Towards Unified Perron-Frobenius Framework for Managing Systemic Risk in Networked Systems

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ABSTRACT: Since along with economic and convenience benefits, interconnectivity brings potential risks and drawbacks, system designers and operators are faced with problem of balancing the relevant systemic risk/benefit tradeoffs. Assuming that system components are exposed to local risk of overload, overstress, etc., this paper quantifies and investigates the effects of local risk sharing among different system components. The positive effect is due to mitigation of the local risks, and the negative effect is due to potential risk exposure cascades. While systemic risk of undesirable cascades can be naturally defined at the micro-level, inherently macroscopic nature of systemic risk suggests a possibility of macro-level description. We propose a Perron-Frobenius based macro-description of systemic risk. We argue that systemic risk of abrupt/discontinuous instabilities should be of higher concern for networked system designers and operators than systemic risk of gradual/continuous instabilities. We also discuss economic efficiency maximization framework addressing this concern.

1 INTRODUCTION

Economic and convenience benefits of interconnectivity drive current explosive emergence and growth of networked systems (Helbing, D. 2013). However, numerous recent systemic failures in various networked systems suggest that market economy may have difficulty in accurate estimation and pricing of the drawbacks and systemic risks of interconnectivity. This paper models and makes recommendations on systemic risk/benefit tradeoff of interconnectivity, where benefits are due to local risk mitigation, and systemic risk is due to risk exposure cascades.

Contagion in networked systems can be naturally modeled as a Markov process with locally interacting components (Dobrushin, R.L. 1971), where system "topology" is encoded as a directed graph with nodes representing system components, and links representing the contagion flow. In applications, some internal node states represent component failure, overload, etc., and thus can be deemed "vulnerable/undesirable" (Majdandzic, A., et al. 2013). This motivates definition of the individual node risk as the probability of this node being in such a state. Following (Lorenz, J., et al. 2009) we define systemic risk as the individual risk averaged over all the nodes in the system.

We assume that increase in the node risk exposure immediately drives downstream adjacent nodes towards higher risk exposure. These "negative externalities" with respect to risk exposure for system components create a possibility of contagion and undesirable cascades. Direct solution of the Kolmogorov equations for the underlying Markov "micro-process" with a large number of interacting components N is computationally infeasible due to astronomically high dimension of the Kolmogorov system. This high dimension of micro-description on the one hand, and systemic risk being inherently a macroscopic phenomenon on the other hand, suggest a possibility of "macro-description" of systemic risk.

Our first step in realizing this possibility is developing mean-field and fluid approximations for the corresponding Kolmogorov microscopic equations for the underlying Markov process. Similar to (Marbukh, V. 2013 & 2014) we obtain an approximate closed system of non-linear, fixed-point equations. For low exposure to the exogenous risk, this system has a "normal/operational" system equilibrium with a low systemic risk. As exogenous risk exposure adiabatically, i.e., "slowly," increases, the approximate equations may experience bifurcation, which results in emergence of "high systemic risk" solution. Following conventional approach (Antunes, N., et al. 2008) we interpret multiple solutions of the approximate equations as describing the metastable, i.e., persistent, system states.

In a particular case of "symmetric" networked systems, dimension of the approximate descriptions are independent of N, and thus these approximations qualify as macro-descriptions. Phase portrait of the corresponding macro-equations determine the system "phase diagram." However, in a general case of heterogeneous networked system, the dimensions of the corresponding approximations grow with N, and thus further dimension reduction is required. We suggest that Perron-Frobenius based "macro-description" is possible in a close proximity to the stability boundary of the normal/operational system equilibrium in the space of system parameters.

Importance of this region is due to economics, which drives networked systems towards the stability boundary of the normal/operational equilibrium. We demonstrate that the proposed macro-description is instrumental in classification and managing of the emerging instability. In particular we suggest that while maximizing economic and convenience benefits of interconnectivity, networked system designers and operators should limit the systemic risk of abrupt/discontinuous instability (Majdandzic, A. 2013) in favor of gradual/continuous instability.

The paper is organized as follows. Section 2 develops approximations for the systemic risks at the microscopic level. These approximations require solving system of non-linear, fixed-point equations of reduced dimension, which is much lower than dimension of the microscopic Kolmogorov system. A particular case of symmetric system, when the reduced dimension does not depend on N, is analyzed in Section 3. Phase portrait of the corresponding macroscopic equations in the space of system parameters provides the global qualitative picture of persistent system behavior. Section 4 argues that onset of systemic instability in real-life heterogeneous systems can be described and managed using Perron-Frobenius theory. Finally, Section 5 summarizes and outlines directions of future research.

2 SYSTEMIC RISK AT MICROSCOPIC LEVEL

This Section describes a microscopic model of risk propagation in networked systems of shared resources. Subsection 2.1 defines the individual and systemic risks. Subsection 2.2 proposes mean-field and fluid approximations for the individual and systemic risks. Subsection 2.3 quantifies the effect of risk sharing on the individual/systemic risk.

2.1 *Definition of systemic risk*

Consider a networked system evolving according to a Markov process $x^N(t)=(x_i(t), i=1,..,N)$ with N locally interacting components $x_i \in X_i$ (Dobrushin, R.L. 1971). System "topology" is characterized by a directed graph, where nodes represent system components, links represent the contagion flow, and some internal node i states $x_i \in X_i^* \subseteq X_i$ are deemed "vulnerable/undesirable," since they represent component failure, overload, etc. (Majdandzic, A., et al. 2013). We use "components" and "nodes" interchangeably, and introduce binary variables $\delta_i = 1$ if node i is in an vulnerable/undesirable state, i.e., $x_i \in X_i^*$, and $\delta_i = 0$ otherwise, i.e., $x_i \notin X_i^*$.

