Simulations of the Hadamard Variance: Probability Distributions and Confidence Intervals

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***Abstract*—A method for simulating power law noise in clocks and oscillators is presented based on modification of the spectrum of white phase noise, then Fourier transforming to the time domain. This approach has been applied successfully to simulation of the Allan variance and the modified Allan variance in both overlapping and non-overlapping forms[1][2]. When significant frequency drift is present in an oscillator, at large sampling times the Allan variance overestimates the intrinsic noise, while the Hadamard variance is insensitive to frequency drift. The simulation method is extended in this paper to predicting the Hadamard variance for the common types of power law noise. Symmetric real matrices are introduced whose traces–the sums of their eigenvalues–are equal to the Hadamard variances, in overlapping or non-overlapping forms, as well as for the corresponding forms of the modified Hadamard variance. We show that the standard relations between spectral densities and Hadamard variance, are obtained with this method. The matrix eigenvalues determine probability distributions for observing a variance at an arbitrary value of the sampling interval τ, and hence for estimating confidence in the measurements. Examples are presented for the common power-law noises.**

I. Introduction

The characterization of clock performance by means of average measures such as Allan variance, Hadamard variance, Theo, and modified forms of such variances, is widely applied within the time and frequency community as well as by most clock and oscillator fabricators. Such variances are measured by comparing the times t_k on a device under test, with the times at regular intervals $k\tau_0$ on a perfect reference, or at least on a better reference. Imperfections in performance of the clock under test are studied by analyzing noise in the time deviation sequence $x_k = t_k - k\tau_0$, or the fractional frequency difference during the sampling interval $\tau = s\tau_0$:

$$\Delta^{(1)}_{k,s} = (x_{k+s} - x_k)/(s\tau_0). \tag{1}$$

The frequency spectrum of fractional frequency differences can usually be adequately characterized by linear superposition of a small set of types of power law noise. The frequency spectrum of the fractional frequency differences of a particular noise type is given by a one-sided spectral density[3]

$$S_y(f) = h_\alpha f^\alpha, \quad f > 0. \tag{2}$$

(The units of $S_y(f)$ are Hz^{-1}.) For the common power-law noise types, α varies in integral steps from +2 down to -2 corresponding respectively to white phase modulation, flicker phase modulation, white frequency modulation, flicker frequency modulation, and random walk of frequency.

Simulation of clock noise can be extremely useful in testing software algorithms that use various windowing functions and Fourier transform algorithms to extract spectral density and stability information from measured time deviations, and especially in predicting the probability for observing a particular value of some clock stability variance. This paper develops a simple simulation method for a time difference sequence that guarantees the average spectral density will have some chosen average power law . Since the Allan variance was introduced fifty years ago, oscillators with significant drift have seen widespread utilization, especially in situations where stability rather than absolute accuragy is important (e.g., for Rubidium Atomic Frequency Standards (RAFS) in GPS satellites). Here, expressions for the Hadamard variances and their modified forms are derived. This approach also leads to predictions of probabilities for observing a variance of a particular type at particular values of the averaging time. A broad class of probability functions naturally arises. These only rarely correspond to chi-squared distributions.

When an oscillator's frequency drifts, usually the long-term behavior of the Allan variance is dominated by the drift, and the oscillator stability is not well characterized by the Allan variance.[4] One could approach this problem by estimating and removing the drift from the measured time series, but the estimation process may itself introduce uncertainties. The Hadamard variance, which is defined (see below) in terms of a third difference of values of the time measurements, is naturally insensitive to drift and is commonly applied to clocks such as those based on RAFS, which are known to suffer from unpredictable frequency drift following launch but which have better stability that Cesium clocks.[23][5]

The present approach to noise simulation was motivated by difficulties in applying methods such as are discussed in references [6], [7], [8] to acceleration noise in spacecraft at very low frequencies[9]. Noisy accelerations of spacecraft can arise from causes such as fluctuations in solar radiation pressure, anisotropic thermal radiation, charged particle drag, etc., and after integrated twice can give rise to a spectral density of displacement noise that diverges at low frequencies faster than any known power law clock noise. For example, an acceleration noise amplitude proportional to f^{-1} integrated twice gives rise to a displacement noise spectral density proportional to f^{-6}. Study of such acceleration noise is important in establishing error budgets of space missions in which spacecraft are tracked by means of electromagnetic signals. This simulation of power law noises for the Allan variance and the Modified Allan Variance has already been published[1][2]. Here we apply the method to simulation of the Hadamard and Modified Hadamard variances.

This paper is organized as follows. Sect. 2 introduces the basic simulation method, and Sect 3 applies the method to the overlapping Hadamard variance. Sect. 4 shows how diagonalization of the averaged squared second-difference operator, applied to the simulated time series, leads to expressions for the probability of observing a value of the variance for some chosen value of the sampling or averaging time. Expressions for the mean squared deviation of the mean of the variance itself are derived in Sect. 6. The approach is used to discuss the modified Hadamard variance in Sect. 7, and the non-overlapping form of the Hadamard variance is treated in Sect. 8.

II. Discrete Time Series

We imagine the noise amplitudes at Fourier frequencies f_m are generated by a set of N normally distributed random complex numbers w_n having mean zero and variance $< w_m^* w_m >= 2\sigma^2$, that would by themselves generate a simulated spectrum for white phase noise. These random numbers are divided by a function of the frequency, $|f_m|^\lambda$, where $2\lambda = 2 - \alpha$ (see Eq.(2)), producing a spectral density that has the desired frequency characteristics. For ordinary power law noise, the exponent λ is a multiple of $1/2$, but it could be anything. The frequency noise is then transformed to the time domain, producing a time series with the statistical properties of the selected power law noise. Variances can be calculated using either the frequency noise or the time series.

In the present paper we discuss applications to calculation of various versions of the Hadamard variance. Of considerable interest are results for the probability of observing a value of the hadamard variance for particular values of the sampling time τ and time series length N. The derivations in this paper are theoretical predictions. A natural frequency cutoff occurs at $f_h = 1/(2\tau_0)$, where τ_0 is the time between successive time deviations. This number is not necessarily related in an obvious way to some hardware bandwidth. The measurements are assumed to be made at the times $k\tau_0$, and the time errors or residuals relative to the reference clock are denoted by x_k. The averaging or sampling time is denoted by $\tau = s\tau_0$, where s is an integer. The total length of time of the entire measurement series is $T = N\tau_0$.

