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Chapter 8

The Orthogonality of Al-Salam–Carlitz Polynomials for Complex Parameters*

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In this chapter, we study the orthogonality conditions satisfied by Al-Salam–Carlitz polynomials $U_n^{(a)}(x;q)$ when the parameters a and q are not necessarily real nor "classical", i.e., the linear functional $\mathbf{u}$ with respect to such a polynomial sequence is quasi-definite and not positive definite. We establish orthogonality on a simple contour in the complex plane which depends on the parameters. In all cases we show that the orthogonality conditions characterize the Al-Salam–Carlitz polynomials $U_n^{(a)}(x;q)$ of degree n up to a constant factor. We

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also obtain a generalization of the unique generating function for these polynomials.

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1. Introduction

The Al-Salam–Carlitz polynomials $U_n^{(a)}(x;q)$ were introduced by Al-Salam and Carlitz in [1] as follows:

$$U_n^{(a)}(x;q) := (-a)^n q^{\binom{n}{2}} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (x^{-1};q)_k}{(q;q)_k} \frac{q^k x^k}{a^k}. \tag{1.1}$$

In fact, these polynomials have a Rodrigues-type formula [2, (3.24.10)]

$$U_n^{(a)}(x;q) = \frac{a^n q^{\binom{n}{2}} (1-q)^n}{q^n w(x;a;q)} \mathscr{D}_{q^{-1}}^n (w(x;a;q)),$$

where

$$w(x;a;q) := (qx;q)_\infty (qx/a;q)_\infty,$$

the q-Pochhammer symbol (q-shifted factorial) is defined as

$$(z;q)_0 := 1, \quad (z;q)_n := \prod_{k=0}^{n-1} (1-zq^k),$$

$$(z;q)_\infty := \prod_{k=0}^{\infty} (1-zq^k), \quad |z|<1,$$

and the q-derivative operator is defined by

$$\mathscr{D}_q f(z) := \begin{cases} \dfrac{f(qz)-f(z)}{(q-1)z} & \text{if } q \neq 1 \text{ and } z \neq 0, \\ f'(z) & \text{if } q = 1 \text{ or } z = 0. \end{cases}$$

Remark 1.1. Observe that by the definition of the q-derivative

$$\mathscr{D}_{q^{-1}} f(z) = \mathscr{D}_q f(qz), \quad \text{and} \quad \mathscr{D}_{q^{-1}}^n f(z) := \mathscr{D}_{q^{-1}}^{n-1}(\mathscr{D}_{q^{-1}} f(z)), \; n = 2,3,\ldots.$$

The expression (1.1) shows us that $U_n^{(a)}(x;q)$ is an analytic function for any complex-valued parameters a and q, and thus can be considered for general $a, q \in \mathbb{C} \setminus \{0\}$.

The classical Al-Salam–Carlitz polynomials correspond to parameters $a < 0$ and $0 < q < 1$. For these parameters, the Al-Salam–Carlitz polynomials are orthogonal on $[a, 1]$ with respect to the weight function w. More specifically, for $a < 0$ and $0 < q < 1$ [2, (14.24.2)],

$$\int_a^1 U_n^{(a)}(x;q)U_m^{(a)}(x;q)(qx, qx/a;q)_\infty d_qx = d_n^2 \delta_{n,m},$$

where

$$d_n^2 := (-a)^n(1-q)(q;q)_n(q;q)_\infty(a;q)_\infty(q/a;q)_\infty q^{\binom{n}{2}},$$

and the q-Jackson integral [2, (1.15.7)] is defined as

$$\int_a^b f(x)d_qx := \int_0^b f(x)d_qx - \int_0^a f(x)d_qx,$$

where

$$\int_0^a f(x)d_qx := a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n.$$

Taking into account the previous orthogonality relation, it is a direct result that if a and q are classical, i.e., a, $q \in \mathbb{R}$, with $a \neq 1$, $0 < q < 1$, all the zeros of $U_n^{(a)}(x;q)$ are simple and belong to the interval $[a, 1]$. This is no longer valid for general a and q complex. In this paper, we show that for general a, q complex numbers, but excluding some special cases, the Al-Salam–Carlitz polynomials $U_n^{(a)}(x;q)$ may still be characterized by orthogonality relations. The case $a < 0$ and $0 < q < 1$ or $0 < aq < 1$ and $q > 1$ is classical, i.e., the linear functional $\mathbf{u}$ with respect to such a polynomial sequence is orthogonal, which is positive definite and in such a case there exists a weight function $\omega(x)$ so that

$$\langle \mathbf{u}, p\rangle = \int_a^1 p(x)\,\omega(x)\,dx, \quad p \in \mathbb{P}[x].$$