In a networked system, excessive load, stress, or damage can be transferred from a component i in a vulnerable state $x_i \in X_i^*$ to the certain combinations of other components, which are not in the vulnerable/undesirable states. This transfer on the one hand provides relief to overloaded, overstressed, or damaged components, but on the other hand may cause undesirable cascades due to the negative externalities. We formalize the negative externalities by assuming that conditional probability of component i being in the undesirable state conditioned on the vector $\delta_{-i}=(\delta_j, j\neq i)$, $E[\delta_i|\delta_{-i}]$ is an increasing function of δ_{-i}:

$$\delta_{-i}^1 \le \delta_{-i}^2 \Rightarrow E[\delta_i|\delta_{-i}^1] \le E[\delta_i|\delta_{-i}^2], \tag{1}$$

where inequality for vectors is interpreted as the corresponding inequalities for all vector components.

To model a possibility of alternate load routing (Stolyar, A.L. & Zhong, Y. 2013), we assume that ability to transfer component i's load to other components is determined by binary function $\chi_i(\delta_{-i})$, which two values: 0 and 1. Specifically, the transfer is possible if $\chi_i(\delta_{-i})=0$, and the transfer is not possible if $\chi_i(\delta_{-i})=1$. Functions $\chi_i(\delta_{-i})$ characterize system topology, e.g., physical connectivity. Under natural assumption of monotonic system structure (Barlow, R.E., & Proshan F. 1965) and for natural load allocation strategies, functions $\chi_i(\delta_{-i})$ are monotonically increasing:

$$\delta_{-i}^1 \le \delta_{-i}^2 \Rightarrow \chi_i(\delta_{-i}^1) \le \chi_i(\delta_{-i}^2). \tag{2}$$

In a particular case when overflowing risk from system component $i=1,..,N$ can be transferred to a single component $j \in J_i$, structural function becomes the following product:

$$\chi_i(\delta_{-i}) = \prod\nolimits_{j\in J_i} \delta_j . \tag{3}$$

We define node i individual risk as the following average:

$$s_i = E[\delta_i \chi_i(\delta_{-i})]. \tag{4}$$

In a particular case (3), individual risk (4) takes the following form:

$$s_i = E\left[\delta_i \prod_{j\in J_i} \delta_j\right]. \tag{5}$$

We define systemic risk as a weighted sum of the individual risks:

$$S = \left(\sum_i w_i s_i\right)\Big/\left(\sum_i w_i\right) \tag{6}$$

with some weights $w_i \geq 0$. In a particular case of the same weights $w_i = w$, $i = 1,..,N$, systemic risk (6) becomes

$$S = N^{-1} \sum_i s_i . \quad (7)$$

Averaging in (4) generally requires knowledge of the joint probability distribution of random variables δ_i for $i = 1,..,N$, which are determined by solution to the corresponding Kolmogorov equations for the Markov process $x^N(t) = (x_i(t), i = 1,..,N)$. Since direct solution of this Kolmogorov system is computationally infeasible due to astronomically high dimension even for moderate size N system, next Subsection proposes some approximates.

2.2 *Approximations of systemic risk*

Following conventional approach (Antunes, N., et al. 2008), consider a mean-field approximation, which assumes that steady states of system components $i = 1,..,N$ are jointly statistically independent:

$$P(x) \approx \prod_{i=1}^{N} P_i(x_i) . \quad (8)$$

Assumption (8) implies that binary random variables δ_j with expectations $\bar{\delta}_i = E[\delta_i]$ are approximately jointly statistically independent for $i = 1,..,N$:

$$P(\delta) \approx \Omega(\delta | \bar{\delta}) := \prod_{i=1}^{N} [\bar{\delta}_i^{\delta_i} (1 - \bar{\delta}_i)^{1-\delta_i}] . \quad (9)$$

Averaging in (4) over (9) results in the following approximate expressions for the individual risks:

$$s_i \approx \bar{\delta}_i \sum_{\delta_{-i}} \chi_i(\delta_{-i}) \prod_{j \neq i} [\bar{\delta}_j^{\delta_j} (1 - \bar{\delta}_j)^{1-\delta_j}] . \quad (10)$$

Further simplification is possible by changing order of function evaluation and averaging in (4):

$$s_i \approx \bar{\delta}_i \varphi_i(\bar{\delta}_{-i}) . \quad (11)$$

In a particular case (3), individual risk (10) takes the following form:

$$s_i \approx \bar{\delta}_i \prod_{j \in J_i} \bar{\delta}_j . \quad (12)$$

After identifying individual risks, the systemic risk can be evaluated according to (6). The rest of this Subsection proposes a mean-field approximation for N probabilities $\bar{\delta}_i = E[\delta_i]$, $i = 1,..,N$ in (10)-(12).

Assuming existence, steady-state probabilities for component i, $P(x_i) = \lim_{t \to \infty} \Pr(x_i(t) = x_i)$ satisfy the following Kolmogorov equations:

$$P(x_i) \sum_{x_i' \in X_i \setminus x_i} \bar{\nu}_i(x_i, x_i') = \sum_{x_i' \in X_i \setminus x_i} P(x_i') \bar{\nu}_i(x_i', x_i) , \quad (13)$$

where

$$\bar{\nu}_i(x_i, x_i') = \sum_{\delta_{-i}} \nu_i[(x_i, \delta_{-i}), x_i'] P(\delta_{-i} | x_i) , \quad (14)$$

and $\nu_i[(x_i, \delta_{-i}), x_i']$ is the transition rate of vector $x^N(t)$ i-th component $x_i(t)$ from state $x_i(t) = x_i$ to state $x_i(t) = x_i'$. Transition rate $\nu_i[(x_i, \delta_{-i}), x_i']$ dependence on vector δ_{-i} reflects interaction between system components.