Noise Sequences. In order that a set of noise amplitudes in the frequency domain represent a real series in the time domain, the amplitudes must satisfy the reality condition

$$w_{-m} = (w_m)^*. \tag{3}$$

N random numbers are placed in $N/2$ real and $N/2$ imaginary parts of the positive and negative frequency spectrum. Thus if $w_m = u_m + iv_m$ where u_m and v_m are independent uncorrelated random numbers, then $(w_m)^* = u_m - iv_m$. Since the frequencies $\pm 1/2\tau_0$ represent essentially the same contribution, $v_{N/2}$ will not appear. We shall assume the variance of the noise amplitudes is such that

$$\big\langle (w_m)^* w_n \big\rangle = \big\langle u^2 + v^2 \big\rangle \delta_{mn} = 2\sigma^2 \delta_{mn}; \quad m \neq 0, N/2. \tag{4}$$

Also, $< w_m^2 >=< u^2 - v^2 + 2iuv >= 0$ for $m \neq N/2$. The index m runs from $-N/2+1$ to $N/2$. In order to avoid division by zero, we shall always assume that the Fourier amplitude corresponding to zero frequency vanishes. This only means that the average of the time residuals in the time series will be zero, and has no effect on any variance that involves time differences.

We perform a discrete Fourier transform of the frequency noise and obtain the amplitude of the k^{th} member of the time series for white PM:

$$x_k = \frac{\tau_0^2}{\sqrt{N}} \sum_{m=-N/2+1}^{N/2} e^{-\frac{2\pi i m k}{N}} w_m. \tag{5}$$

The possible frequencies that occur in the Fourier transform of the time residuals are

$$f_m = \frac{m}{N\tau_0}. \qquad -\frac{N}{2} + 1 \leq m \leq \frac{N}{2}. \tag{6}$$

The simulated time series corresponding to power law noise characterized by $\lambda = 1 - \alpha/2$ is

$$X_k = \frac{\tau_0^2}{\sqrt{N}} \sum_{m=-N/2+1}^{N/2} \frac{|f_0|^\lambda}{|f_m|^\lambda} e^{-\frac{2\pi i m k}{N}} w_m. \tag{7}$$

The constant factor $|f_0|^\lambda$ has been inserted to maintain the physical units of the time series. The parameter f_0 determines the level h_α of the noise. An alternative expression involving a sum over positive frequencies only is:

$$X_k = \frac{2\tau_0^2}{\sqrt{N}} \sum_{m=1}^{N/2-1} \frac{|f_0|^\lambda}{|f_m|^\lambda} \left(\cos\frac{2\pi m k}{N} u_m + \sin\frac{2\pi m j}{N} v_m \right) + \frac{\tau_0^2}{\sqrt{N}} \frac{|f_0|^\lambda}{|f_{N/2}|^\lambda} (-1)^k u_{N/2} \tag{8}$$

To obtain the correct spectral density, we shall assume that the constants introduced in the above expression are related to the strength of the power law noise by

$$\frac{\tau_0^2 |f_0|^\lambda}{\sqrt{N}} = \left(\frac{h_\alpha}{16\pi^2\sigma^2(N\tau_0)} \right)^{1/2}. \tag{9}$$

We then may show that if $2\lambda = 2 - \alpha$ the correct average spectral density is obtained. The simulated time series is

$$X_k = \left(\frac{h_\alpha}{16\pi^2\sigma^2(N\tau_0)} \right)^{1/2} \sum_m \frac{e^{-\frac{2\pi i m k}{N}}}{|f_m|^\lambda} w_m. \tag{10}$$

The average (two-sided) spectral density of the time residuals is obtained by squaring a single term in Eq. (10) and dividing by the spacing $\Delta f = 1/(N\tau_0)$ between successive allowed frequencies:

$$s_x(f_m) = \frac{h_\alpha}{16\pi^2\sigma^2(N\tau_0) f_m^{2\lambda}} \left\langle \frac{w_m w_m^*}{\Delta f} \right\rangle = \frac{h_\alpha}{8\pi^2 f_m^{2\lambda}} \tag{11}$$

The average (two-sided) spectral density of fractional frequency fluctuations is given by the well-known relation[11][12]

$$s_y(f) = (2\pi f)^2 s_x(f)\,, \tag{12}$$

and in agreement with convention, Eq. (2), the one-sided spectral density is

$$S_y(f) = \begin{cases} 0, & f < 0; \\ 2s_y(f) = h_\alpha f^\alpha, & f > 0\,. \end{cases} \tag{13}$$

III. Overlapping Hadamard Variance

Consider the third-difference operator defined by

$$\Delta^{(3)}_{j,s} = \frac{1}{\sqrt{6\tau^2}}\big(X_{j+3s} - 3X_{j+2s} + 3X_{j+s} - X_j\big)\,. \tag{14}$$

The completely overlapping Hadamard variance is formed by averaging the square of this third difference over all possible values of j from 1 to $N-3s$: Thus

$$\sigma^2_{Ho}(\tau) = \frac{1}{N-3s}\Big\langle \sum_{j=1}^{N-3s} \big(\Delta^{(3)}_{j,s}\big)^2 \Big\rangle\,. \tag{15}$$

In terms of the time series, Eq. (10), the third difference can be reduced using elementary trigonometric identities:

$$\begin{aligned} \Delta^{(3)}_{j,s} &= \left(\frac{h_\alpha}{96\tau^2\pi^2\sigma^2 N\tau_0}\right)^{1/2} \sum_m \frac{w_m}{|f_m|^\lambda} e^{-\frac{\pi i m(2j+3s)}{N}} \\ &\times \left(e^{-\frac{3\pi i m s}{N}} - 3e^{\frac{-\pi i m s}{N}} + 3e^{\frac{\pi i m s}{N}} - e^{\frac{3\pi i m s}{N}}\right) \\ &= i\left(\frac{2h_\alpha}{3\tau^2\pi^2\sigma^2 N\tau_0}\right)^{1/2} \sum_m \frac{w_m e^{-\frac{\pi i m(2j+3s)}{N}}}{|f_m|^\lambda} \\ &\times \left(\sin\left(\frac{\pi m s}{N}\right)\right)^3. \end{aligned} \tag{16}$$

We form the averaged square of $\Delta^{(3)}_{j,s}$ by multiplying the expression Eq.(16) by its complex conjugate, then summing over all possible values of j and averaging. In squaring the third difference one of the factors is conjugated; then averaging, only the terms corresponding to the same frequencies in the two factors contribute. the overlapping Hadamard variance is

$$\sigma^2_{Ho}(\tau) = \frac{4h_\alpha}{3N\tau_0\pi^2\tau^2}\sum_m \left(\sin\big(\frac{\pi m s}{N}\big)\right)^6 \frac{1}{|f_m|^{2\lambda}}. \tag{17}$$

