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Topological phases with long-range interactions

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Topological phases of matter are primarily studied in systems with short-range interactions. In nature, however, nonrelativistic quantum systems often exhibit long-range interactions. Under what conditions topological phases survive such interactions, and how they are modified when they do, is largely unknown. By studying the symmetry-protected topological phase of an antiferromagnetic spin-1 chain with $1/r^\alpha$ interactions, we show that two very different outcomes are possible, depending on whether or not the interactions are frustrated. While unfrustrated long-range interactions can destroy the topological phase for $\alpha \lesssim 3$, the topological phase survives frustrated interactions for all $\alpha > 0$. Our conclusions are based on strikingly consistent results from large-scale matrix-product-state simulations and effective-field-theory calculations, and we expect them to hold for more general interacting spin systems. The models we study can be naturally realized in trapped-ion quantum simulators, opening the prospect for experimental investigation of the issues confronted here.

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Since the discovery of topological insulators [1–3], there has been tremendous interest in exploring various topological phases of matter, both theoretically [4,5] and experimentally [6–8]. Topological phases are generally associated with—and derive much of their presumed utility from—stability against *local* perturbations. But precisely what constitutes "local" in this context is a subtle issue; power-law decaying ($1/r^\alpha$) interactions, which are present in many experimental systems, do not necessarily qualify [9–11]. Recent theoretical advances have begun to elucidate the conditions under which long-range interacting systems maintain some degree of locality [12,13], potentially providing some insight into effects of long-range interactions on topological phases of matter. And recently, explicit theoretical evidence of topological order has been found in a variety of long-range interacting systems, including dipolar spins [14] or bosons [15], fermions with long-range pairing [16] and hopping [17,18], and electrons with Coulomb interactions [19]. These results notwithstanding, a complete understanding of how topological phases respond to the addition of long-range interactions is still lacking.

The stability of topological phases to small local perturbations is intimately connected to the existence of a bulk excitation gap [20,21], and the introduction of long-range interactions to a short-range Hamiltonian supporting a topological phase poses several potential challenges to this connection. First, even if the gap remains finite, long-range interactions can change the ground-state correlation decay from exponential to power law [16,18,22,23]. Thus topological phases with local interactions are, at the very least, subject to qualitative changes in their long-distance correlations. Second, the gap can in principle close in the presence of long-range interactions, even when they decay fast enough that the total interaction energy remains extensive [20,24]. Third, long-range interactions have the ability to change the effective dimensionality of the system [25,26], and thus might change the topological properties even if the gap does not close [16,18]. We emphasize that the understanding of these issues is not of strictly theoretical interest. Many of the promising experimental systems for exploring or exploiting topological phases of matter, e.g., dipolar molecules [27–29], magnetic [30] or Rydberg atoms [31], trapped ions [32–37], and atoms coupled to multimode cavities [38], are accurately described as quantum lattice models with power-law decaying interactions. The unique controllability and measurement precision afforded by these systems hold great promise to improve our understanding of topological phases [39–42], but first we must reliably determine when—despite their long-range interactions—they can be expected to harbor the topological phases that have been theoretically explored for short-range interacting systems.

To address these general questions, in this Rapid Communication we study a spin-1 chain with antiferromagnetic Heisenberg interactions, which is a paradigmatic model exhibiting a symmetry-protected topological (SPT) phase [43,44]. Specifically, we consider two extensions of the short-range version of this model by including long-range interactions that decay either as $\mathcal{J}_\alpha(r) = 1/r^\alpha$ or as $\mathcal{J}'_\alpha(r) = (-1)^{r-1}/r^\alpha$, which could be simulated in trapped-ion based experiments for $0 < \alpha < 3$ [45,46]. Based on a combination of large-scale variational matrix-product-state (MPS) simulations and field-theory calculations, we establish and explain a number of important and potentially general consequences of long-range interactions. The $\mathcal{J}'_\alpha(r)$ interactions are unfrustrated, being antiferromagnetic (ferromagnetic) between spins on the opposite (same) sublattice. In this case, numerics and field-theoretic arguments suggest the destruction of the topological phase for $\alpha \lesssim 3$, accompanied by a closing of the bulk excitation gap and spontaneous breaking of a continuous symmetry in one dimension (1D), consistent with other recent findings on the relevance of long-range interactions for $\alpha < D + 2$ in D-dimensional quantum systems [47,48]. The $\mathcal{J}_\alpha(r)$ interactions are frustrated, and, remarkably, do not close the bulk excitation gap for any $\alpha > 0$. In addition, two key properties of the