System (13)-(14) is not closed since conditional distributions $P(\delta_{-i} | x_i)$ are not known. Assumption (8) implies that: $P(\delta_{-i} | x_i) \approx \Omega(\delta_{-i} | \bar{\delta}_{-i})$, and thus

$$\bar{\nu}_i(x_i, x_i') \approx \tilde{\nu}_i(x_i, x_i' | \bar{\delta}_{-i}) , \quad (15)$$

where:

$$\tilde{\nu}_i(x_i, x_i' | \bar{\delta}_{-i}) = \sum_{\delta_{-i}} \nu_i[(x_i, \delta_{-i}), x'] \Omega(\delta_{-i} | \bar{\delta}_{-i}) . \quad (16)$$

Combining (13)-(16) we obtain the following closed system of linear equations for marginal distributions $\tilde{P}_i(x_i | \bar{\delta}_{-i}) \approx P_i(x_i)$:

$$\tilde{P}_i(x_i) \sum_{x_i' \in X_i \setminus x_i} \nu_i(x_i, x_i' | \bar{\delta}_{-i}) = \sum_{x_i' \in X_i \setminus x_i} \tilde{P}_i(x_i') \nu_i(x_i', x_i | \bar{\delta}_{-i}) \quad (17)$$

supplemented with the normalization condition $\sum_{x_i \in X_i} \tilde{P}_i(x_i | \bar{\delta}_{-i}) = 1$ for a given vector $\bar{\delta}_{-i}$. After solving system (17) with respect to $\tilde{P}_i(x_i | \bar{\delta}_{-i})$, we obtain the following closed fixed-point system:

$$\tilde{\delta}_i = \varphi_i(\tilde{\delta}_{-i}) , \quad (18)$$

where $\varphi_i(\tilde{\delta}_{-i}) := \sum_{x_i \in X_i} \tilde{P}_i(x_i | \tilde{\delta}_{-i})$. Solution to (18), $\tilde{\delta} = (\tilde{\delta}_i)$ can be viewed as a mean-field approximation for vector $\bar{\delta} = (\bar{\delta}_i)$.

2.3 *Risk sharing*

The level of risk sharing in the system is determined by sets J_i, $i = 1,..,N$. As we discuss below, it may be beneficial for systemic risk management to transfer component i risk to components $j \in J_i$ with controlled probability $q_i \leq 1$. This results in the following modifications of the individual risk (4):

$$s_i = (1 - q_i) \bar{\delta}_i + q_i E[\delta_i \chi_i(\delta_{-i})] . \quad (19)$$

In a case (3) risk (19) take the following form:

$$s_i = (1 - q_i) \bar{\delta}_i + q_i E\left[\delta_i \prod_{j \in J_i} \delta_j \right] , \quad (20)$$

which under approximation (9) becomes:

$$s_i \approx \left(1 - q_i + q_i \prod_{j \in J_i} \bar{\delta}_j \right) \bar{\delta}_i . \quad (21)$$

Increasing system component i ability to transfer its individual risk to components $j \in J_i$ by broadening set J_i or increasing probability q_i reduces component i individual risk at the cost of increasing the individual risks for components $j \in J_i$. The systemic risk is the result of the interplay of these two trends, and depends on the system parameters. In applications each system component $i = 1,..,N$ is typically characterized by exogenous parameters λ_i and C_i. Parameter λ_i quantifies the component i "exposure to individual risk" or "individual fragility" (Lorenz, J., et al. 2009), which may represent the exogenous load, stress, or damage. Parameter C_i represents component i ability to accommodate this load, resist this stress, or tolerate this damage.

Due to risk sharing, mean-field equations (18) have the following form:

$$\tilde{\delta}_i = \varphi_i(\tilde{\delta}_{-i} | \rho_i, C_i, q_j\rho_j, C_j), \quad (22)$$

$i, j = 1,..,N$, $j \neq i$, where parameter $\rho_i = \lambda_i / C_i$ represents component i normalized "exposure to exogenous risk" or "individual fragility." The right-hand side of (22) is an increasing in $\tilde{\delta}_j$, ρ_j, $q_j\rho_j$ and decreasing in C_i, C_j function. Fluid approximation describes system with high capacity components $C_i \to \infty$ capable of sustaining individual risk below capacity (Stolyar, A.L. & Zhong, Y. 2013). Assuming existence of the following limit:

$$\psi_i(\tilde{\delta}_{-i} | \rho_i, q_j\rho_j) = \lim_{C_i, C_j \to \infty} \varphi_i(\tilde{\delta}_{-i} | \rho_i, C_i, q_j\rho_j, C_j), \quad (23)$$

fluid approximation is:

$$\tilde{\delta}_i = \psi_i(\tilde{\delta}_{-i} | \rho_i, q_j\rho_j), \quad (24)$$

$i, j = 1,..N$, $i \neq j$, where $\psi_i(\tilde{\delta}_{-i} | \rho_i, q_j\rho_j) = 0$ if $\tilde{\delta}_j, \rho_i, q_j\rho_j < 1$, and thus for $\rho_i, q_j\rho_j < 1$ system (24) has "risk free" solution: $S = s_i = \tilde{\delta}_i = 0$.