Note that this is the key for the study of many properties of Al-Salam–Carlitz polynomials I and II. Thus, our goal is to establish orthogonality conditions for most of the remaining cases for which the linear form $\mathbf{u}$ is quasi-definite, i.e., for all $n, m \in \mathbb{N}_0$

$$\langle \mathbf{u}, p_n p_m\rangle = k_n \delta_{n,m}, \quad k_n \neq 0.$$

We believe that these new orthogonality conditions can be useful in the study of the zeros of Al-Salam–Carlitz polynomials. For general

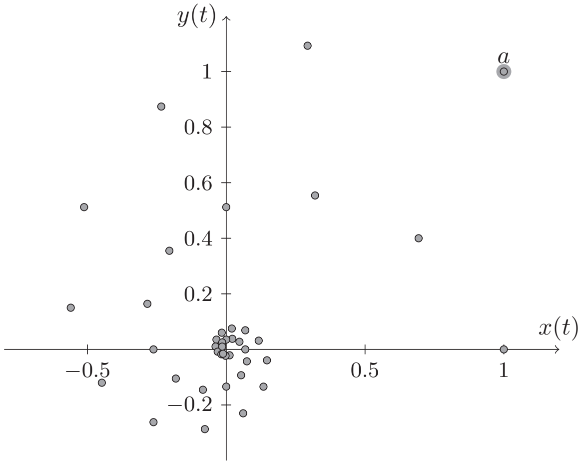

Fig. 1. Zeros of $U_{30}^{(1+i)}(x; \frac{4}{5}\exp(\pi i/6))$.

$a, q \in \mathbb{C} \setminus \{0\}$, the zeros are not confined to a real interval, but they distribute themselves in the complex plane as we can see in Fig. 1. Throughout this paper denote $p := q^{-1}$.

2. Orthogonality in the Complex Plane

Theorem 2.1. *Let $a, q \in \mathbb{C}$, $a \neq 0, 1$, $0 < |q| < 1$. The Al-Salam–Carlitz polynomials are the unique polynomials* (*up to a multiplicative constant*) *satisfying the property of orthogonality*

$$\int_a^1 U_n^{(a)}(x;q)U_m^{(a)}(x;q)w(x;a;q)d_qx = d_n^2\delta_{n,m}. \tag{2.1}$$

Remark 2.2. If $0 < |q| < 1$, the lattice $\{q^k : k \in \mathbb{N}_0\} \cup \{aq^k : k \in \mathbb{N}_0\}$ is a set of points which are located inside on a single contour that goes from 1 to 0, and then from 0 to a, through the spirals

$$S_1 : z(t) = |q|^t \exp(it \arg q), \quad S_2 : z(t) = |a||q|^t \exp(it \arg q + i \arg a),$$

where $0 < |q| < 1$, $t \in [0, \infty)$, which we can see in Fig. 2. Taking into account (2.1), we need to avoid the $a = 1$ case. For the $a = 0$ case, we cannot apply Favard's result [3], because in such a case this polynomial sequence fulfills the recurrence relation (see [2])

$$U_{n+1}^{(0)}(x;q) = (x - q^n)U_n^{(0)}(x;q), \quad U_0^{(0)}(x;q) = 1.$$

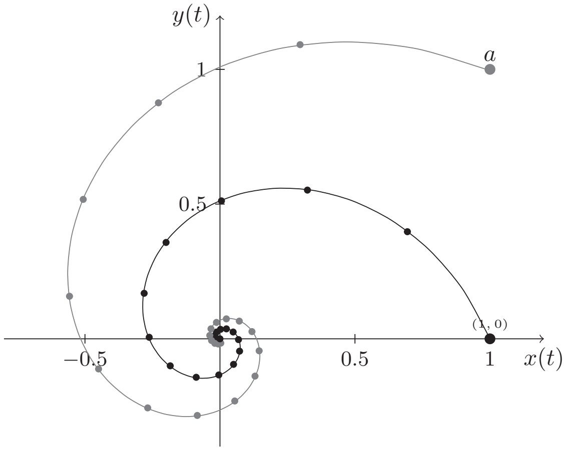

Fig. 2. The lattice $\{q^k : k \in \mathbb{N}_0\} \cup \{(1+i)q^k : k \in \mathbb{N}_0\}$ with $q = 4/5\exp(\pi i/6)$.