The spacing between frequencies is $1/(N\tau_0) = df$; in the limit of large N the sum over frequencies passes to an integral:

$$\sigma^2_{Ho}(\tau) = \frac{4h_\alpha}{3\pi^2\tau^2}\int_{-f_h}^{f_h} \frac{df}{|f|^{2\lambda}}\Big(\sin(\pi f\tau)\Big)^6. \tag{18}$$

Writing this as a single-sided integral in terms of the spectral density,

$$\sigma^2_{Ho}(\tau) = \frac{8}{3\pi^2\tau^2}\int_0^{f_h} \frac{S_y(f)df}{f^2}\Big(\sin(\pi f\tau)\Big)^6. \tag{19}$$

Returning to the discussion of Eq. (17), for convenience we introduce the abbreviation

$$K = \frac{8h_\alpha}{3\pi^2\tau^2(N\tau_0)}\,. \tag{20}$$

If we write the sum in terms of positive frequencies only, a factor of 2 comes in except for the most positive frequency and so

$$\sigma^2_{Ho}(\tau) = K\Bigg(\sum_{m>0}^{N/2-1} \frac{\Big(\sin\big(\frac{\pi m s}{N}\big)\Big)^6}{f_m^{2\lambda}} + \frac{\Big(\sin\frac{\pi s}{2}\Big)^6}{2(f_{N/2})^{2\lambda}}\Bigg). \tag{21}$$

The influence of the second term in Eq. (21) is very small except when s is very small; we shall therefore neglect it in the remainder of this paper.

Similar arguments lead to known expressions for the non-overlapping version of the Hadamard variance and the modified Hadamard variance. Proofs of these statements can be given provided no windowing or undersampling is applied to the time series. These forms of the variance will be discussed in later sections.

IV. Limit of Large Sampling Times

We evaluate the integral in Eq.(19) for each of the common power-law noises in the limit of large sampling times, $\tau = s\tau_0$. The spectral density is $S_y(f) = h_\alpha f^\alpha$. Contributions to the integrals typically have oscillating terms that become small in this limit. Table I lists the limiting values of the overlapping hadamard variance with such oscillating terms omitted. γ is the Euler's constant in $\sigma^2_{Ho}(\tau)$ for $\alpha = 1$.

$S_y(f)$	$\sigma^2_{Ho}(\tau)$	Mod $\sigma^2_{Ho}(\tau)$	λ
$h_2 f^2$	$\frac{5h_2 f_h}{6\pi^2\tau^2}$	$\frac{5h_2}{12\pi^2\tau^3}$	0
$h_1 f$	$\frac{h_1}{6\pi^2\tau^2}(5\gamma$ $+5\ln(f_h\pi\tau) + \frac{\ln(48)}{2})$	$\frac{3h_1 \ln\left(\frac{4}{3}\right)}{\pi^2\tau^2}$	$\frac{1}{2}$
h_0	$\frac{h_0}{2\tau}$	$\frac{2h_0}{9\tau}$	1
$\frac{h_{-1}}{f}$	$\frac{1}{2}h_{-1}\ln\left(\frac{256}{27}\right)$	$\frac{8}{3}h_{-1}\ln\left(\frac{27\times 3^{3/8}}{32}\right)$	$\frac{3}{2}$
$\frac{h_{-2}}{f^2}$	$\frac{1}{3}h_{-2}\pi^2\tau$	$\frac{2\pi^2}{9}h_{-2}\tau$	2

Table I
Asymptotic expressions for the Hadamard and Modified Hadamard Variances, in the limit of large sampling times $\tau = s\tau_0$.

As is the case for the overlapping Allan variance, the difference between White PM and Flicker PM cannot be distinguished by the dependence on τ. Figure 1 plots the overlapping Hadamard variance for flicker PM, N=1024. Several simulation runs are shown as well as the average variance for $S_y(f) = h_1 f$.

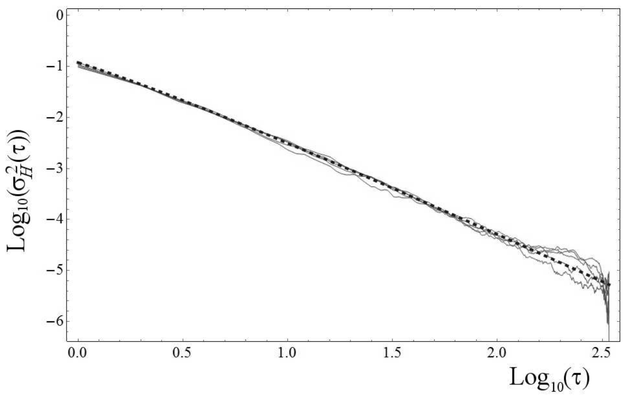

Figure 1. Overlapping Hadamard variance for N=1024, for flicker PM. h_1 has been set equal to unity and $\tau_0 = 1$. Several actual simulation runs are plotted; the theoretical average is shown as a dashed line.

V. Non-Overlapping Hadamard Variance

In the completely overlapping form, the sum in Eq.(15) runs from 1 to $N-3s$; thus some values of the time series would be used more than once. In order to avoid this problem a non-overlapping form is defined as in Eq. (15) but with the sum over j skipping repeated data items. Consider the sum

$$\frac{1}{s}\sum_{l=j}^{j+s-1}\left(\Delta_{l,s}^{(3)}\right)^2. \tag{22}$$

The sum has s terms and uses each data item from j to $j+4s-1$ exactly once. This block of data is of length $4s$. In general $4s$ is incommensurable with N so there will exist some data items that will not be included in such blocks. Let M_{max} be the maximum integer such that

$$4(M_{max}+1)s \le N. \tag{23}$$

Then if we define the non-overlapping form of the Hadamard variance as

$$\sigma_{Hno}(\tau)^2 = \frac{1}{(M_{max}+1)s^2}\left\langle \sum_{M=0}^{Mmax}\sum_{l=1+4Ms}^{4(M+1)s}\left(\Delta_{l,s}^{(3)}\right)^2\right\rangle, \tag{24}$$

we will have left out part of one block of data, but each item of data in the sum will appear with equal weight in Eq. (24). Although an estimate of the variance could be improved by incorporating the partial block of data and appropriately modifying the normalization constant, we shall develop the theory ignoring such contributions because the ensemble average of each term in Eq. (24) is independent of l.

Squaring the third difference in Eq. (16), we write one of the factors as a complex conjugate and average over the random numbers. Then the only terms that contribute to the double sum are those for equal frequencies, and

$$\left\langle\left(\Delta_{k,s}^{(3)}\right)^2\right\rangle = \frac{8}{6}\frac{h_\alpha}{\pi^2\tau^2(N\tau_0)}\sum_{m>0}\frac{\left(\sin\frac{\pi ms}{N}\right)^6}{|f_m|^{2\lambda}}. \tag{25}$$

Thus the average of both overlapping and non-overlapping forms of the Hadamard variance are the same, but as will be shown the probability distributions and confidence intervals are different.