Further we assume that right-hand sides of (22) and (24) are continuous functions for all $(\tilde{\delta}_i, \rho_i, C_i)$, and thus, Brouwer fixed-point theorem guarantees that these systems have at least one equilibrium for any set of system parameters. Since systems (22) and (24) are generally non-linear, this equilibrium may not be unique. Following common practice (Antunes, N., et al. 2008) we assume that globally stable solutions to (22) and (24) describe system steady state emerging as $t \to \infty$ for any initial system state. Multiple locally stable solutions describe metastable or persistent system states. On the "fast" time scale system state converges to the "closest" metastable state. Mixing of these metastable states occurs on the "slow" time scale and can be described by the embedded Markov chain (Marbukh, V. 1993).

3 SYSTEMIC RISK IN SYMMETRIC SYSTEM

Consider a symmetric system with component-independent system parameters: $\rho_i = \rho$, $C_i = C$, $q_i = q$, $|J_i| = m$. We also assume that for binomial distribution

$$P(\delta_i) = \mathrm{Bi}(k | N, \tilde{\delta}) := \frac{N!}{k!(N-k)!} \tilde{\delta}^k (1-\tilde{\delta})^{N-k}, \quad (25)$$

right-hand sides in (22) and (24) do not depend on $i = 1,..,N$. In this case systems (22) and (24) preserve symmetric distribution (25), where $\tilde{\delta}$ satisfies the following mean-field equation:

$$\tilde{\delta} = \varphi(\tilde{\delta} | m, q; \rho, C) \quad (26)$$

and fluid approximation

$$\tilde{\delta} = \psi(\tilde{\delta} | m, q; \rho) \quad (27)$$

respectively, where $\psi(\tilde{\delta} | m, q; \rho) \equiv 0$ for $\rho < 1$. In symmetric case expression (21) for systemic and individual risks takes the following form:

$$S = s = (1 - q + q\tilde{\delta}^m)\tilde{\delta}. \quad (28)$$

Thus the optimal level of resource sharing $m = m^{opt}$ and risk transfer probability $q = q^{opt}$ minimize expression (28) subject to constraint (22) or (24) for mean-field or fluid approximation respectively.

Further in this Section we consider a specific case of symmetric system, where risk sharing can be modeled as increase in the normalized risk exposure

$$\tilde{\rho} = [1 + q\omega(\tilde{\delta} | m)]\rho, \quad (29)$$

where function $\omega_m(\tilde{\delta} | m)$ is increasing in both $\tilde{\delta}$ and m, and $\omega(0 | m) \equiv \omega(\tilde{\delta} | 0) \equiv 0$. Under fluid approximation system components are capable of accommodating the entire risk below capacity (Stolyar, et al. 2013), and thus

$$\tilde{\delta} = \max(0,\ 1 - 1/\tilde{\rho}). \quad (30)$$

Subsections 3.1 and 3.2 analyze systemic risk in symmetric system under mean-field and fluid approximations respectively.

3.1 *Mean-field approximation*

Figure 1 sketches typical solution structure of equation (26) (Marbukh, V. 2013 & 2014).

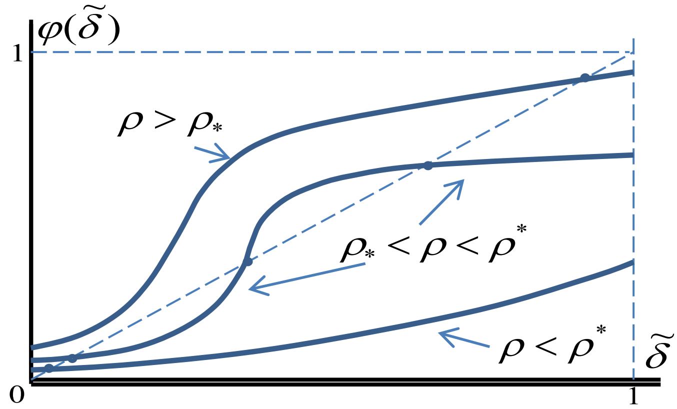

Figure 1. Solution to equation (26) for different ρ.

For sufficiently small and sufficiently large exogenous risk exposure ρ: $\rho < \rho_*$ and $\rho > \rho^*$, equation (26) has unique globally stable solution $\tilde{\delta}_*$ and $\tilde{\delta}^*$ describing low and high systemic risk stable system equilibria respectively. For intermediate values of ρ: $\rho_* < \rho < \rho^*$, these two solutions ρ_* and ρ^* co-exist as locally stable, and describe metastable, i.e., persistent system equilibria.

Figure 2 sketches "long-term" risks (28) vs. adiabatically, i.e., "slowly," changing exogenous normalized risk exposure ρ for different values of m.

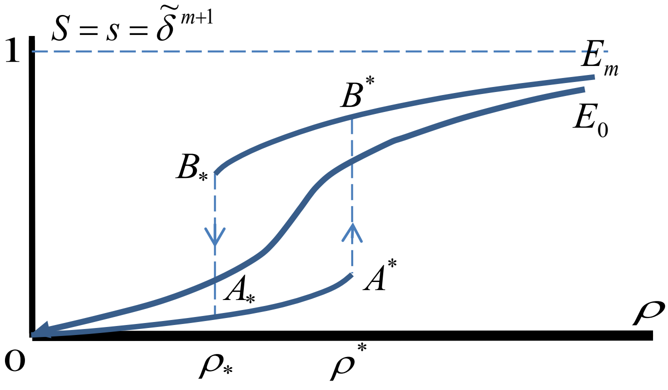

Figure 2. Systemic/individual risk vs. ρ.

