mathrm{g}^{(s)}$.

Appendix B: Coordinate independence

In this appendix we derive Eq. (5.15), and thus the coordinate independence of $(\Delta\hat{P})^2$, without the assumption of a single coordinate patch covering K. This assumption is not valid, for example, when the interior of K is not homeomorphic to $\mathbb{R}^4$. More generally, it is often simply more convenient to use multiple coordinate patches.

For the purposes of this proof, however, we avoid explicitly juggling multiple coordinate transition maps by considering each to be locally induced by a global diffeomorphism $\varphi^{(s)} : M \to M$ that depends continuously on the parameter $s \in [0, 1]$. This induces a pushforward $\varphi_*^{(s)}$ of a contravariant tensor field or (in the same direction) a pullback of the inverse diffeomorphism $((\varphi^{(s)})^{-1})^*$ of a covariant field, which we will also denote by $\varphi_*^{(s)}$. Note if restricted to a coordinate patch $\varphi_*^{(s)}$ is related to the transformation $L^\mu_{\alpha'}(s)$ of Sec. V by $(\varphi_*^{(s)} v)_{\alpha'} = L^\mu_{\alpha'}(s) v_\mu$. (And notice that in the classic index analysis of Sec. V, the prime serves double duty, denoting both this s-dependent coordinate transformation, and its $s = 0$ instance.)

To further obviate the need for explicit coordinate charts, we use here Penrose's abstract index notation [50], denoted by Latin indices. Like coordinate component indices, the number of such subscripted/superscripted indices indicate tensor ranks, and repeated indices indicate tensor contractions. But Penrose's abstract indices do not index coordinate components, rather they signify entire tensors. For example, $\mathrm{g}_{\mu\nu} \in \mathbb{R}$ while $\mathrm{g}_{ab} \in T^*M \otimes T^*M$. Thus Penrose's abstract indices provide all the convenience of indices without any of the commitment to coordinates. In these terms, our objective is to show

$$\begin{aligned} &\int_K d\mu_{\varphi_*^{(s)} \mathrm{g}^{(s)}} \varphi_*^{(s)} T^{ab} \frac{d}{ds} \varphi_*^{(s)} \mathrm{g}_{ab}^{(s)} \bigg|_{s=0} \\ &\quad = \int_K d\mu_{\mathrm{g}^{(s)}} T^{ab} \frac{d}{ds} \mathrm{g}_{ab}^{(s)} \bigg|_{s=0} . \end{aligned} \tag{B1}$$

Assuming that $\nabla_a \hat{T}^{ab} = 0$, we have

$$\begin{aligned}
\int_K d\mathring{\mu}'\,\hat{T}'^{ab}\,\frac{d}{ds}\Big|_0 \varphi_*^{(s)}\mathrm{g}_{ab}^{(s)} &= \int_K d\mathring{\mu}'\,\hat{T}'^{ab}\,\frac{d}{ds}\Big|_0 \phi_*^{(s)}\mathrm{g}_{ab}^{(s)\prime} \\
&= \int_K d\mathring{\mu}'\,\hat{T}'^{ab}\left[\left(\frac{d}{ds}\Big|_0 \mathrm{g}_{ab}^{(s)}\right)' + \frac{d}{ds}\Big|_0 \phi_*^{(s)}\mathrm{g}_{ab}^{(0)\prime}\right] \\
&= \int_K d\mathring{\mu}'\,\hat{T}'^{ab}\left[\left(\frac{d}{ds}\Big|_0 \mathrm{g}_{ab}^{(s)}\right)' + \nabla'_a X'_b + \nabla'_b X'_a\right] \\
&= \int_K d\mathring{\mu}\,\hat{T}^{ab}\left[\frac{d}{ds}\Big|_0 \mathrm{g}_{ab}^{(s)} + \nabla_a X_b + \nabla_b X_a\right] \\
&= \int_K d\mathring{\mu}\,\left[\hat{T}^{ab}\,\frac{d}{ds}\Big|_0 \mathrm{g}_{ab}^{(s)} + 2(\nabla_a(\hat{T}^{ab}X_b) - (\nabla_a\hat{T}^{ab}X_b))\right] \\
&= \int_K d\mathring{\mu}\,\hat{T}^{ab}\,\frac{d}{ds}\Big|_0 \mathrm{g}_{ab}^{(s)} + 2\int_K d\mathring{\mu}\,\nabla_a(\hat{T}^{ab}X_b) \\
&= \int_K d\mathring{\mu}\,\hat{T}^{ab}\,\frac{d}{ds}\Big|_0 \mathrm{g}_{ab}^{(s)} + 2\int_{\partial K} d\lambda\, n_a\hat{T}^{ab}X_b\ ,
\end{aligned} \tag{B2}$$