Proof of Theorem 2.1. Let $0 < |q| < 1$, and $a \in \mathbb{C}$, $a \neq 0, 1$. We are going to express the q-Jackson integral (2.1) as the difference of the two infinite sums and apply the identity

$$\sum_{k=0}^{M} f(q^k)\mathscr{D}_{q^{-1}} g(q^k)q^k = \frac{f(q^M)g(q^M) - f(q^{-1})g(q^{-1})}{q^{-1}-1} - \sum_{k=0}^{M} g(q^{k-1})\mathscr{D}_{q^{-1}} f(q^k)q^k. \qquad (2.2)$$

Let $n \geq m$. Then, for one side, since $w(q^{-1};a;q) = 0$, and using the identities [2, (14.24.7) and (14.24.9)], one has

$$\begin{aligned}
&\sum_{k=0}^{\infty} U_m^{(a)}(q^k;q)U_n^{(a)}(q^k;q)w(q^k;a;q)q^k \\
&\quad = \frac{a(1-q)}{q^{2-n}} \lim_{M\to\infty} \sum_{k=0}^{M} \mathscr{D}_{q^{-1}}[w(q^k;a;q)U_{n-1}^{(a)}(q^k;q)]U_m^{(a)}(q^k;q)q^k \\
&\quad = aq^{n-1} \lim_{M\to\infty} U_m^{(a)}(q^M;q)U_{n-1}^{(a)}(q^M;q)w(q^M;a;q) \\
&\qquad + aq^{n-1}(q^m-1) \lim_{M\to\infty} \sum_{k=0}^{M-1} w(q^k;a;q)U_{n-1}^{(a)}(q^k;q)U_{m-1}^{(a)}(q^k;q)q^k.
\end{aligned}$$

Following an analogous process as before, and since $w(aq^{-1};a;q)=0$, we have

$$\begin{aligned}\sum_{k=0}^{\infty} & U_m^{(a)}(aq^k;q)U_n^{(a)}(aq^k;q)w(aq^k;a;q)aq^k \\ &= aq^{n-1}\lim_{M\to\infty} U_m^{(a)}(aq^M;q)U_{n-1}^{(a)}(aq^M;q)w(aq^M;a;q) \\ &\quad + aq^{n-1}(q^m-1)\lim_{M\to\infty}\sum_{k=0}^{M-1} w(aq^k;a;q)U_{n-1}^{(a)}(aq^k;q)U_{m-1}^{(a)}(aq^k;q)aq^k.\end{aligned}$$

Therefore, if $m<n$, and since m is finite, one can first repeat the previous process $m+1$ times obtaining

$$\begin{aligned}\sum_{k=0}^{\infty} & U_m^{(a)}(q^k;q)U_n^{(a)}(q^k;q)w(q^k;a;q)q^k \\ &= \lim_{M\to\infty}\sum_{\nu=1}^{m+1}(-aq^n)^\nu q^{-\nu(\nu+1)/2}(q^{-m+\nu-1};q)_\nu \\ &\qquad \times U_{m-\nu+1}^{(a)}(q^M;q)U_{n-\nu}^{(a)}(q^M;q)w(q^M;a;q),\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{\infty} & U_m^{(a)}(aq^k;q)U_n^{(a)}(aq^k;q)w(aq^k;a;q)aq^k \\ &= \lim_{M\to\infty}\sum_{\nu=1}^{m+1}(-aq^n)^\nu q^{-\nu(\nu+1)/2}(q^{-m+\nu-1};q)_\nu \\ &\qquad \times U_{m-\nu+1}^{(a)}(aq^M;q)U_{n-\nu}^{(a)}(aq^M;q)w(aq^M;a;q).\end{aligned}$$