VI. Modified Overlapping Hadamard Variance

The Hadamard variance suffers from the same difficulty as does the Allan Variance–the variances for both white phase noise and flicker PM are proportional to τ^{-2} for large τ, thus the Hadamard variance cannot distinguish between white PM and flicker PM. David Allan solved this problem by inventing the modified variance, which involves averaging s differences before squaring, then averaging the result. Similarly we define a Modified Hadamard variance as:

$$\mathrm{Mod}\ \sigma_H^2 = \left\langle\left(\frac{1}{s}\sum_{l=j}^{j+s-1}\Delta_{j,s}^{(3)}\right)^2\right\rangle, \tag{26}$$

where the average is taken over the ensemble of values of the random number distributions and over all possible values of j. For the overlapping form, using elementary trigonometric identities the expression reduces to

$$\mathrm{Mod}\ \sigma_H^2 = \frac{64\tau_0^4\sigma^2}{3\tau^2 N s^2}\sum_m\frac{|f_0|^{2\lambda}}{|f_m|^{2\lambda}}\left(\sin\frac{\pi ms}{N}\right)^6\left(\frac{\sin\frac{\pi ms}{N}}{\sin\frac{\pi m}{N}}\right)^2. \tag{27}$$

In the limit of sufficiently densely spaced frequencies, the sum passes to a single-sided integal:

$$\mathrm{Mod}\ \sigma_H^2 = \frac{8}{3}\int_0^{f_h}\frac{S_y(f)df}{(\pi s\tau f)^2}\frac{(\sin\pi\tau f)^8}{(\sin\pi\tau_0 f)^2} \tag{28}$$

In general this integral is difficult to evaluate. In Table I we give the results in the limit

$$\tau/\tau_0 \to \infty. \tag{29}$$

As is the case for the modified Allan variance, the modified Hadamard variance distinguishes between white pm and flicker pm.

In the non-overlapping case, the desire is to use each data item only once, but average over s values of the third difference before squaring. The average over s values of the third difference, from Eq. (16) is

$$\frac{1}{s}\sum_{l=j}^{j+s-1}\Delta_{k,s}^{(3)} = i\sqrt{\frac{K}{12s^2}}\sum_{l=j}^{j+s-1}\sum_m\frac{w_m}{|f_m|^\lambda} \times\left(\sin\left(\frac{\pi ms}{N}\right)\right)^3 e^{\frac{\pi im}{N}(2l+3s)}. \tag{30}$$

In this sum each item in the time series occurs exactly once. Therefore as in the case of the non-overlapping variance, the modified Hadamard variance can be constructed in terms of blocks of non-overlapping data. Here the sum over l is a geometric series giving:

$$\frac{1}{s}\sum_{l=j}^{j+s-1}\Delta_{k,s}^{(3)} = i\sqrt{\frac{K}{12s^2}}\sum_{l=j}^{j+s-1}\sum_m\frac{w_m}{|f_m|^\lambda} \times\frac{\left(\sin\left(\frac{\pi ms}{N}\right)\right)^4}{\sin(\frac{\pi m}{N})}e^{\frac{\pi im}{N}(1+s)}. \tag{31}$$

Squaring and averaging, again writing one factor in terms of a complex conjugate, as in the above cases only terms of equal frequencies contribute and then summing only over positive frequencies, the modified Hadamard variance is the same as that given in Eq. (27), which can also be written

$$\text{Mod } \sigma_H^2 = \frac{8K}{3s^2} \sum_{m>0} \frac{\left(\sin \frac{\pi m s}{N}\right)^8}{|f_m|^{2\lambda}\left(\sin \frac{\pi m}{N}\right)^2} . \tag{32}$$

In the next section we discuss the different probability functions that arise.

VII. Eigenvalues and Probabilities

In the present section we shall develop expressions for the probability of observing a particular value A_o for the overlapping Hadamard variance in a single measurement, or in a single simulation run. A_o is a random variable representing a possible value of the overlapping variance. We use a subscript "o" to denote the completely overlapping case. To save writing, we introduce the following abbreviations:

$$F_m^j = \frac{\left(\sin\left(\frac{\pi m s}{N}\right)\right)^3}{|f_m|^\lambda} \sin\left(\frac{\pi m(2j+3s)}{N}\right)$$
$$G_m^j = -\frac{\left(\sin\left(\frac{\pi m s}{N}\right)\right)^3}{|f_m|^\lambda} \cos\left(\frac{\pi m(2j+3s)}{N}\right) \tag{33}$$

The dependence on s is suppressed, but is to be understood. We write the third difference in terms of a sum over positive frequencies only, keeping in mind that the most positive and the most negative frequencies only contribute a single term since $\sin(\pi(j+s)) = 0$. The imaginary contributions cancel, and from Eq. (16) we obtain

$$\Delta_{j,s}^{(3)} = \sqrt{K} \sum_{m>0} \left(F_m^j \frac{u_m}{\sigma} + G_m^j \frac{v_m}{\sigma}\right) . \tag{34}$$

There is no term in $v_{N/2}$. Then from Eq. (15) the overlapping Hadamard variance is given by

$$\sigma_y^2(\tau) = \frac{K}{N-3s} \sum_{j=1}^{N-3s} \sum_{m>0} \left((F_m^j)^2 + (G_m^j)^2\right) . \tag{35}$$

To compute the probability that a particular value A_o is observed for the Hadamard variance, given all the possible values that the random variables $u_1, v_1, ...u_{N/2}$ can have, we form the integral

$$P(A_o) = \int \delta\left(A_o - \frac{1}{N-3s} \sum_j \left(\Delta_{j,s}^{(3)}\right)^2\right) \times \prod_{m>0} \left(e^{-\frac{u_m^2+v_m^2}{2\sigma^2}} \frac{du_m dv_m}{2\pi\sigma^2}\right) . \tag{36}$$

The Dirac delta function constrains the averaged third difference to the specific value A_o while the normally distributed random variables $u_1, v_1, ...u_m, v_m, ...u_{N/2}$ range over their values. There is no integral for $v_{N/2}$. Inspecting this probability and Eq. (34) for the third difference indicates that we could dispense with the factors of σ^{-1} and work with normally distributed random variables having variance unity. Henceforth we set $\sigma = 1$.