where $\phi^{(s)} = \varphi^{(s)} \circ (\varphi^{(0)})^{-1}$, X'_a generates $\phi^{(s)}$, $d\lambda$ is the surface element induced on the boundary of K, n_a is the corresponding surface normal, and primes denote $\phi_*^{(0)}$. Then neglecting the boundary term, as explained in Sec. V, we achieve the desired diffeomorphism-independence.

Note that $\phi^{(s)}$ above corresponds to $\phi^{(s)}$ in the proof of Theorem 4.2 in [5]. That theorem implies that $\nabla_a\hat{T}^{ab}$ must vanish if the above integral is invariant with respect to the transformation $\phi_*^{(s)}$ of the metric. The above result is a generalization of the converse.

Appendix C: Commutators and vacuum correlation functions for the electromagnetic field

The field operators for the free electric and magnetic fields can be written as (we use cgs Gaussian units with $c = 1$)

$$\hat{\mathbf{E}}(\mathbf{x},t) = \hat{\mathbf{E}}^{(+)}(\mathbf{x},t) + \hat{\mathbf{E}}^{(-)}(\mathbf{x},t)\ , \tag{C1}$$

$$\hat{\mathbf{B}}(\mathbf{x},t) = \hat{\mathbf{B}}^{(+)}(\mathbf{x},t) + \hat{\mathbf{B}}^{(-)}(\mathbf{x},t)\ , \tag{C2}$$

where the positive- and negative-frequency parts of the fields are given by

$$\hat{\mathbf{E}}^{(+)} = \hat{\mathbf{E}}^{(-)\dagger} = i\sum_\sigma \int \frac{d^3k}{(2\pi)^3}\,\sqrt{2\pi\hbar\omega}\,\hat{a}_{\mathbf{k}\sigma}\mathbf{e}_{\mathbf{k}\sigma}e^{i\omega(\mathbf{n}\cdot\mathbf{x}-t)}\ , \tag{C3}$$

$$\hat{\mathbf{B}}^{(+)} = \hat{\mathbf{B}}^{(-)\dagger} = i\sum_\sigma \int \frac{d^3k}{(2\pi)^3}\,\sqrt{2\pi\hbar\omega}\,\hat{a}_{\mathbf{k}\sigma}\mathbf{n}\times\mathbf{e}_{\mathbf{k}\sigma}e^{i\omega(\mathbf{n}\cdot\mathbf{x}-t)}\ . \tag{C4}$$

Here $\mathbf{k} = \omega\mathbf{n}$ is the wave vector ($\omega = |\mathbf{k}|$ is the angular frequency and $\mathbf{n}$ a unit vector), and $\hat{a}_{\mathbf{k}\sigma}$ and $\mathbf{e}_{\mathbf{k}\sigma}$ are the annihilation operator and unit (transverse) polarization vector for the plane-wave mode with wave vector $\mathbf{k}$ and polarization σ. The creation and annihilation operators satisfy the canonical commutator,

$$\left[\hat{a}_{\mathbf{k}\sigma}, \hat{a}^\dagger_{\mathbf{k}'\sigma'}\right] = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\delta_{\sigma\sigma'}\ . \tag{C5}$$

The Hamiltonian for the electromagnetic field is

$$\hat{H} = \int d^3x\, {:}\hat{T}^{00}{:} = \frac{1}{8\pi}\int d^3x\, {:}\hat{\mathbf{E}}\cdot\hat{\mathbf{E}} + \hat{\mathbf{B}}\cdot\hat{\mathbf{B}}{:} = \frac{1}{2\pi}\int d^3x\,\hat{\mathbf{E}}^{(-)}\cdot\hat{\mathbf{E}}^{(+)} = \sum_\sigma\int\frac{d^3k}{(2\pi)^3}\,\hbar\omega a^\dagger_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}\ . \tag{C6}$$

where the integral extends over all space.