Hence, since the difference of both limits, term by term, goes to 0 since $|q|<1$, then

$$\int_a^1 U_n^{(a)}(x;q)U_m^{(a)}(x;q)(qx,qx/a;q)_\infty d_qx = 0.$$

For $n=m$, following the same idea, we have

$$\begin{aligned}\int_a^1 & U_n^{(a)}(x;q)U_n^{(a)}(x;q)w(x;a;q)d_qx \\ &= \frac{a(q^n-1)}{q^{1-n}}\sum_{k=0}^{\infty}(w(q^k;a;q)(U_{n-1}^{(a)}(q^k;q))^2q^k \\ &\qquad - aw(aq^k;a;q)(U_{n-1}^{(a)}(aq^k;q))^2q^k)\end{aligned}$$

$$= (-a)^n (q;q)_n q^{\binom{n}{2}} \sum_{k=0}^{\infty} (w(q^k;a;q)q^k - a\ w(aq^k;a;q)q^k)$$

$$= (-a)^n (q;q)_n (q;q)_\infty\, q^{\binom{n}{2}} \sum_{k=0}^{\infty} ((q^{k+1}/a;q)_\infty - a(aq^{k+1};q)_\infty) \frac{q^k}{(q;q)_k},$$

since it is known that in this case [2, (14.24.2)]

$$\int_a^1 U_n^{(a)}(x;q) U_n^{(a)}(x;q) w(x;a;q) d_q x$$
$$= (-a)^n (q;q)_n (q;q)_\infty (a;q)_\infty (q/a;q)_\infty q^{\binom{n}{2}}.$$

Due to the normality of this polynomial sequence, i.e., $\deg U_n^{(a)}(x;q) = n$ for all $n \in \mathbb{N}_0$, the uniqueness is straightforward, thus the result holds. □

From this result, and taking into account that the squared norm for the Al-Salam–Carlitz polynomials is known, we obtained the following consequence for which we could not find any reference.

Corollary 2.3. *Let $a, q \in \mathbb{C} \setminus \{0\}$, $|q| < 1$. Then*

$$\sum_{k=0}^{\infty} ((q^{k+1}/a;q)_\infty - a(aq^{k+1};q)_\infty) \frac{q^k}{(q;q)_k} = (a;q)_\infty (q/a;q)_\infty.$$

The following case, which is just the Al-Salam–Carlitz polynomials for the $|q| > 1$ case, is commonly called the Al-Salam–Carlitz II polynomials.

Theorem 2.4. *Let $a, q \in \mathbb{C}$, $a \neq 0, 1$, $|q| > 1$. Then, the Al-Salam–Carlitz polynomials are unique (up to a multiplicative constant) satisfying the property of orthogonality given by*

$$\int_a^1 U_n^{(a)}(x;q^{-1}) U_m^{(a)}(x;q^{-1}) (q^{-1}x;q^{-1})_\infty (q^{-1}x/a;q^{-1})_\infty d_{q^{-1}} x$$
$$= (-a)^n (1-q^{-1})(q^{-1};q^{-1})_n (q^{-1};q^{-1})_\infty$$
$$\times (a;q^{-1})_\infty (q^{-1}/a;q^{-1})_\infty\, q^{-\binom{n}{2}} \delta_{m,n}. \tag{2.3}$$

Proof. Let us denote q^{-1} by p; then $0 < |p| < 1$. For $a \in \mathbb{C}$, $a \neq 0, 1$. Then, by using the identity (2.2) replacing $q \mapsto p$, and taking into account that $w(aq;a;p) = w(q;a;p) = 0$ and [2, (14.24.9)], for $m < n$ one has

$$\begin{aligned}
&\sum_{k=0}^{\infty} aw(ap^k;a;p)U_m^{(a)}(ap^k;p)U_n^{(a)}(ap^k;p)p^k \\
&\quad = ap^{n-1}\lim_{M\to\infty} U_m^{(a)}(ap^M;p)U_{n-1}^{(a)}(ap^M;p)w(ap^M;a;p) \\
&\qquad + ap^{n-1}(1-p^m)\lim_{M\to\infty}\sum_{k=0}^{M-1} aw(ap^k;a;p)U_{n-1}^{(a)}(ap^k;p)U_{m-1}^{(a)}(ap^k;p)p^k.
\end{aligned}$$