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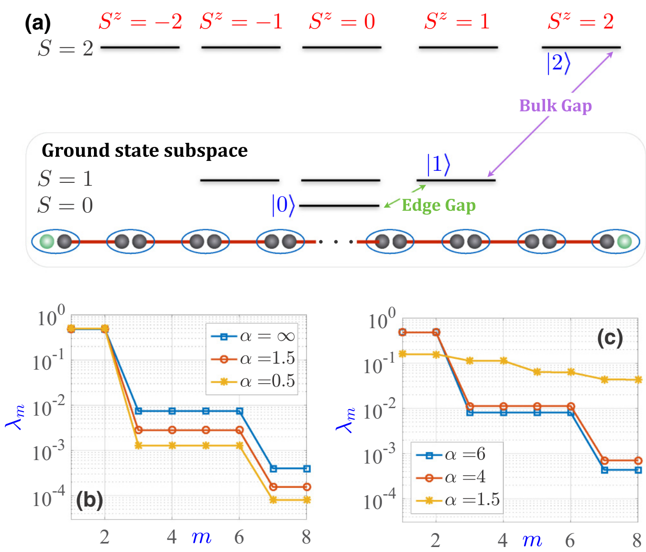

FIG. 1. (a) Low-lying energy levels of the Haldane chain for even L. The entanglement structure of ground states is shown at the bottom. The ground states in the total $S^z = 0,1,2$ subspace are named $|0\rangle,|1\rangle,|2\rangle$ and have energies E_0,E_1,E_2. (b), (c) The mth largest value λ_m ($m = 1,2,\ldots,8$) of the ground-state entanglement spectrum for H_α (b) and H'_α (c) using finite-size MPS calculations with $L = 200$. We choose the $|1\rangle$ state to avoid extra entanglement between edge spins. For H'_α, the entanglement spectrum for $1.5 \leqslant \alpha \leqslant 4$ will exhibit a smooth crossover between the $\alpha = 1.5$ and $\alpha = 4$ cases due to the finite system size, but we expect a sharp transition at some $\alpha_c \lesssim 3$ in the thermodynamic limit. The exact pair degeneracies in $\{\lambda_m\}$ are a result of the spatial-inversion symmetry protecting the topological phase [44,49].

SPT phase, a doubly degenerate entanglement spectrum [49] and a nonvanishing string-ordered correlation [50], are both preserved. However, because of the long-range interactions, spin-spin correlations and the edge-excitation amplitudes only decay exponentially within some intermediate distance scale, after which they decay algebraically. We expect these qualitative changes to be quite general, occurring in other long-range interacting systems in which the topological phase survives.

Model. We consider a spin-1 chain with either frustrated or unfrustrated long-range Heisenberg interactions:

$$H_\alpha = \sum_{j,r>0} \mathcal{J}_\alpha(r)\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+r},\ H'_\alpha = \sum_{j,r>0} \mathcal{J}'_\alpha(r)\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+r}. \quad (1)$$

With only nearest-neighbor interactions ($\alpha \to \infty$), $H_\infty = H'_\infty$ is usually called the *Haldane chain*, which has been extensively studied theoretically [51–53], numerically [54–58], and experimentally [59,60]. The low-lying states of the Haldane chain are shown in Fig. 1(a) for an open boundary chain with even size L. The unique ground state has total spin $S = 0$. The first set of excited states has $S = 1$ ($\hbar = 1$), contains spin excitations only near the edge of the chain, and is separated from the ground state by an energy gap (*edge gap*) that is exponentially small in L and topologically protected. Consequently, these excited states belong to a degenerate ground-state subspace in the thermodynamic ($L \to \infty$) limit. The second set of excited states all have $S = 2$, contain spin excitations in the bulk of the chain, and have an energy gap (*bulk gap*) that converges to a finite value when $L \to \infty$. The entanglement structure of the four ground states is close to that of the Affleck-Kennedy-Lieb-Tasaki (AKLT) states [61] shown at the bottom of Fig. 1(a), where each spin-1 is decomposed into two spin-1/2's, pairs of spin-1/2's on neighboring sites form singlets, and the system is finally projected back onto the spin-1's. The four quasidegenerate ground states correspond to the four states formed by the two unpaired spin-1/2's at the edge.