The positive- and negative-frequency parts of the fields have the free-field commutators,

$$\big[\hat{E}_j^{(+)}(\mathbf{x},t),\hat{E}_k^{(-)}(\mathbf{x}',t')\big]=\big[\hat{B}_j^{(+)}(\mathbf{x},t),\hat{B}_k^{(-)}(\mathbf{x}',t')\big]=2\pi\hbar\left(-\delta_{jk}\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x_j\partial x_k}\right)\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{\omega}e^{i\omega[\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')-(t-t')]}\;, \tag{C7}$$

$$\big[\hat{E}_j^{(+)}(\mathbf{x},t),\hat{B}_k^{(-)}(\mathbf{x}',t')\big]=-\big[\hat{B}_j^{(+)}(\mathbf{x},t),\hat{E}_k^{(-)}(\mathbf{x}',t')\big]=2\pi\hbar\,\epsilon_{jkl}\frac{\partial^2}{\partial t\,\partial x_l}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{\omega}e^{i\omega[\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')-(t-t')]}\;. \tag{C8}$$

The integral on the right evaluates to

$$\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{\omega}e^{i\omega[\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')-(t-t')]}=-\frac{i}{4\pi}G(\mathbf{x}-\mathbf{x}',t-t')+\frac{1}{2\pi^2}D(\mathbf{x}-\mathbf{x}',t-t')\;, \tag{C9}$$

where

$$G(\mathbf{x},t)=-4\pi\,\mathrm{Im}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{\omega}e^{i\omega(\mathbf{n}\cdot\mathbf{x}-t)}=\frac{\delta\big(t-|\mathbf{x}|\big)-\delta\big(t+|\mathbf{x}|\big)}{|\mathbf{x}|} \tag{C10}$$

is the difference between retarded and advanced Green functions, i.e., the solution of the homogeneous wave equation for an incoming spherical wave that reflects off the origin and becomes an outgoing spherical wave, and

$$D(\mathbf{x},t)=2\pi^2\,\mathrm{Re}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{\omega}e^{i\omega(\mathbf{n}\cdot\mathbf{x}-t)}=\mathrm{p.v.}\frac{1}{-t^2+|\mathbf{x}|^2}=\mathrm{p.v.}\frac{1}{(\Delta s)^2} \tag{C11}$$

is the principal value of the inverse of the invariant interval. From these follow the free-field commutators and vacuum correlators [51]:

$$\begin{aligned}\big[\hat{E}_j(\mathbf{x},t),\hat{E}_k(\mathbf{x}',t')\big]&=\big[\hat{B}_j(\mathbf{x},t),\hat{B}_k(\mathbf{x}',t')\big]\\&=2i\,\mathrm{Im}\big(\big[\hat{E}_j^{(+)}(\mathbf{x},t),\hat{E}_k^{(-)}(\mathbf{x}',t')\big]\big)\\&=i\hbar\left(\delta_{jk}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x_j\partial x_k}\right)G(\mathbf{x}-\mathbf{x}',t-t')\;,\end{aligned} \tag{C12}$$

$$\begin{aligned}\big[\hat{E}_j(\mathbf{x},t),\hat{B}_k(\mathbf{x}',t')\big]&=-\big[\hat{B}_j(\mathbf{x},t),\hat{E}_k(\mathbf{x}',t')\big]\\&=2i\,\mathrm{Im}\big(\big[\hat{E}_j^{(+)}(\mathbf{x},t),\hat{B}_k^{(-)}(\mathbf{x}',t')\big]\big)\\&=-i\hbar\,\epsilon_{jkl}\frac{\partial^2}{\partial t\partial x_l}G(\mathbf{x}-\mathbf{x}',t-t')\;,\end{aligned} \tag{C13}$$

$$\begin{aligned}\frac{1}{2}\langle 0\big|[\hat{E}_j(\mathbf{x},t)\hat{E}_k(\mathbf{x}',t')+\hat{E}_k(\mathbf{x}',t')\hat{E}_j(\mathbf{x},t)]\big|0\rangle&=\frac{1}{2}\langle 0\big|[\hat{B}_j(\mathbf{x},t)\hat{B}_k(\mathbf{x}',t')+\hat{B}_k(\mathbf{x}',t')\hat{B}_j(\mathbf{x},t)]\big|0\rangle\\&=\mathrm{Re}\big(\big[\hat{E}_j^{(+)}(\mathbf{x},t),\hat{E}_k^{(-)}(\mathbf{x}',t')\big]\big)\\&=\frac{\hbar}{\pi}\left(-\delta_{jk}\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x_j\partial x_k}\right)D(\mathbf{x}-\mathbf{x}',t-t')\;,\end{aligned} \tag{C14}$$