Following the same idea from the previous result, we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} w(p^k;a;p)U_m^{(a)}(p^k;p)U_n^{(a)}(p^k;p)p^k \\
&\quad = ap^{n-1}\lim_{M\to\infty} U_m^{(a)}(p^M;p)U_{n-1}^{(a)}(p^M;p)w(p^M;a;p) \\
&\qquad + ap^{n-1}(1-p^m)\lim_{M\to\infty}\sum_{k=0}^{M-1} w(p^k;a;p)U_{n-1}^{(a)}(p^k;p)U_{m-1}^{(a)}(p^k;p)p^k.
\end{aligned}$$

Therefore, the property of orthogonality holds for $m < n$. Next, if $n = m$, we have

$$\begin{aligned}
&\int_a^1 U_n^{(a)}(x;p)U_n^{(a)}(x;p)w(x;a;p)\,d_px \\
&\quad = \frac{a(p^n-1)}{p^{1-n}}\sum_{k=0}^{\infty}(aw(ap^k;a;p)(U_{n-1}^{(a)}(ap^k;p))^2p^k \\
&\qquad - w(p^k;a;p)(U_{n-1}^{(a)}(p^k;p))^2p^k) \\
&\quad = (-a)^n(p;p)_n p^{\binom{n}{2}}\left(\sum_{k=0}^{\infty} aw(ap^k;a;p)p^k - w(p^k;a;p)p^k\right) \\
&\quad = (-a)^n\ (q^{-1};q^{-1})_n(p;p)_\infty p^{\binom{n}{2}}\sum_{k=0}^{\infty}\frac{q^k(a(p^{k+1}a;p)_\infty - (p^{k+1}/a;p)_\infty)}{(p;p)_k} \\
&\quad = (-a)^n(q^{-1};q^{-1})_n(p;p)_\infty(a;p)_\infty(p/a;p)_\infty p^{\binom{n}{2}}.
\end{aligned}$$

Using the same argument as in Theorem 2.1, the uniqueness holds, so the claim follows. □

Remark 2.5. Observe that in the previous theorems if $a = q^m$, with $m \in \mathbb{Z}$, $a \neq 0$, after some logical cancellations, the set of points where we need to calculate the q-integral is easy to compute. For example, if $0 < aq < 1$ and $0 < q < 1$, one obtains the sum [2, (14.25.2), p. 537].

Remark 2.6. The $a = 1$ case is special because it is not considered in the literature. In fact, the linear form associated with the Al-Salam–Carlitz polynomials $\mathbf{u}$ is quasi-definite and fulfills the Pearson-type distributional equations

$$\mathscr{D}_q[(x-1)^2\mathbf{u}] = \frac{x-2}{1-q}\mathbf{u} \quad \text{and} \quad \mathscr{D}_{q^{-1}}[q^{-1}\mathbf{u}] = \frac{x-2}{1-q}\mathbf{u}.$$

Moreover, the Al-Salam–Carlitz polynomials fulfill the three-term recurrence relation [2, (14.24.3)]

$$xU_n^{(a)}(x;q) = U_{n+1}^{(a)}(x;q) + (a+1)q^n U_n^{(a)}(x;q) - aq^{n-1}(1-q^n)U_{n-1}^{(a)}(x;q), \tag{2.4}$$

where $n = 0, 1, \ldots,$ with initial conditions $U_0^{(a)}(x;q) = 1$, $U_1^{(a)}(x;q) = x - a - 1$.

Therefore, we believe that it will be interesting to study such a case for its peculiarity because the coefficient $q^{n-1}(1-q^n) \neq 0$ for all n, so one can apply Favard's result.

2.1. The $|q| = 1$ case

In this section, we only consider the case where q is a root of unity. Let N be a positive integer such that $q^N = 1$; then, due to the recurrence relation (2.4) and following the same idea that the authors did in [4, Section 4.2], we apply the following process:

(1) The sequence $(U_n^{(a)}(x;q))_{n=0}^{N-1}$ is orthogonal with respect to the Gaussian quadrature

$$\langle \mathbf{v}, p \rangle := \sum_{s=1}^{N} \gamma_1^{(a)} \cdots \gamma_{N-1}^{(a)} \frac{p(x_s)}{(U_{N-1}^{(a)}(x_s))^2},$$

where $\{x_1, x_2, \ldots, x_N\}$ are the zeros of $U_N^{(a)}(x;q)$ for such value of q.