We use variational MPS calculations [62–65] to determine the ground-state entanglement structure of H_α and H'_α in Figs. 1(b) and 1(c). For $\alpha > 0$ ($\alpha > 3$), the ground-state entanglement spectrum of H_α (H'_α), defined as the eigenvalues of the left/right half chain's reduced density matrix, is dominated by the two largest degenerate eigenvalues $\lambda_1 = \lambda_2 \approx 0.5$. This can be understood heuristically as the result of cutting a spin-1/2 singlet in the AKLT state, and suggests the survival of the topological Haldane phase. For H'_α with $\alpha \lesssim 3$, the entanglement spectrum has an entirely different structure, and we will study the related ground-state properties below.

Effective field theory. The low-energy physics of the Haldane chain can be understood via field-theoretic analysis due to Haldane [52] and Affleck [66]; here, we build on their work to provide a field-theoretic treatment of the long-range interacting model. We begin by decomposing the spin operators into staggered and uniform fields, $\boldsymbol{n}(2i + \frac{1}{2}) = (\boldsymbol{S}_{2i} - \boldsymbol{S}_{2i+1})/2$ and $\boldsymbol{l}(2i + \frac{1}{2}) = (\boldsymbol{S}_{2i} + \boldsymbol{S}_{2i+1})/2$. The intuition behind this decomposition is that the classical ground state of both H_α and H'_α is Néel ordered for any $\alpha > 0$, with $\boldsymbol{n}^2(x) = 1$ and $\boldsymbol{l}(x) = 0$. We therefore expect that in the quantum ground state $\boldsymbol{n}^2(x) \approx 1$, while $\boldsymbol{l}(x) \approx 0$ represents small quantum fluctuations in the direction of $\boldsymbol{n}(x)$. Importantly, we expect that only long-wavelength fluctuations of $\boldsymbol{n}(x)$ and $\boldsymbol{l}(x)$ will be important at low energy. In momentum space, we can write $H_\alpha \approx \int dq[\omega(q)|\boldsymbol{n}(q)|^2 + \Omega(q)|\boldsymbol{l}(q)|^2]$ and $H'_\alpha \approx \int dq[\Omega(q)|\boldsymbol{n}(q)|^2 + \omega(q)|\boldsymbol{l}(q)|^2]$ [67], with

$$\omega(q) = 2\sum_{r=1}^{\infty} \mathcal{J}'_\alpha(r)\cos qr, \quad \Omega(q) = 2\sum_{r=1}^{\infty} \mathcal{J}_\alpha(r)\cos qr. \quad (2)$$

For any $\alpha > 0$, $\omega(q)$ is analytic at small q and can be expanded as $\omega_0 + \omega_2 q^2 + O(q^4)$, whereas $\Omega(q)$ is nonanalytic at small q with an expansion $\Omega_0 + \Omega_2 q^2 + \lambda|q|^{\alpha-1} + O(q^4)$. The coefficients $\omega_{0,2}$, $\Omega_{0,2}$, and λ depend on α, but their exact values are not important for the following analysis. Physically, the analyticity (nonanalyticity) of the spectrum arises because the long-range interactions interfere destructively (constructively) for the staggered field. Keeping only the lowest nontrivial order in q for the dispersion of both $\boldsymbol{n}(q)$ and $\boldsymbol{l}(q)$ turns out to be sufficient for obtaining qualitatively correct behavior of the excitation gap. Therefore, we keep only the 0th-order term in the dispersion of $\boldsymbol{l}(q)$, and the next-leading term in the dispersion of $\boldsymbol{n}(q)$ [for $\boldsymbol{n}(q)$, the 0th-order term only adds a constant to the Hamiltonian due to the constraint $\boldsymbol{n}^2(x) = 1$]. Thus for $\alpha > 0$ ($\alpha > 3$) the Hamiltonian H_α (H'_α) is approximately given by (ignoring the order-unity coefficients) $H_\alpha \sim H'_\alpha \sim \int dq[q^2|\boldsymbol{n}(q)|^2 + |\boldsymbol{l}(q)|^2]$. When the zero-temperature partition function is expressed as a coherent-spin-state path integral, the action is quadratic in the field $\boldsymbol{l}$ and it can be integrated out [68,69]. The remaining path integral over the staggered field $\boldsymbol{n}$ is a (1+1)D $O(3)$ nonlinear sigma model, with Lagrangian density [nonlinear constraint $\boldsymbol{n}^2(x) = 1$ implied]