Curve $0E_0$ corresponds to sufficiently low level of risk sharing m, when equation (26) has unique globally stable equilibrium $\tilde{\delta}$ for all ρ. For sufficiently high risk sharing level m and intermediate load $\rho_*(m) < \rho < \rho^*(m)$, equation (26) has two locally stable equilibria $\tilde{\delta}_*$ and $\tilde{\delta}^*$. As load ρ "slowly" increases or decreases, the risks $S = s$ follow curve $0A_*(m)A^*(m)B^*(m)E_m$ or $E_mB^*(m)B_*(m)A_*(m)0$ respectively. Curves $0A_*(m)$ and $E_mB^*(m)$ correspond to the globally stable "low" and "high" risk system equilibria respectively. Branches $A_*(m)A^*(m)$ and $B_*(m)B^*(m)$ correspond to the co-existing "low" and "high" metastable system equilibria respectively for intermediate load $\rho_*(m) < \rho < \rho^*(m)$. Discontinuities at the critical loads $\rho_*(m)$ and $\rho^*(m)$ as well as the hysteresis loop $A_*(m)A^*(m)B^*(m)B_*(m)$ indicate discontinuous, i.e., the first order phase transition.

Increase in resource sharing level m increases "spread" between the "low" and "high" risk metastable equilibria. Figure 3 demonstrates this dual effect of risk sharing level m on the systemic risk (28).

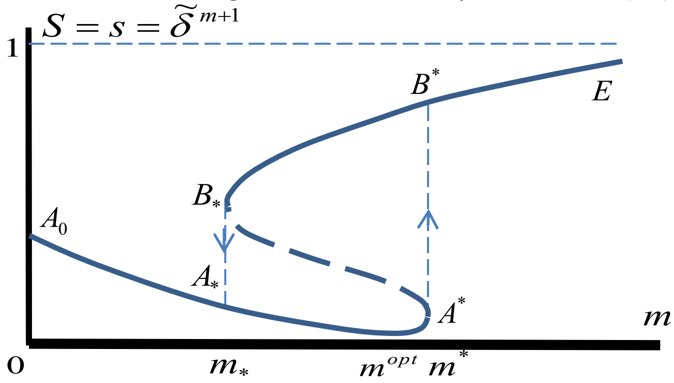

Figure 3. Systemic/individual risk vs. risk sharing level m.

As m "slowly" increases (decreases), the systemic/individual risks $S = s$ follow curve $A_0A_*A^*B^*E$ $(EB^*B_*A_*A_0)$. Branches $A_0A_*A^*$, EB^*B_* and hysteresis loop $A_*A^*B^*B_*A_*$ in Figure 3 correspond to branches OA_*A^*, EB^*B_* and hysteresis loop $A_*A^*B^*B_*A_*$ in Figure 2 respectively.

3.2 *Fluid approximation*

Figure 4 shows solution to fluid approximation (27) for $\gamma < 1$ and $q = 1$, where

$$\gamma := q \lim_{x \downarrow 0} d\omega(x|m)/dx. \qquad (31)$$

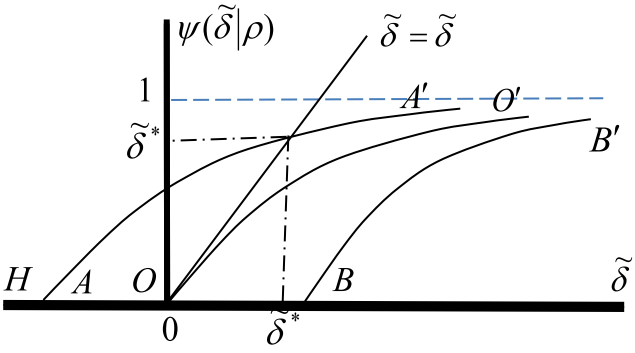

Figure 4. Solution to fluid approximation (27) for $\gamma < 1$.

Curves $HAOBB'$, $HAOO'$, and HAA' sketch right-hand side of (27) for $\rho < 1$, $\rho = 1$, and $\rho > 1$ respectively. Figure 5 sketches steady-state systemic/individual risks (28) under fluid approximation (27) for $\gamma < 1$ and $q = 1$.

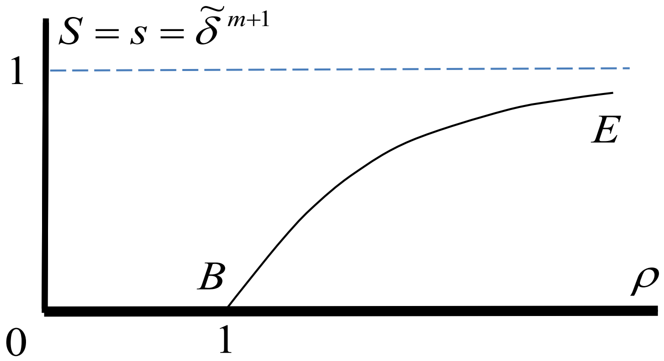

Figure 5. Steady risks under fluid approximation for $\gamma < 1$.

Figure 6 shows solution to fluid approximation (27) for $\gamma > 1$ and $q = 1$.