(2) Since $\langle \mathbf{v}, U_n^{(a)}(x;q)U_n^{(a)}(x;q)\rangle = 0$, we need to modify such a linear form. Next, we can prove that the sequence $(U_n^{(a)}(x;q))_{n=0}^{2N-1}$ is orthogonal with respect to the bilinear form

$$\langle p, r \rangle_2 = \langle \mathbf{v}, pq \rangle + \langle \mathbf{v}, \mathscr{D}_q^N p \mathscr{D}_q^N r \rangle,$$

since $\mathscr{D}_q U_n^{(a)}(x;q) = (q^n - 1)/(q-1)U_{n-1}^{(a)}(x;q)$.

(3) Since $\langle U_{2N}^{(a)}(x;q), U_{2N}^{(a)}(x;q)\rangle_2 = 0$, and taking into account the above results, we consider the linear form

$$\langle p, r\rangle_3 = \langle \mathbf{v}, pq\rangle + \langle \mathbf{v}, \mathscr{D}_q^N p \mathscr{D}_q^N r\rangle + \langle \mathbf{v}, \mathscr{D}_q^{2N} p \mathscr{D}_q^{2N} r\rangle.$$

(4) Therefore one can obtain a sequence of bilinear forms such that the Al-Salam–Carlitz polynomials are orthogonal with respect to them.

3. A Generalized Generating Function for Al-Salam–Carlitz Polynomials

For this section, we are going to assume $|q| > 1$, or $0 < |p| < 1$. Indeed, by starting with the generating functions for Al-Salam–Carlitz polynomials [2, (14.25.11) and (14.25.12)], we derive generalizations using the connection relation for these polynomials.

Theorem 3.1. *Let $a, b, p \in \mathbb{C} \setminus \{0\}$, $|p| < 1$, $a, b \neq 1$. Then*

$$U_n^{(a)}(x;p) = (-1)^n (p;p)_n p^{-\binom{n}{2}} \sum_{k=0}^{n} \frac{(-1)^k a^{n-k} (b/a;p)_{n-k} p^{\binom{k}{2}}}{(p;p)_{n-k}(p;p)_k} U_k^{(b)}(x;p). \tag{3.1}$$

Proof. If we consider the generating function for Al-Salam–Carlitz polynomials [2, (14.25.11)]

$$\frac{(xt;p)_\infty}{(t, at;p)_\infty} = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{n}{2}}}{(p;p)_n} U_n^{(a)}(x;p) t^n,$$

and multiply both sides by $(bt;p)_\infty/(bt;p)_\infty$, we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{n}{2}}}{(p;p)_n} U_n^{(a)}(x;p) t^n = \frac{(bt;p)_\infty}{(at;p)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{n}{2}}}{(p;p)_n} U_n^{(b)}(x;p) t^n. \tag{3.2}$$

If we now apply the q-binomial theorem [2, (1.11.1)]

$$\frac{(az;p)_\infty}{(z;p)_\infty} = \sum_{k=0}^{\infty} \frac{(ap;p)_n}{(p;p)_n} z^n, \quad 0 < |p| < 1, \quad |z| < 1,$$

to (3.2), and then collect powers of t, we obtain

$$\sum_{k=0}^{\infty} t^k \sum_{m=0}^{k} \frac{(-1)^m a^{k-m}(b/a;p)_{k-m} p^{\binom{m}{2}}}{(p;p)_{k-m}(p;p)_m} U_m^{(b)}(x;p)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{n}{2}}}{(p;p)_n} U_n^{(a)}(x;p) t^n.$$