$$\mathcal{L}(x) \approx \frac{1}{g}(|\partial \boldsymbol{n}/\partial t|^2 - v_s^2|\partial \boldsymbol{n}/\partial x|^2). \tag{3}$$

Here, g is an effective (α- and short-distance-cutoff-dependent) coupling strength, and the spin-wave velocity v_s is also α dependent. This model is gapped and disordered [51].

To investigate the ground-state properties of Eq. (3), we can remove the constraint $\boldsymbol{n}^2(x) = 1$, while phenomenologically introducing a mass gap Δ_α and a renormalized spin-wave velocity v_α (the parameters Δ'_α and v'_α will be used to describe the Lagrangian for H'_α) [57,58]. Transforming to momentum space, we thereby arrive at a free-field Lagrangian density

$$\mathcal{L}(q) \propto |\partial \boldsymbol{n}/\partial t|^2 - \left(\Delta_\alpha^2 + v_\alpha^2 q^2\right)|\boldsymbol{n}(q)|^2. \tag{4}$$

This Lagrangian leads to ground-state correlations $\mathcal{C}_{ij} = \langle S_i^z S_j^z\rangle_0$ [where $\langle\cdots\rangle_m$ denotes the expectation value in the state $|m\rangle$ defined in Fig. 1(a)] that decays as

$$\mathcal{C}_{ij} \propto (-1)^r \int \frac{e^{iqr}\,dq}{\sqrt{\Delta_\alpha^2 + v_\alpha^2 q^2}} \propto (-1)^r K_0(r/\xi_\alpha). \tag{5}$$

Here, $\xi_\alpha \equiv v_\alpha/\Delta_\alpha$ (or $\xi'_\alpha \equiv v'_\alpha/\Delta'_\alpha$ for H'_α) defines the correlation length, and $K_0(x)$ is a modified Bessel function, which behaves as $K_0(x) \sim \exp(-x)/\sqrt{x}$ for large x.

For $\alpha < 3$, the nonanalytic $|q|^{\alpha-1}$ term in H'_α dominates the dispersion of $\boldsymbol{n}(q)$ at small q, and Eqs. (3) and (4) are not valid. To analyze this case, we write down the renormalization group (RG) flow equation for the coupling strength g under the scaling transformation $x \to xe^{-l}$ to one-loop order [68,70],

$$\frac{dg}{dl} = \frac{\alpha - 3}{2}g + \frac{g^2}{4\pi}. \tag{6}$$

For $\alpha < 3$, an unstable fixed point appears at $g^* = 2\pi(3-\alpha)$, and for a bare coupling $g < g^*$ the RG flow is towards a weak-coupling ordered state at $g = 0$ [68]. The bare coupling, and therefore the value of α at which this phase transition occurs, is difficult to determine *a priori*. But we nevertheless expect (and confirm numerically) that for $\alpha < \alpha_c$, with $2 < \alpha_c < 3$, the gap will close as the system spontaneously breaks the continuous SU(2) symmetry of H'_α [48,71].

Comparison with numerics. Using finite-size MPS calculations, we have obtained the bulk excitation gap $E_2 - E_1$ and the correlation length [fitted using Eq. (5)] for both H_α and H'_α. As shown in Figs. 2(a) and 2(b), we see consistent results with the field-theory predictions. For H_α, the gap remains open for all $\alpha > 0$, and the correlation length decreases together with α due to both an increase of the bulk gap, and a decrease of the spin-wave velocity (as a result of a weakened Néel order for longer-range interactions). To the contrary, for H'_α, the gap decreases quickly as the interactions become longer ranged, and the correlation length diverges when α decreases to around 3, suggesting the disappearance of the topological phase at $\alpha \lesssim 3$ [72]. Calculation of the string-ordered correlation $\mathcal{S}_{ij} \equiv \langle S_i^z S_j^z \prod_{i<k<j}(-1)^{S_k^z}\rangle_0$ of both H_α and H'_α at $\alpha = 1.5$ [Fig. 2(c)] provides further evidence that the topological phase survives for H_α, but not for H'_α, for $0 < \alpha \lesssim 3$.