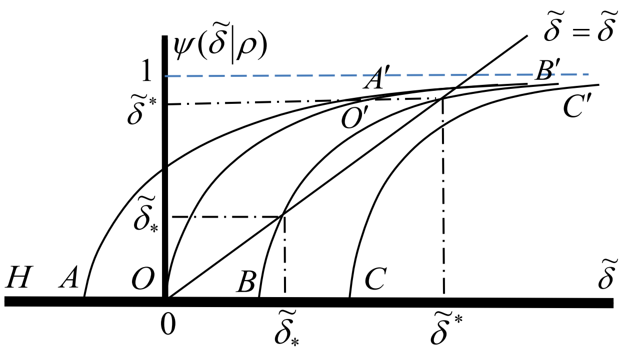

Figure 6. Solution to fluid approximation (27) for $\gamma > 1$.

Curves $HAOBCC'$, $HAOBB'$, $HAOO'$, and HAA' sketch right-hand side of (27) for $\rho < \rho_*$, $\rho_* < \rho < 1$, $\rho = 1$, and $\rho > 1$ respectively. For a sufficiently light or heavy exogenous risk exposure $\rho < \rho_*$ or $\rho > 1$, equation (27) has a unique fixed point $\tilde{\delta} = 0$ or $\tilde{\delta} = \tilde{\delta}^*$ respectively. For an intermediate exogenous load $\rho_* < \rho < 1$, these both fixed points coexist as locally stable, and are separated by an unstable fixed point.

Figure 7 sketched the persistent systemic/individual risks (28) under fluid approximation (27) for $\gamma > 1$ and $q = 1$.

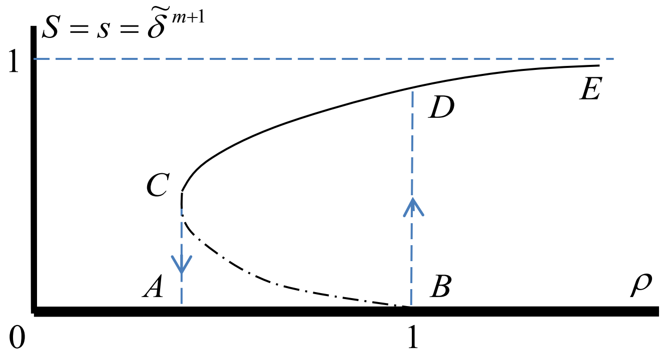

Figure 7. Persistent risks under fluid approximation for $\gamma > 1$.

For $\rho \le 1$ fixed point $\tilde{\delta} = 0$ describes the desirable "risk-free" equilibrium. Fixed point $\tilde{\delta} = \tilde{\delta}^*$, represented by curve CDE, describes the "high-risk" equilibrium. Curve CB represents unstable equilibrium fixed point separating stable equilibrium points $\tilde{\delta} = 0$ and $\tilde{\delta} = \tilde{\delta}^*$. While the "risk-free" and "high-risk" equilibria are globally stable for "light" and "heavy" exogenous risk exposure ρ, these equilibria are metastable for "intermediate" exogenous risk exposure ρ. As ρ "slowly" increases from 0 to ∞, or decreases from ∞ to 0, risk follows curve $0ABDE$, or $EDCA0$ respectively. Hysteresis loop $BDCAB$ is indicative of the discontinuous "phase transition" and metastability.

4 TOWARDS SYSTEMIC RISK MANAGEMENT

This Section argues that analysis in Subsection 3.2 can be extended to describe and manage onset of systemic instability in general heterogeneous systems. Subsection 4.1 demonstrates that under fluid approximation systemic instability occurs in one dimension described by Perron-Frobenius theory. Subsection 4.2 quantifies beneficial effect of risk sharing in networked systems on the system ability to sustain exogenous risks. Subsection 4.3 demonstrates how proposed in Subsection 4.1 systemic risk characterization can be used for balancing benefits and downsides of risk sharing.

4.1 *Perron-Frobenius risk characterization*

Mean-field system (22) has solution $\tilde{\delta}_* \approx 0$, which describes normal/operational, low-risk regime: $S \approx 0$ for $\rho_i < 1$, $i = 1,..,N$. It can be shown that solution $\tilde{\delta}_*$ is unique for sufficiently small ρ_i. As exogenous parameters ρ_i adiabatically, i.e., "slowly," increase, low-risk equilibrium may disappear or loose stability, and a "high-risk equilibrium" may emerge. Onset of this instability can be described using Perron-Frobenius theory (Nussbaum, R.D. 2012) as follows. Let $K = \{x_i | x_i \ge 0, i = 1,..,N\}$ be standard cone in N- dimensional Euclidian space. For $x, y \in K$ let $x \le y$ if $x_i \le y_i$ for $i = 1,..,N$; $x < y$ means that $x \le y$ and $x \ne y$. It can be shown under some additional technical assumptions, that negative externalities with respect to individual risks (1) imply monotone mapping (22): $\varphi : K \to K$, i.e., $0 \le x \le y$ implies $0 \le \varphi_i(x_{-i}) \le \varphi_i(y_{-i})$ for any set of system parameters. Thus system (22) falls within domain of non-linear Perron-Frobenius theory.

Due to limited space, consider fluid approximation (24), which has risk-free solution $\delta_i = 0$ for parameter vector $\rho := (\rho_1,.., \rho_N) \in U$, where the system operational region is $U = \{\rho_i < 1,\ i = 1,..,N\}$. Economics incentivizes system operator(s) to keep the system within its operational region U close to the corner point $\rho_i = 1$, $i = 1,..,N$ since both increase of the capacities C_i and decrease of the loads λ_i reduce the system operator(s) profit. However, unavoidable uncertainties in the exogenous parameters $\rho = (\rho_1,.., \rho_N)$ create inherent tradeoff between economic benefits of keeping system close to the boundary of the operational region on the one hand and risk of system crossing the stability boundary on the other hand.

