Impact of Decision and Allocation Rules on Selection Decisions

Dennis D. Leber
National Institute of Standards and Technology
Gaithersburg, MD 20910, U.S.A.

Jeffrey W. Herrmann
University of Maryland
College Park, MD 20742, U.S.A.

Abstract

The quality of a selection decision is a function of the decision rule used and the data collected to support the decision. When physical measurements are the basis of the decision data, the measurement sampling scheme controls measurement uncertainty and influences decision quality. Faced with a fixed experimental budget for measurement collection over multiple attributes, a decision-maker must decide how to allocate this budget to best support the selection decision. We expand on previous work in this area of sample allocation in multiple attribute selection decisions to compare the quality of allocation rules derived under various estimation and decision rules. The decision rules considered in this work include the selection of the best (expected value) and multinomial selection with allocation rules developed based on maximum likelihood and Bayesian inference. We derive allocation rules for each of these cases and illustrate their performance through computational experiments.

Keywords
Ranking and Selection, Multiple Attribute Decision Making, Experimental Design, Sample Allocation, Measurement Uncertainty

1. Introduction
Procurement decisions are selection decisions that are often based on a large number of competing performance measures. The value of the performance measures for each alternative may not be known, but must be evaluated through experimentation. For example, when the U.S. Customs and Border Protection chooses a radiation detection system to install at U.S.-based international airports, the ability of the considered systems to identify an array of radiological and nuclear materials of interest must be evaluated in a laboratory setting before a system is selected.

We define a multiple attribute selection problem as a decision in which a decision-maker must select a single alternative from a finite set of alternatives and each alternative is described by several characteristics important to the decision-maker. A common conceptual model of the multiple attribute selection problem entails the decision-maker placing a decision value on each alternative by considering the desirability of the alternatives' important characteristics and choosing the alternative that provides the greatest decision value [1-3]. We illustrate this model in Figure 1, where m alternatives, $a_1,\ldots,a_m$, are each described by k important characteristics (attributes). The values of the attributes, $\mu_{i1},\ldots,\mu_{ik}$, are used by the decision-maker in developing a decision value ξ_i for each of the $i = 1,\ldots,m$ alternatives. The decision-maker then invokes a selection process to select an alternative, a_s.

The field of decision analysis provides a large body of work to aid in operationalizing the conceptual model displayed in Figure 1. Decision analysis topics of particular relevance include guidance on generating alternatives, formulating mathematical models of decision-makers' preferences, and methods to combine multiple attribute values in creating decision values. Section 2 includes a brief review of these topics.

Often, the true attribute value is not known and must be estimated based on a limited number, n_{ij}, of measurements: $x_{ij1},\ldots,x_{ijn_{ij}}$. We denote the measured value used in estimating the jth attribute from alternative a_i with the random

variable $X_{ij} = \mu_{ij} + \varepsilon_j$ where μ_{ij} is the true attribute value and ε_j is the random measurement error. Since ε_j is a result of the measurement technique used to measure the jth attribute, we assume that the error distribution for attribute j is the same across the m alternatives.
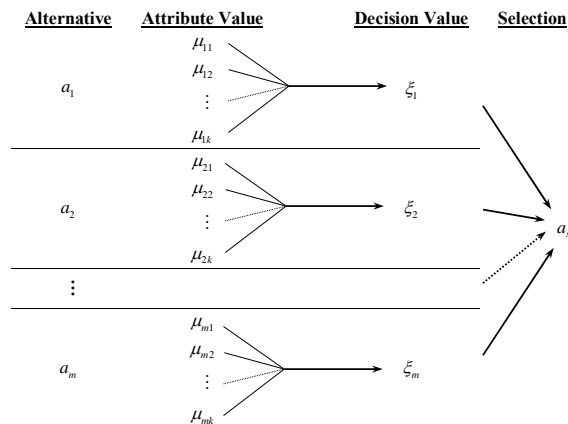

Figure 1: Conceptual model of a multiple attribute selection problem

The uncertainty in the measured values creates uncertainty about the attribute values, and subsequently, uncertainty about the decision values. Thus, the decision-maker may fail to make a correct selection by failing to select the alternative that would have been selected had the true attribute and decision values been known. In general, the uncertainty in the true attribute values can be reduced by increasing the number of measurements used in their estimation, which, in turn, reduces the uncertainty in the decision values and conceivably increases the likelihood of making a correct selection. When the decision-maker is provided a fixed measurement budget, B, that must not be exceeded, the challenge becomes how to allocate this budget across both the alternatives and across the attributes to provide the greatest probability of making the correct selection. This work addresses the problem of allocating a fixed budget across both the alternatives and the attributes in the multiple attribute selection problem.