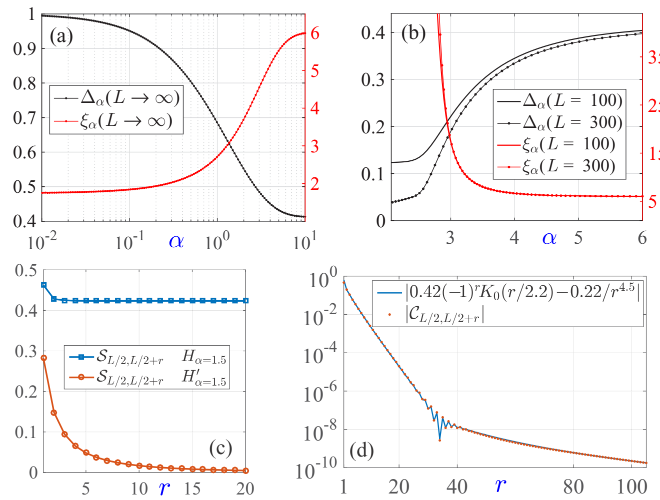

FIG. 2. (a) Bulk gap Δ_α and ground-state correlation length ξ_α in the $L \to \infty$ limit, obtained by finite-size scaling for $200 \leqslant L \leqslant 500$. (b) Bulk gap Δ'_α and ξ'_α with $L = 100$ and $L = 300$. (c) Ground-state string-ordered correlation function $\mathcal{S}_{ij}$ for H_α and H'_α with $\alpha = 1.5$ and $L = 300$. For various α and $200 \leqslant L \leqslant 500$, we consistently find that $\mathcal{S}_{ij}$ quickly saturates to a finite value for H_α at all $\alpha > 0$, but vanishes at large distance for H'_α at $\alpha \lesssim 3$. (d) Ground-state spin-spin correlation $\mathcal{C}_{ij}$ for $\alpha = 0.5$ and $L = 500$. This choice of $\alpha = 0.5$ is arbitrary, but assists in a clear presentation of the coexisting exponential and $1/r^{\alpha+4}$ power-law decays.

We now analyze the effects of terms beyond leading order in q that have been ignored in our field-theory treatment. Including the higher-order analytic terms, such as the $O(q^4)$ term, will result in negligible corrections to the correlation functions that decay in distance faster than Eq. (5) [57]. However, even for $\alpha > 3$, inclusion of the nonanalytic $O(|q|^{\alpha-1})$ term will add a power-law tail to the correlation functions, which will dominate over Eq. (5) at long distance. In the Supplemental Material, we show by a more involved field-theory calculation that, for $H_\alpha, \mathcal{C}_{ij}$ decays as $1/r^{\alpha+4}$ at large r. Our MPS calculations using $L = 500$ spins [Fig. 2(d)] show remarkable agreement with the field-theory predictions, even capturing the oscillations in $|\mathcal{C}_{ij}|$ occurring at intermediate distance where the short-range and long-range contributions to the correlation functions are of comparable magnitude and interfere. A power-law tail in $\mathcal{C}_{ij}$ should also exist for H'_α, but the increased correlation length prevents us from observing its existence clearly for $\alpha > 3$.