To evaluate this risk consider system (24) for $\rho_i = 1 + \beta_i$, where $\beta_i = \beta_{i0}\varepsilon$, $\beta_{i0} > 0$, and $\varepsilon \downarrow 0$. Up to terms of order $o(\varepsilon)$ as $\varepsilon \downarrow 0$, system (24) takes the following form:

$$\delta_i = \sum_{j \ne i} a_{ij}\delta_j + b_i \beta_{i0}\varepsilon, \tag{32}$$

where $a_{ij} = \lim_{\varepsilon \to 0}[\partial \psi_i(\delta_{-i}|\rho)/\partial \delta_j]_{\delta=0, \rho_i = 1+\beta_{i0}\varepsilon} \ge 0$ and $b_i = [\partial \psi_i(\delta_{-i}|\rho)/\partial \rho_i]_{\delta=0, \rho=1} \ge 0$ due to negative externalities with respect to the risk exposure. Assuming for simplicity matrix $A = (a_{ij})_{i,j=1}^{N}$ to be irreducible, sufficient condition for existence of a "small" solution to linear system (32), $\delta_i = O(\varepsilon)$ as $\varepsilon \downarrow 0$ is:

$$\gamma(A) \le 1, \tag{33}$$

where γ is the Perron-Frobenius eigenvalue of matrix A. Criterion (33) is a generalization of one-dimensional condition (31). It can be shown that similar to considered in Subsection 3.2 one-dimensional case, condition $\gamma(A) > 1$ implies abrupt or discontinuous systemic instability.

We suggest that an abrupt/discontinuous systemic instability is more dangerous than a gradual/continuous one for the following reasons. It has been argued (Scheffer, M., et al. 2009), that gradual/continuous systemic instabilities are often preceded by warning signals, such as critical slowing down and anomalously large fluctuations. These "warning signals" may be used for prediction of the upcoming continuous systemic instabilities and taking the relevant corrective actions. On the other hand, abrupt/discontinuous instabilities occur unexpectedly, i.e., without warning signs, and are typically associated with existence of metastable, i.e., persistent, undesirable equilibria (Majdandzic, A., et al. 2013).

Typically networked system evolution depends on controlled parameters to be manipulated in attempt to optimize certain economic performance criterion. Balancing economic efficiency with systemic risks can be naturally formulated as a constrained optimization problem. Due to "higher priority" of abrupt/discontinuous systemic instabilities, we propose systemic risk aware economic efficiency maximization with condition (33) playing role of the optimization constraint. To account for uncertainty in the exogenous parameters, constraint (33) should hold with certain probability, which reflects the system risk aversion. Due to difficulty of evaluation of the Perron-Frobenius eigenvalue $\gamma(A)$, future research should investigate a possibility of using in (33) an upper bound for γ (Nussbaum, R.D. 2012), e.g.,

$$\max_{i=1,..,N} \sum_j a_{ij} \le 1. \qquad (34)$$

4.2 *Benefits of risk sharing: sustainable region*

Fluid approximation (24) for a system without risk sharing, i.e., for $q_i = 0$, $i = 1,..,N$, yields unique risk-free solution with $S = 0$ if vector of exogenous parameters $\rho = (\rho_i)$ stays within the operational region $U = \{\rho_i < 1,\ i = 1,..,N\}$. Resource sharing allows system to accommodate some imbalances in the risk exposures resulted from unavoidable uncertainties due to variability of the exogenous conditions, hardware breakdowns, etc. To quantify the sustainable region T comprised of all vectors $\rho = (\rho_i)$ which system with risk sharing can accommodate, we quantify the inefficiency of accommodating component i individual risk by component $j \ne i$ by parameters $\chi_{ij} \ge \chi_{ii} = 1$.

If portion q_{ij} of component $i = 1,.., N$ exogenous risk ρ_i is transferred to components $j \ne i$, and

$$q_i = \sum_{j \in J_i} q_{ij} \le 1, \qquad (35)$$

then the system can completely absorb the exogenous risk if

$$\tilde{\rho}_i = (1 - q_i)\rho_i + C_i^{-1} \sum_{j \in J_i} q_{ji} \chi_{ji} C_j \rho_j \le 1. \qquad (36)$$

Given system topology determined by sets J_i, the system sustainable region T is comprised from all vectors $\rho = (\rho_i)$ such that inequalities (35)-(36) can be satisfied for some probabilities $q_{ij} \ge 0$. Sustainable region T subsumes the operational region $U = \{\rho_i < 1,\ i = 1,.., N\} \subset T$, and thus region $T \setminus U$ quantifies benefits of risk sharing in the networked system with respect to system ability to accommodate the "worst-case" imbalances in the risk exposure by different system components. Note that benefits of resource sharing under scenario with probabilistic uncertainty can be quantified by the corresponding statistical multiplexing gain.

As an example consider a networked system with two components: $N = 2$. The corresponding sustainable region, plotted in Figure 8, is

$$\rho_i + \chi_{ji}(C_j / C_i)[\rho_j - 1]^+ \le 1 \qquad (37)$$

for $i = 1,2$, $j = 3 - i$, where $[x]^+ := \max(0, x)$.