$$\begin{aligned}\frac{1}{2}\langle 0\big|[\hat{E}_j(\mathbf{x},t)\hat{B}_k(\mathbf{x}',t')+\hat{B}_k(\mathbf{x}',t')\hat{E}_j(\mathbf{x},t)]\big|0\rangle&=-\frac{1}{2}\langle 0\big|[\hat{B}_j(\mathbf{x},t)\hat{E}_k(\mathbf{x}',t')+\hat{E}_k(\mathbf{x}',t')\hat{B}_j(\mathbf{x},t)]\big|0\rangle\\&=\mathrm{Re}\big(\big[\hat{E}_j^{(+)}(\mathbf{x},t),\hat{B}_k^{(-)}(\mathbf{x}',t')\big]\big)\\&=\frac{\hbar}{\pi}\,\epsilon_{jkl}\frac{\partial^2}{\partial t\,\partial x_l}D(\mathbf{x}-\mathbf{x}',t-t')\;.\end{aligned} \tag{C15}$$

The field quadratures (7.29) and (7.32) for a nearly plane-wave mean field can be evaluated in the Fourier domain as follows ($a=1,2$):

$$\begin{aligned}\hat{\mathcal{X}}_a=\frac{1}{4\pi}\int d^4x\,E_a(\mathbf{x},t)\hat{E}_y(\mathbf{x},t)&=\frac{1}{4\pi}\int dt\,d^3x\,E_a^{(-)}(\mathbf{x},t)\hat{E}_y^{(+)}(\mathbf{x},t)+\text{H.c.}\\&=\frac{1}{2}\tau\int\frac{d^3k}{(2\pi)^3}\,\hbar\omega\sum_{\sigma,\sigma'}\alpha_{a;\mathbf{k}\sigma}\hat{a}_{\mathbf{k}\sigma'}(\mathbf{e}_y\cdot\mathbf{e}_{\mathbf{k}\sigma}^*)(\mathbf{e}_y\cdot\mathbf{e}_{\mathbf{k}\sigma'})+\text{H.c.}\\&=\frac{1}{2}\tau\int\frac{d^3k}{(2\pi)^3}\,\hbar\omega\,\alpha_{a,\mathbf{k}}\hat{a}_{\mathbf{k}y}+\text{H.c.}\;.\end{aligned} \tag{C16}$$

In the first line we discard counter-rotating terms that average to nearly zero over the temporal integral; in the second line, we insert the Fourier transforms of the field operators and the wave forms and do the temporal integral over the duration τ for which the mean fields are turned on; in the third line, we use the fact that $\alpha_{a;\mathbf{k}\sigma}$ has support only for $\mathbf{k}$ pointing nearly in the x direction, with polarization nearly along the y direction, to restrict the two sums over polarization to y linear polarization.

Using Eq. (7.43), we can write

$$\hat{\mathcal{X}}_1 = \sqrt{\frac{\hbar C}{2}}(\hat{b} + \hat{b}^\dagger) \; , \qquad \hat{\mathcal{X}}_2 = \sqrt{\frac{\hbar C}{2}}(-i\hat{b} + i\hat{b}^\dagger) \; , \tag{C17}$$

i.e., $\hat{b} = (\hat{\mathcal{X}}_1 + i\hat{\mathcal{X}}_2)/\sqrt{2\hbar C}$, where

$$\hat{b} = \frac{\tau}{\sqrt{2\hbar C}} \int \frac{d^3k}{(2\pi)^3}\, \hbar\omega\, \alpha^*_{1,\mathbf{k}} \hat{a}_{\mathbf{k}y} \tag{C18}$$

[the constant C is given in Eq. (7.44)]. One can verify that the quadrature components obey the commutation relation (7.22) or, equivalently, that $\hat{b}$ and $\hat{b}^\dagger$ satisfy the canonical bosonic commutation relation, $[\hat{b}, \hat{b}^\dagger] = 1$.

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