Taking into account this expression, the result follows. □

Theorem 3.2. *Let $a, b, p \in \mathbb{C} \setminus \{0\}$, $|p| < 1$, $a, b \neq 1$, $t \in \mathbb{C}$, $|at| < 1$. Then*

$$(at;p)_\infty \, {}_1\phi_1 \left(\begin{matrix} x \\ at \end{matrix} ; p, t \right) = \sum_{k=0}^{\infty} \frac{p^{k(k-1)}}{(p;p)_k} \, {}_1\phi_1 \left(\begin{matrix} b/a \\ 0 \end{matrix} ; p, atp^k \right) U_k^{(b)}(x;p) t^k, \tag{3.3}$$

where

$${}_r\phi_s \left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} ; p, z \right)$$

$$= \sum_{k=0}^{\infty} \frac{(a_1;p)_k (a_2;p)_k \cdots (a_r;p)_k}{(b_1;p)_k (b_2;p)_k \cdots (b_s;p)_k} \frac{z^k}{(p;p)_k} (-1)^{(1+s-r)k} p^{(1+s-r)\binom{k}{2}},$$

is the unilateral basic hypergeometric series.

Proof. We start with a generating function for Al-Salam–Carlitz polynomials [2, (14.25.12)]

$$(at;q)_\infty \, {}_1\phi_1 \left(\begin{matrix} x \\ at \end{matrix} ; q, t \right) = \sum_{k=0}^{\infty} \frac{q^{n(n-1)}}{(q;q)_n} V_n^{(a)}(x;q) t^n$$

and (3.1) to obtain

$$(at;p)_\infty \, {}_1\phi_1 \left(\begin{matrix} x \\ at \end{matrix} ; p, t \right)$$

$$= \sum_{n=0}^{\infty} t^n (-1)^n p^{\binom{n}{2}} \sum_{k=0}^{n} \frac{(-1)^k a^{n-k} (b/a;p)_{n-k} p^{\binom{k}{2}}}{(p;p)_{n-k}(p;p)_k} U_k^{(b)}(x;p).$$

We reverse the order of summations, shift the n variable by a factor of k, and use the basic properties of the q-Pochhammer symbol, and [2, (1.10.1)]. Observe that we can reverse the order of summation since our sum is of the form

$$\sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} c_{n,k} U_k^{(a)}(x;p),$$

where

$$a_n = t^n, \quad c_{n,k} = \frac{(-1)^k a^{n-k} (b/a;p)_{n-k} p^{\binom{k}{2}}}{(p;p)_{n-k}(p;p)_k}.$$

In this case, one has

$$|a_n| \le |t|^n, \quad |c_{n,k}| \le K(1+n)^{\sigma_1} |a|^n,$$

and $|U_n^{(a)}(x;p)| \le (1+n)^{\sigma_2}$, where K_1, σ_1, and σ_2 are positive constants independent of n. Therefore, if $|at| < 1$, then

$$\left| \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} c_{n,k} U_k^{(a)}(x;p) \right| < \infty,$$

and this completes the proof. $\square$

As we saw in Section 2, the orthogonality relation for Al-Salam–Carlitz polynomials for $|q| > 1$, $|p| < 1$, and $a \neq 0, 1$ is

$$\int_{\Gamma} U_n^{(a)}(x;p) U_m^{(a)}(x;p) w(x;a;p) d_p x = d_n^2 \delta_{n,m}.$$

Taking this result in mind, the following result follows.

Theorem 3.3. *Let* $a, b, p \in \mathbb{C}\setminus\{0\}$, $t \in \mathbb{C}$, $|at| < 1$, $|p| < 1$, $m \in \mathbb{N}_0$. *Then*

$$\int_a^1 {}_1\phi_1 \left(\begin{matrix} q^{-x} \\ at \end{matrix} ; q, t \right) U_m^{(b)}(q^{-x};p)(q^{-1}x;q^{-1})_\infty (q^{-1}x/a;q^{-1})_\infty dq^{-1}$$

$$= \big(-bt \big)^m q^{3\binom{m}{2}} (b;p)_\infty (p/b;p)_\infty \, {}_1\phi_1 \left(\begin{matrix} b/a \\ 0 \end{matrix} ; q, atq^m \right).$$

Proof. From (3.3), we have $x \mapsto p^x$ and multiply both sides by $U_m^{(b)}(x;p) w(x;a;p)$, and by using the orthogonality relation (2.3), the desired result holds. $\square$

Note that the applications of connection relations to the rest of the known generating functions for Al-Salam–Carlitz polynomials [2, (14.24.11) and (14.25.12)] leave these generating functions invariant.

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