The exponent involving the random variables is a quadratic form that can be written in matrix form by introducing the $N-1$ dimension column vector U (the zero frequency component is excluded)

$$U^T = [u_1,\, v_1,\, ...u_m,\, v_m, ...v_{N/2-1}, u_{N/2}] . \tag{37}$$

Then

$$\frac{1}{2}\sum_{m>0}(u_m^2 + v_m^2) = \frac{1}{2}U^T U = \frac{1}{2}U^T \mathbf{1} U, \tag{38}$$

where $\mathbf{1}$ represents the unit matrix. The delta-function in Eq. (36) can be written in exponential form by introducing one of its well-known representations, an integral over all angular frequencies ω[13]:

$$P(A_o) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega\left(A_o - \frac{1}{N-3s}\sum_j \left(\Delta_{j,s}^{(3)}\right)^2\right)} \times \prod_{m>0}\left(e^{-\frac{u_m^2+v_m^2}{2\sigma^2}} \frac{du_m dv_m}{2\pi\sigma^2}\right) . \tag{39}$$

The contour of integration goes along the real axis in the complex ω plane.

The squared third difference is a complicated quadratic form in the random variables $u_1, v_1, ...u_m, v_m, ...u_{N/2}$. If this quadratic form could be diagonalized without materially changing the other quadratic terms in the exponent, then the integrals could be performed in spite of the imaginary factor i in the exponent. To accomplish this we introduce a column vector C^j that depends on j, m, s, N and whose transpose is

$$(C^j)^T = [F_1^j, G_1^j, ...F_m^j, G_m^j, ...G_{N/2-1}^j, F_{N/2}^j] . \tag{40}$$

The column vector has $N-1$ real components. It contains all the dependence of the third difference on frequency, averaging time τ, and on the particular power law noise. We use indices $\{m, n\}$ as matrix (frequency) indices. The (scalar) third difference operator in (34) can be written very compactly as a matrix product

$$\Delta_{j,s}^{(3)} = \sqrt{K}(C^j)^T U = \sqrt{K} U^T C^j . \tag{41}$$

Then the quantity to be averaged is

$$\frac{1}{N-3s}\sum_j \left(\Delta_{j,s}^{(3)}\right)^2 = U^T\left(\frac{K}{N-3s}\sum_j C^j (C^j)^T\right) U. \tag{42}$$

The matrix

$$H_o = \frac{K}{N-3s}\sum_j C^j (C^j)^T \tag{43}$$

is real and symmetric. H_o is also Hermitian and therefore has real eigenvalues. A real symmetric matrix can be diagonalized by an orthogonal transformation[14][15], which we denote by O. Although we shall not need to determine this orthogonal transformation explicitly, it could be found by first finding the eigenvalues ϵ and eigenvectors ψ_ϵ of H_o, by solving the equation

$$H_o \psi_\epsilon = \epsilon \psi_\epsilon . \tag{44}$$

The transformation O is a matrix of dimension $(N-1)\times(N-1)$ consisting of the components of the normalized eigenvectors placed in columns. Then

$$H_oO = OE\,, \tag{45}$$

where E is a diagonal matrix with entries equal to the eigenvalues of the matrix H_o. Then since the transpose of an orthogonal matrix is the inverse of the matrix,

$$O^TH_oO = E\,. \tag{46}$$

The matrix H_o is thus diagonalized, at the cost of introducing a linear transformation of the random variables:

$$\frac{1}{N-2s}\sum_j\left(\Delta^{(2)}_{j,s}\right)^2 = U^TH_oU = U^TOO^TH_oOO^TU = (U^TO)E(O^TU)\,. \tag{47}$$

We introduce $N-1$ new random variables by means of the transformation:

$$V = O^TU\,. \tag{48}$$

Then the term in the exponent representing the Gaussian distributions is

$$-\frac{1}{2}U^T\mathbf{1}U = -\frac{1}{2}U^TO1O^TU = -\frac{1}{2}V^T\mathbf{1}V = -\frac{1}{2}\sum_{n=1}^{N-1}V_n^2\,. \tag{49}$$

The Gaussian distributions remain basically unchanged.

Further, the determinant of an orthogonal matrix is ±1, because the transpose of the matrix is also the inverse, and the total volume element for the space of random numbers is unchanged:

$$dV_1dV_2...dV_{N-1} = dU_1dU_2...dU_{N-1}. \tag{50}$$

After completing the diagonalization,

$$\frac{1}{N-3s}\sum_j\left(\Delta^{(3)}_{j,s}\right)^2 = \sum_i\epsilon_iV_i^2\,. \tag{51}$$

The probability is therefore

$$P(A_o) = \int\frac{d\omega}{2\pi}e^{i\omega\left(A_o-\sum_k\epsilon_kV_k^2\right)}\prod_i\left(e^{-\frac{V_i^2}{2}}\frac{dV_i}{\sqrt{2\pi}}\right). \tag{52}$$

An eigenvalue of zero will not contribute to this probability since the random variable corresponding to a zero eigenvalue just integrates out.

Let the eigenvalue ϵ_i have multiplicity μ_i, which means that the eigenvalue ϵ_i is repeated μ_i times. Integration over the random variables then gives a useful form for the probability:

$$P(A_o) = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{e^{i\omega A_o}}{\prod_k(1+2i\epsilon_k\omega)^{\mu_i/2}}\,. \tag{53}$$

Finally the contour integral may be deformed and closed in the upper half complex plane where it encloses the singularities of the integrand. This is discussed in detail in Ref.[1] and will not be repeated here. If $A_o<0$ the contour may be closed in the lower half plane where there are no singularities, so in this case $P(A_o<0)=0$.

Properties of the eigenvalues. First, it is easily checked that the probability is correctly normalized by using properties of the delta-function.

$$\int P(A_o)dA_o = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{\int e^{i\omega A_o}dA_o}{\prod_i(1+2i\epsilon_i\omega)^{\mu_i/2}} = \int_{-\infty}^{+\infty}\frac{\delta(\omega)d\omega}{\prod_i(1+2i\epsilon_i\omega)^{\mu_i/2}} = \int_{-\infty}^{+\infty}d\omega\delta(\omega) = 1\,. \tag{54}$$

Next let us calculate the trace of H_o. Since the trace is not changed by an orthogonal transformation,

$$\mathrm{Trace}(O^TH_oO) = \mathrm{Trace}(H_oOO^T) = \mathrm{Trace}(H_oOO^{-1}) = \mathrm{Trace}(H_o) = \sum_i\epsilon_i\,. \tag{55}$$

The sum of the diagonal elements of H_o equals the sum of the eigenvalues of H_o. If we then explicitly evaluate the sum of the diagonal elements of H_o we find from Eqs. (55) and (33)

$$\sum_i\epsilon_i = \frac{K}{N-3s}\sum_j\mathrm{Trace}\big(C^j(C^j)^T\big) = \frac{K}{N-3s}\sum_j\sum_{m>0}\left(\left(F_m^j\right)^2+\left(G_m^j\right)^2\right) = K\sum_{m>0}\frac{\left(\sin\frac{\pi ms}{N}\right)^6}{|f_m|^{2-\alpha}} = \sigma_y^2(\tau)\,. \tag{56}$$