Edge-excited states. We expect the influence of long-range interactions on the edge- and bulk-excited states to be strong at small α; because the topological phase of H'_α does not survive for $\alpha \lesssim 3$, we will focus on H_α from now on. Edges can be introduced into the field theory by replacing the two end spin-1's with spin-1/2's, represented by $\boldsymbol{\tau}_L$ ($\boldsymbol{\tau}_R$) for the left (right) edge, resulting in an edge-bulk coupling Hamiltonian $H_c = \sum_{i=2}^{L-1} \boldsymbol{S}_i \cdot [\boldsymbol{\tau}_L/(i-1)^\alpha + \boldsymbol{\tau}_R/(L-i)^\alpha]$ [57]. For the edge-excited state $|1\rangle$ [Fig. 1(a)], $\boldsymbol{\tau}_{L,R}$ are polarized in the $+z$ direction, and we expect $\langle S_i^z\rangle$ to decay away from the ends. Solving the free theory defined by Eq. (4) and treating H_c using standard first-order perturbation theory [57], we find that $\langle n^z(x)\rangle_1 \propto \int dq \quad \{\exp[iq(L-x)] - \exp[iq(x-1)]\}/(\Delta_\alpha^2 + v_\alpha^2 q^2) \propto$

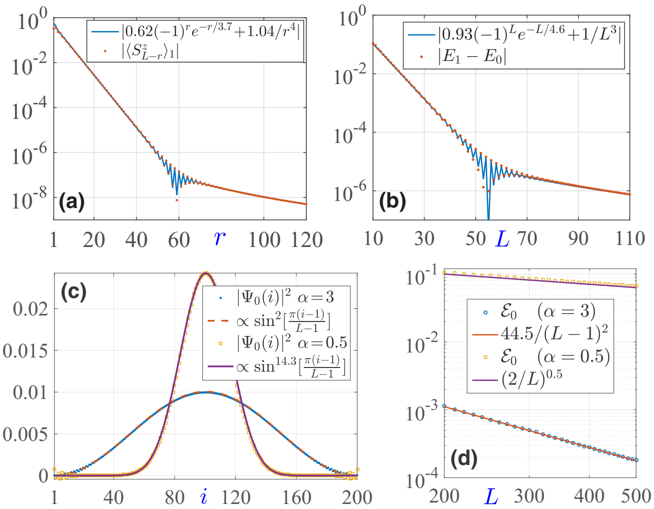

FIG. 3. (a) Distribution of an edge excitation in state $|1\rangle$ for $L=500$ and $\alpha=2$. (b) Edge gap $|E_1-E_0|$ as a function of the chain size L for $\alpha=3$. (c) Lowest-energy magnon probability density distribution for $L=200$ and $\alpha=3.0,0.5$. (d) The finite-size correction to the lowest magnon excitation energy [see Eq. (7)]. For $\alpha=3$, we obtain $v_\alpha=2.18$ and $v_\alpha/\Delta_\alpha\approx 4.51$, in good agreement with the $\xi_\alpha\approx 4.55$ obtained in Fig. 2.

$\exp[-(L-x)/\xi_\alpha]-\exp[-(x-1)/\xi_\alpha]$ for even L. In addition, $\langle l^z(x)\rangle_1$ contributes a power-law correction $1/(x-1)^{\alpha+2}+1/(L-x)^{\alpha+2}$ for x far away from both ends [73]. Our numerical calculation of $\langle S^z(x)\rangle_1$, shown in Fig. 3(a), agrees well with a sum of these two contributions, clearly exhibiting an exponential followed by $1/r^{\alpha+2}$ decay.

The edge gap $|E_1-E_0|$ can be obtained by using a path integral to integrate out the $\boldsymbol{n}$ field [57], resulting in an effective edge-edge Hamiltonian $\propto(-1)^L\exp(-L/\xi_\alpha)\boldsymbol{\tau}_L\cdot\boldsymbol{\tau}_R$. This scaling is confirmed, at relatively small L, by the numerical results in Fig. 3(b). However, the numerics also reveal that at large L the edge gap receives a long-range correction given by $1/L^\alpha$. This remarkably simple result, including the unity prefactor, can be understood as follows. The edge-excited states behave differently from the bulk-excited states due to *correlations* between the orientations of $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$, and therefore $\langle\boldsymbol{S}_i\cdot\boldsymbol{S}_j\rangle_1-\langle\boldsymbol{S}_i\cdot\boldsymbol{S}_j\rangle_0$ is very small unless i and j are very close to 0 and L, respectively. Thus we have $E_1-E_0\approx L^{-\alpha}\sum_{i<j}(\langle\boldsymbol{S}_i\cdot\boldsymbol{S}_j\rangle_1-\langle\boldsymbol{S}_i\cdot\boldsymbol{S}_j\rangle_0)=1/L^\alpha$, where the last equality is a sum rule following from the total spin of the ground ($S=0$) and edge-excited ($S=1$) states.