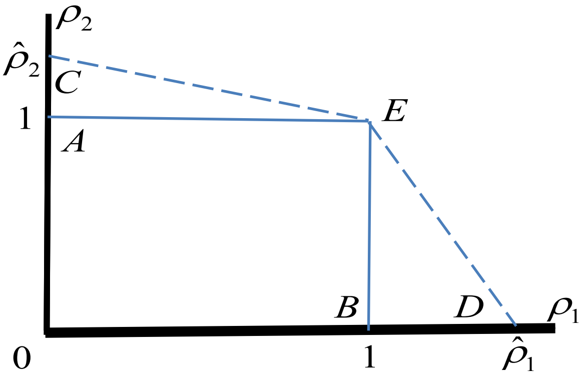

Figure 8. Operational and sustainable regions for $N = 2$.

Resource sharing expands operational region $U = OAEBO$ into sustainable region $T = OACEDBO$ by adding triangles ACE & BDE.

When $\chi_{ij} = 1$; $i, j = 1,.., N$, i.e., the "non-native" risk can be accommodated as efficiently as the "native" one, risk sharing has no downside. However, in a typical situation when $\chi_{ij} > \chi_{ii} = 1$, $i \ne j$, i.e., risk transfer amplifies the aggregate risk in the system, negative externalities due to resource sharing create risk of abrupt/discontinuous instability. Generally, this risk can be controlled by limiting degree of resource sharing. Next Subsection briefly discusses mitigation systemic risk of abrupt/discontinuous instabilities by controlling probabilities q_i.

4.3 *Controlling systemic risk of abrupt instability*

In addition to keeping exogenous fragilities at the corner point $\rho_i \approx 1$, $i = 1,.., N$ of the operational region $U = \{\rho_i < 1,\ i = 1,.., N\}$, economics also incentivizes system operators to keep probabilities q_i close to one to ensure that system is capable of accommodating at least some unavoidable imbalances in risk exposure by different system components. At the same time, as Subsection 4.1 suggests, system designer(s)/operator(s) should avoid abrupt/discontinuous systemic instability by enforcing condition (33). In this subsection we specify and illustrate condition (33) on the example of considered in the previous Subsection heterogeneous networked system. Specifically, our goal is to gain some insight into the tradeoff between desire to take advantage of risk sharing on the one hand and avoiding a possibility of abrupt /discontinuous instabilities by enforcing restriction (33) on the other hand.

Specifically, we assume that component i individual risk can be transferred to components $j \in J_i$ with controlled probability $q_i(\delta_j) \in [0,1]$. Probabilities $q_i(\delta_j)$ satisfy self-consistency conditions, e.g., $q_i(\delta_j) = 0$ if $\prod_{j \in J_i} \delta_j = 1$ and $q_i(0) = \sum_{j \in J_i} q_{ij}$, where q_{ij} is the probability that component i individual risk is transferred to a component $j \in J_i$ when $\delta_i = 1$ and $\delta_j = 0$, $\forall j \in J_i$. It is easy to see that the corresponding matrix A in condition (33) is

$$A = (C_j C_i^{-1} \chi_{ji} q_{ji})_{i,j=1}^{J} \tag{38}$$

and condition (34) becomes

$$\max_i C_i^{-1} \sum_{j=1}^{J} C_j \chi_{ji} q_{ji} \le 1. \tag{39}$$

Probabilities q_{ij} determine the system economic efficiency, e.g., quantified by the sustainable region T, given by (35)-(36). Constrained maximization of the selected economic efficiency criterion over probabilities q_{ij} subject to (33), (38) may be viewed as a systemic risk-aware performance optimization framework. "Robustness" of this framework is due to conditions (33), (38) and (39) being independent on the exogenous parameters ρ_i, which are typically subject to uncertainty and variability. In a particular case $N = 2$ condition (33), (38) takes the following form:

$$q_{12} q_{21} \le (\chi_{12} \chi_{21})^{-1}. \tag{40}$$

When $\chi_{12} = \chi_{21} = 1$, i.e., risk sharing does not amplify the aggregate risk, constraint (40) is not binding. When $\chi_{12}\chi_{21} > 1$, i.e., this amplification takes place, constraint (40) is binding, and particular selection of probabilities q_{ij} satisfying (40) depends on the specific scenario. For example, if risk exposure for component $i = 1$ is more economically beneficial than for component $j = 2$, then the natural selection of probabilities q_{ij} is $q_{12} = 1$ and $q_{21} = (\chi_{12}\chi_{21})^{-1}$.

5 CONCLUSION AND FUTURE RESEARCH

This paper has discussed interplay of the benefits and drawbacks of risk sharing in networked systems. The benefits are due to mitigation of local risk imbalances, and the drawbacks are due to contagion and risk propagation. The contagion is a result of negative externalities with respect to local risk exposure, when risk sharing among system components amplifies the aggregate risk in the system. The contagion may produce a continuous or discontinuous systemic instabilities, which can be classified and controlled by applying Perron-Frobenius theory to risk propagation equations.

Economics driving system towards stability boundary of the normal/operational equilibrium in combination with inherent uncertainties alplify risk of systemic instability. We have suggested that balancing economic benefits with systemic risks should eliminate or at least mitigate systemic risk of abrupt/discontinuous instability in favor of gradual/continuous instability. This contention is based on predictability and implications of different kinds of systemic instabilities.

Numerous questions deserve further investigation, e.g., accuracy of the proposed mean-field and fluid approximations. More broadly, future work should address the practicality of the proposed framework at the system design and operational stages. This may include controlling both the contagion on the existing network as well as controlling the network structure/topology. Of particular interest is a potential ability of online measurements of the corresponding Perron-Frobenius eigenvalue and other "macroscopic" parameters to be used as a part of "early warning system" of the approaching systemic instabilities.

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