Every term labeled by j contributes the same amount. We obtain the useful result that *the overlapping Hadamard variance is equal to the sum of the eigenvalues of the matrix H_o.* Similar results have been established for the various types of the Allan variance.[1]

Distribution of eigenvalues. The equation for eigenvalues $H_o\psi_\mu = \epsilon\psi_\mu$ produces many zero eigenvalues, especially when τ is large. The dimension of the matrix H_o is therefore much larger than necessary. We have performed extensive numerical calculations for many different values of N, which indicate that for the completely overlapping Hadamard variance, the eigenvalue equation has a total of $N-1$ eigenvalues, but only $N-3s$ non-zero eigenvalues; the number of significant eigenvalues is in fact equal to the number of terms $N-3s$ in the sum over j in the equations:

$$\frac{K}{N-3s}\sum_n\sum_j^{N-3s}(C_m)^j(C_n)^j\psi_n = \epsilon\psi_m\,. \tag{57}$$

The factorized form of H_o, that arises on squaring the difference operator in Eq. (43), permits the reduction of the size of the matrix that is to be diagonalized. We introduce the quantities

$$\phi^j_\mu = \sum_n(C_n)^j\psi_{n\mu}\,. \tag{58}$$

We use the Greek index μ to label a non-zero eigenvalue and the index ν to label a zero eigenvalue. The eigenvalue equation becomes

$$\frac{K}{N-3s}\sum_j (C_m)^j \phi_\mu^j = \epsilon \psi_{m\mu}\,. \tag{59}$$

Multiply by $(C_m)^l$ and sum over the frequency index m. Then

$$\frac{K}{N-3s}\sum_{m,j} (C_m)^l (C_m)^j \phi_\mu^j = \epsilon \phi_\mu^l\,. \tag{60}$$

This is an eigenvalue equation with reduced dimension $N-3s$ rather than $N-1$, since the number of possible values of j is $N-3s$. The eigenvalue equation can therefore be written in terms of a reduced matrix H_{red}, given by

$$(H_{red})^{lj} = \frac{K}{N-3s}\sum_m (C_m)^l (C_m)^j\,. \tag{61}$$

The indices l, j run from 1 to $N-3s$. Eigenvalues generated by Eq. (60) are all non-zero. To prove this, multiply Eq. (60) by ϕ_μ^l and sum over l. We obtain

$$\frac{K}{N-3s}\sum_m \left(\sum_l (C_m)^l \phi_\mu^l\right)^2 = \epsilon \sum_l \left(\phi_\mu^l\right)^2. \tag{62}$$

The eigenvalue cannot be zero unless

$$\sum_l (C_m)^l \phi_\mu^l = 0 \tag{63}$$

for every m. The number of such conditions however is larger than the number $N-3s$ of variables, so the only way this can be satisfied is if $\phi_\mu^l = 0$, a trivial solution. Therefore to obtain normalizable eigenvectors from Eq. (60), the corresponding eigenvalues must all be positive. This is true even though some of these conditions may be trivially satisfied if the factor $\sin(\pi m s/N)$ vanishes, which happens sometimes when

$$ms = MN \tag{64}$$

where M is an integer. Every time a solution of Eq. (64) occurs, two of Eqs. (63) relating components of ϕ_μ^l are lost. Suppose there were n solutions to Eq. (64); then the number of conditions lost would be $2n$. The number of variables is $N-3s$ and the number of conditions left in Eq. (63) would be $N-1-2n$. The excess of conditions over variables is thus

$$N-1-2n-(N-3s) = 3s-2n-1\,. \tag{65}$$

It can be shown that under all circumstances $3s-2n-1>0$. We temporarily drop the subscript o since the remainder of the results in this section are valid for any of the variances. If the eigenvalues are found and the appropriate matrix is diagonalized, we may compute the probability for observing a value of the variance, denoted by the random variable A, by

$$P(A) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega\left(A - V^T E V\right)} \prod_i \left(\frac{e^{-V_i^2/2} dV_i}{\sqrt{2\pi}}\right) = \int \frac{d\omega}{2\pi} \frac{e^{i\omega A}}{\prod (1+2i\epsilon_i \omega)^{\mu_i/2}}\,. \tag{66}$$

Case of a single eigenvalue. If a single eigenvalue occurs once only, the general probability expression, Eq. (66), has a single factor in the denominator, and evaluation of the integral gives:

$$P(A) = \frac{1}{\sqrt{2\pi\sigma_y^2(\tau)}} \frac{e^{-A/(2\sigma_y^2(\tau))}}{\sqrt{A}} \tag{67}$$

This is a chi-squared distribution with exactly one degree of freedom.

Case of two distinct non-zero eigenvalues. For the overlapping variance, when s has its maximum value $N/2-1$ there are two unequal eigenvalues. The probability integral can be performed by closing the contour in the upper half plane and gives the expression

$$P(A) = \frac{1}{2\sqrt{\epsilon_1\epsilon_2}} e^{-\frac{A}{4}\left(\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}\right)} I_0\left(\frac{A}{4}\left(\frac{1}{\epsilon_2}-\frac{1}{\epsilon_1}\right)\right). \tag{68}$$

where I_0 is the modified Bessel function of order zero.[16][17] The probability is correctly normalized. It differs from a chi-squared distribution in that the density does not have a singularity at $A=0$.

Case of a single root ϵ repeated three times. The probability integral can be evaluated by integrating by parts, with the result

$$P(A) = \frac{1}{\sqrt{2\pi\epsilon^3}} A e^{-A/2\epsilon}\,. \tag{69}$$

Evaluation of the contour integral when there are more than two distinct eigenvalues is discussed in [1]. If an eigenvalue occurs an even number $2n$ times, the corresponding singularity becomes a pole of order n and a chi-squared probability distribution may result; this has only been observed to occur for white PM.

In Figure 2 we plot a histogram of the values of the Hadamard variancefor flicker PM for the case $N=1024$, $\tau_0 = 1$, $\tau = 340$, for the overlapping case. A histogram of the results of 5000 independent runs of the noise simulation process is also shown. For $N = 1024$, the maximum value of τ is $341\tau_0$. In this case there are four distinct eigenvalues. Evaluation of the contour integral was discussed in detail in Ref. [1]. The probability density is given by

$$P(A) = \frac{1}{\pi}\int_{r_1}^{r_2} \frac{\sqrt{r_1 r_2 r_3 r_4}\, e^{-yA} dy}{\sqrt{(y-r_1)(r_2-y)(r_3-y)(r_4-y)}} - \frac{1}{\pi}\int_{r_3}^{r_4} \frac{\sqrt{r_1 r_2 r_3 r_4}\, e^{-yA} dy}{\sqrt{(y-r_1)(y-r_2)(y-r_3)(r_4-y)}}\,. \tag{70}$$

For this case, the average is $\tau = 5.29\times10^{-6}$; the confidence limits within which there is a 50% probability of finding the variance are 1.48×10^{-6} and 6.46×10^{-6}.