In the following section we provide a brief overview of some of the literature relevant to this problem. In Section 3 we further define the problem and provide the estimation and selection rules used. In Section 4 we derive several allocation rules. In Sections 5 we describe computational experiments used to evaluate the allocation rules with results provided in Section 6. We conclude with a summary and discussion of future work in Section 7.

2. Literature Review

We find this work to be at the intersection of several areas which include statistical experiment design, ranking and selection, and multiple attribute decision analysis. With sample size and sample allocation a fundamental question in the statistical design of experiments, the underlying principles are paramount in guiding this work. Of particular interest is the design of comparative experiments. Box, Hunter, and Hunter [4] and Montgomery [5] provide extensive guidance for the principles and methods of statistical design of experiments.

At the core of this current work, we are faced with a decision problem, and more specifically, a multiple attribute decision problem. A large body of literature exists on normative multiple attribute value and utility analysis, e.g., [1-3]. We leverage these works for guidance on combining multiple attribute values to form a univariate decision value.

Ranking and selection methods are used to compare a finite number of alternatives whose performance measures are generated by a stochastic process. The study of ranking and selection first gained traction in the 1950s in the statistics community. In 1979 Gupta and Panchapakesan [6] published the first modern text on the subject with Bechhofer et al. [7] publishing a more recent text in 1995. During this time, the field of computer simulation, and in particular discrete event simulation, began advancing the work of ranking and selection and now accounts for much of the research in the area. Kim and Nelson [8] provide an extensive overview of the recent developments in ranking and selection with a focus on the indifferent zone (IZ) allocation procedure for selecting the alternative with the largest expected value. The IZ procedure determines how often each alternative is observed (simulated) while guaranteeing a specified probability of correct selection provided that the true performance of the "best" alternative

exceeds that of its closest competitor by an amount the experimenter feels is worth detecting. The IZ procedure has no predefined limit on the number of observations. Butler et al. [9] apply IZ to a multiple attribute decision problem.

Rather than selecting the alternative with the largest expected performance value, another often employed selection rule, multinomial selection, selects the alternative with the largest estimated probability of being the best on any given trial. Miller et al. [10] provide an efficient computational approach for implementing the multinomial selection procedure. And recently Tollesfson et al. [11] provided optimal algorithms for the multinomial selection procedure.

Unlike the IZ allocation procedure, the Optimal Computing Budget Allocation (OCBA) procedure derives a sample allocation based on a fixed budget. OCBA is the Ranking and Selection procedure most relevant to our work. Chen and Lee have published many articles on the subject with a comprehensive collection of ideas presented in a recent text [12]. While similar to our work in that the budget is constrained, the OCBA approach considers the allocation across multiple alternatives with a single performance measure, while our work is focused on the allocation across both alternatives and multiple attributes. We use many of the ideas of OCBA in our development.

3. Multiple Attribute Selection Problem with Measurement Uncertainty

Expanding on the concepts introduced in Section 1, the random measured value, X_{ij}, used in estimating attribute j from alternative a_i adheres to a probability distribution, denoted $F_j\left(\mu_{ij},\theta_{ij}\right)$, that depends upon the attribute's true value, μ_{ij}, and other distributional parameters, θ_{ij}, to include the uncertainty associated with the measurement technique. Upon observing the n_{ij} measurements used to estimate attribute j, a decision-maker may describe the attribute value by a single point, such as a sample mean, or a distribution, such a Bayesian posterior distribution. We denote this attribute value description as the probability distribution $G_j\left(\mu_{ij},n_{ij},\varphi_{ij}\right)$ that depends upon the attribute's true value, μ_{ij}, the number of observed measurements, n_{ij}, and other distributional parameters, φ_{ij}, to include the uncertainty associated with the measurement technique. The multiple attribute decision model, $f\left(\cdot\right)$, is used to combine the attribute values, leading to a decision value for each alternative a_i. The decision-maker may describe the decision values as a single point or a distribution. We denote the decision value description as the probability distribution $H\left(\xi_i,\gamma_i\right)$ that depends upon the alternative's true decision value, ξ_i and other distributional parameters, γ_i, to include the uncertainty associated with the measurement techniques and the number of observed measurements for each attribute. Based on the decision-maker's description of the decision values, an alternative, a_s, is selected according to a selection rule that takes into account the information that is generated from the measurements. Note that a_s is random because it depends upon the random measurements. The conceptual model illustrated in Figure 1 is expanded in Figure 2 to include the measurement processes and distributional descriptions.
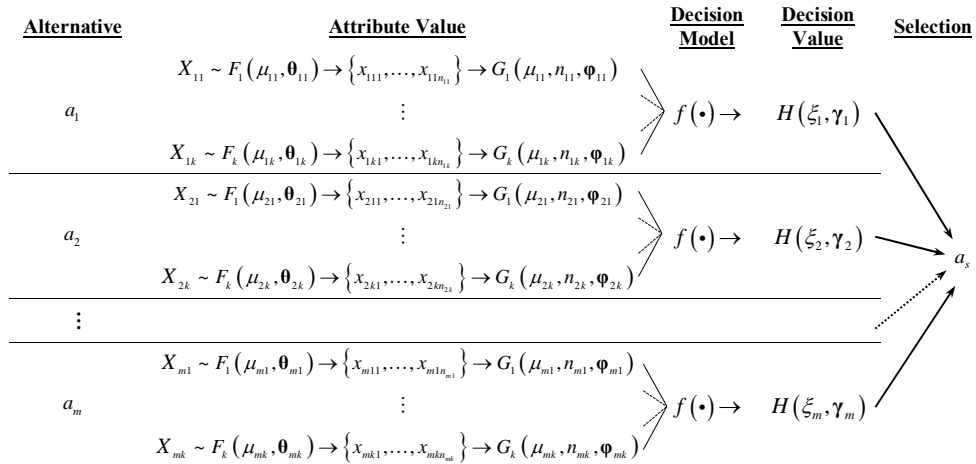