Bulk-excited states. As in the short-range Haldane chain, the elementary bulk excitations of H_α are spin-1 magnons [55–57]. Physically, the magnon represents fluctuations in the staggered magnetization, and, from Eq. (4), these fluctuations have a dispersion relation $\epsilon_\alpha(q)=\sqrt{\Delta_\alpha^2+(v_\alpha q)^2}\approx\Delta_\alpha+q^2v_\alpha^2/(2\Delta_\alpha)$ (valid at small q). The lowest-energy magnon wave function $\Psi_0(x)$ can be extracted from the numerics using the relation $|\Psi_0(i)|^2\approx|\langle S_i^z\rangle_2-\langle S_i^z\rangle_1|$. The presence of long-range interactions gives the magnon an additional potential energy due to the edge-bulk coupling Hamiltonian H_c, and $\Psi(x)$ can be approximately described by the following Schrödinger equation (with Dirichlet boundary condition at $x=1,L$),

$$\frac{v_\alpha^2}{2\Delta_\alpha}\frac{\partial^2\Psi(x)}{\partial x^2}+\frac{1}{2}\left[\frac{1}{(x-1)^\alpha}+\frac{1}{(L-x)^\alpha}\right]\Psi(x)=\mathcal{E}\Psi(x). \quad (7)$$

The kinetic (potential) energy always scales as $1/L^2$ ($1/L^\alpha$); therefore, for $\alpha>2$ and large L, the potential energy can be ignored. The ground-state energy $\mathcal{E}_0\approx v_\alpha^2\pi^2/(2\Delta_\alpha L^2)$ and probability density $|\Psi_0(x)|^2\approx(2/L)\sin^2(\pi x/L)$ are then identical to those of a particle in a box, as confirmed numerically in Figs. 3(c) and 3(d). The relation $E_2-E_1\approx\Delta_\alpha+v_\alpha^2\pi^2/(2\Delta_\alpha L^2)$ allows us to obtain both v_α and Δ_α through finite-size scaling [Fig. 2(b)], and we confirm that the correlation length determined by $\xi_\alpha=v_\alpha/\Delta_\alpha$ agrees with that obtained by fitting $\mathcal{C}_{ij}$ using Eq. (5). For $\alpha<2$, the potential energy dominates the kinetic energy for large L, and the potential can be approximated as harmonic around $x=L/2$. Thus $|\Psi_0(x)|^2$ resembles a Gaussian [Fig. 3(c)], and a simple scaling analysis predicts a width $\gamma\propto L^{1-\alpha/2}$. In the large-$L$ limit, $|\Psi_0(x)|^2$ becomes sharply peaked at $x=L/2$ and, from Eq. (7), we expect the bulk gap to scale as $\Delta_\alpha+(2/L)^\alpha$, which is clearly observed in Fig. 3(d). Since $E_2-E_1=2$ when $\alpha=0$, it follows that $\Delta_{\alpha\to 0}=1$, consistent with Fig. 2(a).

Outlook. The stability of the topological Haldane phase to $1/r^\alpha$ interactions for all $\alpha>0$ is favorable for trapped-ion based experiments, as stronger couplings can be achieved for smaller α [36,37]. Moreover, because the correlation length *shrinks* for longer-range interactions, a relatively small number of ions will suffice to suppress finite-size effects. Probing the topological phase by measuring both $\mathcal{C}_{ij}$ and $\mathcal{S}_{ij}$ with single-site resolution is nearly impossible in typical condensed-matter systems, but is quite straightforward in ion traps [74]. Based on the generality of our field-theory analysis, we speculate that for generic lattice models, the tails in the power-law interactions can possibly destroy the topological phase only when long-range interactions are unfrustrated and $\alpha<D+2$. Experimentally, unfrustrated long-range interactions can be easily implemented by generating a $1/r^\alpha$ ferromagnetic interaction [71]. We hope that our work can serve as a springboard for future studies on how distinct topological phases behave in the presence of long-range interactions.

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