VIII. Modified Non-Overlapping Hadamard Variance

The usual form of the Hadamard variance does not distinguish between white PM and flicker PM. These noise types are distinguished by the Modified variance defined by first averaging third differences over s consecutive values and then performing the remaining averages over the ensemble of

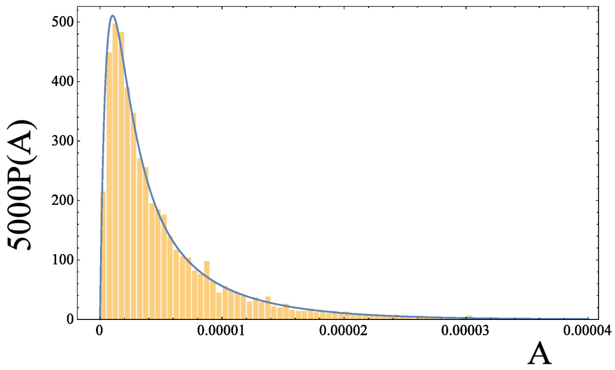

Figure 2. Probability Density for $N = 1024$, $\tau = 340\tau_0$, for flicker PM. h_1 has been set equal to unity and $\tau_0 = 1$, normalized to correspond to 5000 simulation runs. A histogram of the values obtained in the independent simulation runs is shown with 80 bins in the range 0 to 0.00004

random numbers. The modified Hadamard variance is defined as

$$\text{Mod }\sigma_H^2(\tau) = \left\langle \left(\frac{1}{s} \sum_{l=j}^{j+s-1} \Delta_{l,s}^{(3)} \right)^2 \right\rangle . \quad (71)$$

The sum over l in Eq. (71) utilizes a block of data corresponding to indices from $l = j$ to $l = j+4s-1$, with each data item included exactly once. The next block of data to be averaged would include data labeled from $j + 4s$ to $j + 8s - 1$, and in general from $j + 4Ms$ to $j + 4(M + 1)s - 1$, where M is an integer. Starting from $j = 1$, there will be some maximum value of M such that

$$4s(M_{max} + 1) \leq N. \quad (72)$$

If the equality is satisfied, the data stream consists of complete blocks; if not, there will be a partial block for which the sum over s cannot be completed; we then discard the data from such an incomplete block and work only with complete blocks, for which each data item is included exactly once.

Consider a single block, starting from j as in Eq. (71) above. Then it is straightforward to show that

$$\frac{1}{s} \sum_{l=j}^{l+s-1} \Delta_{j,s}^{(3)} = \frac{i}{s} \sqrt{K_1} \sum_m \frac{w_m}{|f_m|^\lambda} \frac{\left(\sin \frac{\pi m s}{N}) \right)^4}{\sin \frac{\pi m}{N}} \times e^{\frac{-i\pi m(2j+4s-1)}{N}} . \quad (73)$$

where

$$K_1 = \frac{2h_\alpha}{3\pi^2\tau^2(N\tau_0)} . \quad (74)$$

and we have set $\sigma = 1$. Averaging the square of this quantity over the random variables, using Eq. 4, then writing the sum over positive frequencies, we obtain

$$\left\langle \left(\frac{1}{s} \sum_{l=j}^{j+s-1} \Delta_{j,s}^{(3)} \right)^2 \right\rangle = \frac{4K}{3s^2} \sum_{m>0} \frac{1}{|f_m|^{2\lambda}} \frac{\left(\sin \frac{\pi m s}{N} \right)^8}{\left(\sin \frac{\pi m}{N} \right)^2} , \quad (75)$$

which is the same as for the overlapping case. If there are M blocks of non-overlapping data in the average, then the result will still be given by Eq. (75) since the average entails division by M. If the spacing of frequencies is dense enough to pass to an integral, we obtain

$$\text{Mod }\sigma_H^2(\tau) = \frac{8}{3\pi^2\tau^2} \int_0^{f_h} \frac{S_y(f)}{f^2} \frac{\left(\sin(\pi f \tau) \right)^8}{(s \sin(\pi f \tau_0))^2} . \quad (76)$$

To derive expressions for the probability of observing a particular value of the Hadamard variance, we replace j by $1 + 4Ms$ and write the sum in Eq. (71) over positive frequencies before averaging:

$$\frac{1}{s} \sum_{l=1+4Ms}^{4Ms+s} \Delta_{l,s}^{(3)} = \frac{\sqrt{4K_1}}{s} \sum_{m>0} (F_m u_m + G_m v_m) . \quad (77)$$

where

$$F_m^M = \frac{\left(\sin \frac{\pi m s}{N} \right)^4}{|f_m|^\lambda \sin \frac{\pi m}{N}} \sin q; \quad (78)$$

$$G_m^M = -\frac{\left(\sin \frac{\pi m s}{N} \right)^4}{|f_m|^\lambda \sin \frac{\pi m}{N}} \cos q, \quad (79)$$

and where $q = \pi m(1 + 8Ms + 4s)/N$.

With these new definitions for F_m^M and G_m^M, we define a new vector C^M such that

$$(C^M)^T = \{F_1^M, G_1^M, F_2^M, G_2^M, ..., F_{N/2}\} . \quad (80)$$

Using Eq. (37) for the random numbers leads to

$$\frac{1}{s} \sum_{l=1+4Ms}^{4Ms+s} \Delta_{l,s}^{(3)} = \sqrt{\frac{4K_1}{s^2}} C^T U = \sqrt{\frac{4K_1}{s^2}} U C^T . \quad (81)$$

Then the quantity to be averaged is

$$\frac{1}{M_{max}+1} \sum_M \left(\frac{1}{s} \sum_{l=1+4Ms}^{4Ms+s} \Delta_{j,s}^{(3)} \right)^2 = \frac{4K_1}{s^2(M_{max}+1)} \sum_M U^T C^M (C^M)^T U. \quad (82)$$

Defining the matrix

$$(H_H)_{mn} = \left(\frac{4K_1}{s^2(M_{max}+1)} \sum_M C^M (C^M)^T \right)_{mn}, \quad (83)$$

the probability of observing a value A of the variance will be

$$P(A) = \int \frac{d\omega}{2\pi} e^{i\omega(A - U^T H_H U/s^2)} \times \prod_{m>0} \left(e^{-\frac{u_m^2+v_m^2}{2\sigma^2}} \frac{du_m dv_m}{2\pi\sigma^2} \right) . \quad (84)$$