Figure 2: Multiple attribute selection problem model including measurement processes, value estimations, and distributional representations

3.1. Assumptions

We make the following assumptions in this work:

1. The set of m distinct alternatives, $\{a_1,\ldots,a_m\}$ is provided, where m is a finite positive integer such that all alternatives can be assessed.

2. Also provided is a decision model, $\xi_i = f\left(\mu_{i1},\ldots,\mu_{ik}\right) = \sum_{j=1}^{k}\lambda_j v_j\left(\mu_{ij}\right)$, that reflects the decision-maker's preference structure and combines the multiple attribute values to produce a decision value, ξ_i, for each alternative a_i. The decision model is a multiple attribute linear value model with linear individual value functions, $v_j\left(\mu_{ij}\right) = \mu_{ij}$. The attribute weights, λ_j, are defined such that $\sum_{j=1}^{k}\lambda_j = 1$.

3. Each alternative is described by $k \geq 2$ attributes. Separate and independent measurement processes are used in obtaining measurement data for each attribute. The measurement data are collected in a single experimental effort (single-stage). The random measurement error associated with the measurement processes, ε_j, are independent and identically distributed (i.i.d.) where the $\varepsilon_j \sim N\left(0,\sigma_j^2\right)$, and σ_j^2 is known. It follows that $X_{ij} \sim N\left(\mu_{ij},\sigma_j^2\right)$, and the measurements $x_{ij1},\ldots,x_{ijn_{ij}}$ are i.i.d. samples from this distribution. Note that in the field of metrology, it is not uncommon for the error associated with a continuous measurand to be modeled with a normal distribution and for the variance of this distribution to be well characterized and assumed known [13].

4. The total fixed experimental budget, $B \geq mk$, shall not be exceeded and the cost of each measurement is equivalent. Thus, B is the upper bound on the number of measurements that can be performed.

3.2. Estimation Approach

The decision-maker uses the limited number of measurements, $x_{ij1},\ldots,x_{ijn_{ij}}$, to estimate the true value, μ_{ij}, of attribute j of alternative a_i in support of the selection decision. We considered two approaches for this estimation process: maximum likelihood estimation and Bayesian posterior distribution.

Under the assumption that the random measured value, X_{ij}, is distributed according to a normal distribution with unknown mean, μ_{ij}, and known variance, σ_j^2, the maximum likelihood estimator (MLE) for μ_{ij} is the sample mean of the n_{ij} measurements, $\bar{X}_{ij} = \frac{1}{n_{ij}}\sum_{l=1}^{n_{ij}}x_{ijl}$ [14]. Further, because the measurements $x_{ij1},\ldots,x_{ijn_{ij}}$ are assumed to be i.i.d., $\bar{X}_{ij} \sim N\left(\mu_{ij},\frac{\sigma_j^2}{n_{ij}}\right)$. In Section 4.1 we use the MLE of μ_{ij}, and its properties, to derive an allocation rule that aims to maximize the probability that the decision-maker makes a correct selection.

Under the Bayesian paradigm for estimation, we assume that before collecting any measurement data, the decision-maker's knowledge of the unknown true attribute value, μ_{ij}, can be described by the conjugate normal prior distribution $N\left(\mu_{0ij},\tau_{0ij}^2\right)$. Upon observing the normally distributed measurement data, the decision-maker's knowledge of μ_{ij} is updated and presented by the normally distributed posterior distribution [15] as described in Equation (1).

$$p\left(\mu_{ij} \mid x_{ij1},\ldots,x_{ijn_{ij}}\right) \sim N\left(\frac{\sigma_j^2\mu_{0ij} + n_{ij}\tau_{0ij}^2\bar{X}_{ij}}{\sigma_j^2 + n_{ij}\tau_{0ij}^2}, \frac{\sigma_j^2\tau_{0ij}^2}{\sigma_j^2 + n_{ij}\tau_{0ij}^2}\right) \quad i = 1,\ldots,m, \; j = 1,\ldots,k \tag{1}$$