Diagonalization of the matrix H_H leads in the usual way to the expression for the probability in terms of the eigenvalues ϵ_i:

$$P(A) = \int_0^{f_h} \frac{d\omega}{2\pi} \frac{e^{i\omega A}}{\prod_i \sqrt{1 + 2i\omega\epsilon_i}} . \quad (85)$$

The eigenvalue equation will be

$$H_H\psi^{(\epsilon)} = \epsilon\psi^{(\epsilon)}\,, \tag{86}$$

or

$$\frac{4K_1}{s^2(M_{max}+1)}\sum_{j,M} C_i^M (C_j^M)^T \psi_j^{(\epsilon)} = \epsilon\psi_i^{(\epsilon)}\,. \tag{87}$$

The number of eigenvalues can be investigated by reducing the order of the matrix. Let

$$\phi^{M(\epsilon)} = \sum_n C_j^M \psi_j^{(\epsilon)}\,. \tag{88}$$

Then Eq. (86) becomes

$$\frac{4K_1}{s^2(M_{max}+1)}\sum_{M} C_i^M \phi^{M(\epsilon)} = \epsilon\psi_i^{(\epsilon)}\,. \tag{89}$$

Multiply by C_i^L and sum over the frequency index. This gives

$$\sum_M (H_{red})^{LM}\phi^{M(\epsilon)} = \epsilon\phi^{L(\epsilon)} \tag{90}$$

where the reduced matrix has dimension $M_{max}+1$ and is given by

$$(H_{red})^{LM} = \frac{4K_1}{s^2(M_{max}+1)}\sum_i C_i^L C_i^M\,. \tag{91}$$

Multiply Eq. (90) by $\phi^{L\epsilon}$ and sum over L. The result is

$$\frac{4K_1}{s^2(M_{max}+1)} = \epsilon\sum_L (\phi^{L\epsilon})^2 \tag{92}$$

IX. A radar variance

The analysis methods developed in this paper can be extended to other variances. For example, these methods can be applied to cases in which there is dead time between measurements of average frequency during the sampling intervals. Suppose for example that the measurements consist of intervals of length $\tau = s\tau_0$ during which an average frequency is measured, separated by dead time intervals of length $D-\tau$ during which no measurements are available, with the possibility of significant drift during the dead times. Let the index j label the measurement intervals with $j = 1, 2, ...N$, and let $D = d\tau_0$ with d an integer. A variance can be defined in terms of the difference between the average frequency in the j^{th} interval and that in the interval labeled by $j + r$:

$$\Delta^{(2)}_{j,r,s} = \frac{1}{\sqrt{2}}\big(\overline{y}_{j+r,s} - \overline{y}_{j,s}\big)\,, \tag{93}$$

where $\overline{y}_{j,s}$ is the average frequency in the interval j of length $s\tau_0$. The average fractional frequency during the measurement interval τ is

$$\overline{y}_{r,d,s} = \frac{1}{\tau}(X_{rd+s} - X_{rd})\,. \tag{94}$$

To eliminate drift during the dead time, a second difference of frequencies can be used:

$$\Delta^{(3)}_{j,r,d,s} = \frac{1}{\sqrt{6}}(\overline{y}_{j+2r,d,s} - 2\overline{y}_{j+r,d,s} + \overline{y}_{j,d,s})\,. \tag{95}$$

this is a third difference in the times. Using trigonometric identities it can be reduced to

$$\Delta^{(3)}_{j,r,d,s} = 8i\sqrt{K_1}\sum_m \frac{w_m}{|f_m|^\lambda}\sin\frac{\pi ms}{N} \times\left(\sin\frac{\pi mrd}{N}\right)^2 e^{-\pi i m(s+2jd+2rd)/N}\,. \tag{96}$$

Then an appropriate variance can be defined as

$$\Psi(\tau,D)^2 = \left\langle \big(\Delta^{(2)}_{j,r,d,s}\big)^2 \right\rangle. \tag{97}$$

Performing the average and writing the result in terms of a sum over positive frequencies,

$$\Psi(\tau,D)^2 = \frac{4h_\alpha}{3\pi^2\tau^2(N\tau_0)} \times\sum_{m>0}\frac{1}{|f_m|^{2\lambda}}\big(\sin\frac{\pi ms}{N}\big)^2\big(\sin\frac{\pi mrd}{N}\big)^4\,. \tag{98}$$

If the measurements are sufficiently densely spaced that it is possible to pass to an integral, this can be shown to reduce to

$$\Psi(\tau,D)^2 = \frac{8}{3}\int_0^{f_h} df\,\frac{S_y(f)}{(\pi f\tau)^2}\big(\sin(\pi f r D)\big)^4\big(\sin(\pi f\tau)\big)^2\,. \tag{99}$$

When $D = \tau$ and $r = 1$ there is no real dead time and this variance reduces to the ordinary Hadamard variance.

X. Summary and Conclusion

In this paper a method of simulating time series for the common power-law noises has been developed and applied to several variances used to characterize clock stability. These include overlapping and non-overlapping forms of the Hadamard variance, and the Modified Hadamard variance. It was shown that diagonalization of quadratic forms for the average variances leads to expressions for the probabilities of observing particular values of the variance for a given sampling time $\tau = s\tau_0$. The probabilities are expressed in terms of integrals depending on the eigenvalues of matrices formed from squares of the second differences that are used to define the variance. The eigenvalues are usually distinct; only for white PM have eigenvalues been observed (after much calculation) to occur with multiplicities other than unity; we do not have a proof of this. Generally speaking, the number of eigenvalues is equal to the number of terms occurring in the sum used to define averages of the second-difference operator. The probability distribution $P(A)$ for some variance A is useful in estimating the $\pm 25\%$ confidence interval about the average variance. The number of eigenvalues gets smaller as the sampling time τ gets larger.

It is well-known that a chi-squared distribution with n degrees of freedom occurs for a variable that is the sum of squares of n normally distributed random variables. It has been shown that, with this method, probabilities for the Hadamard variance, using the present simulation method, are not always–in fact are rarely–chi-squared distributions. This is because the frequency dependence, inserted to make the time series obey the chosen noise power law, disrupts the distribution of eigenvalues in most cases. Other methods of simulating power law noise have been published; the present approach differs from that of [6]. The present approach respects all the standard expressions for spectral density and the relationships between Hadamard variance and spectral density for the common power-law noises. It also yields reasonable simulation results for power-law noises that diverge more rapidly than flicker noise at low frequencies.

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