Assumption 2 provides $\xi_i = \sum_{j=1}^{k}\lambda_j v_j\left(\mu_{ij}\right)$ for each alternative a_i. Let $\tilde{\xi}_i$ be the random variable whose probability distribution is the posterior distribution of the decision value ξ_i. This distribution describes the decision-maker's knowledge of the true decision value, ξ_i, for alternative a_i, after observing n_{ij} measurements for the estimation of each of the k attributes. The distribution of $\tilde{\xi}_i$ is presented in Equation (2).

$$\tilde{\xi}_i \sim N\left(\tilde{\mu}_i, \tilde{\tau}_i^2\right) \text{ where } \tilde{\mu}_i = \sum_{j=i}^{k} \lambda_j \frac{\sigma_j^2 \mu_{0ij} + n_{ij}\tau_{0ij}^2 \overline{X}_{ij}}{\sigma_j^2 + n_{ij}\tau_{0ij}^2} \text{ and } \tilde{\tau}_i^2 = \sum_{j=1}^{k} \lambda_j^2 \frac{\sigma_j^2 \tau_{0ij}^2}{\sigma_j^2 + n_{ij}\tau_{0ij}^2} \qquad (2)$$

We used the Bayesian posterior distribution of the decision value ξ_i in Sections 4.2 and 4.3 to derive allocation rules that aim to maximize the probability that the decision-maker makes a correct selection.

3.3. Selection Process

Kim and Nelson [8] describe four classes of comparisons as they relate to ranking and selection problems: selecting the alternative with the largest or smallest expected performance measure (selection of the best), comparing all alternatives against a standard (comparison with a standard), selecting the alternative with the largest probability of actually being the best performer (multinomial selection), and selecting the system with the largest probability of success (Bernoulli selection). Kim and Nelson further note that in developing an experimental approach for each class, a constraint is imposed on either the probability of correct selection or on the overall experimental budget. That is, some procedures (e.g., indifference zone procedures) attempt to find a desirable alternative with a guarantee on the probability of correct selection with no regards to the experimental budget, and other procedures (e.g., OCBA procedures) attempt to maximize the probability of correct selection while adhering to an experimental budget constraint. In this work, we considered two selection procedures: selection of the best and multinomial selection, while adhering to an overall experimental budget.

3.3.1. Selection of the Best

We implemented a selection of the best procedure by selecting the alternative that has the largest expected decision value. Under the maximum likelihood estimation procedure, it follows from the invariant property of maximum likelihood estimators [14] that the maximum likelihood estimator of ξ_i is $\overline{Y}_i = \sum_{j=1}^{k} \lambda_j \overline{X}_{ij}$. Furthermore, $\overline{Y}_i \sim N\left(\xi_i, \sum_{j=1}^{k} \frac{\lambda_j^2 \sigma_j^2}{n_{ij}}\right)$. We select the alternative a_s where $s = \arg\max_i \overline{Y}_i$. Under the Bayesian estimation procedure, the decision-maker's knowledge of ξ_i is described by the distribution of $\tilde{\xi}_i$, provided by Equation (2). We select the alternative a_s where $s = \arg\max_i \tilde{\mu}_i$.

3.3.2. Multinomial Selection

Multinomial selection procedures were originally designed for experiments with a categorical response [8]. Goldsman [16] suggested a more general perspective for the field of computer simulation. Given m competing alternatives, it is assumed that there is an unknown probability vector $\mathbf{p} = \left(p_1, \ldots, p_m\right)$ such that $0 \le p_i \le 1$ and $\sum_{i=1}^{m} p_i = 1$. The p_i are the probabilities that alternative a_i "*wins*" on any given trial, where winning is the observation of a most desirable criteria of goodness (e.g., the largest decision value). $\mathbf{p}$ thus defines an m-nomial probability distribution for winning over the set of alternatives. The goal of a multinomial selection procedure is to identify the alternative with the largest p_i.

We used this idea and implemented a multinomial selection procedure for the Bayesian estimation approach where $p_i = P\left(\tilde{\xi}_i > \tilde{\xi}_r, \forall r = 1, \ldots, m, r \ne i\right)$. Because it is difficult to calculate p_i, we used Monte Carlo simulation to estimate it. From each of the distributions for $\tilde{\xi}_i$, $i = 1, \ldots, m$, we draw a single realization and note the alternative with the largest realized value among the m values. We repeat this process a large number of times, N, (e.g., $N = 1000$) and tabulate the relative frequency, $\hat{p}_i$, that alternative a_i provided the largest realized value. As $N \to \infty$, $\hat{p}_i \to P\left(\tilde{\xi}_i > \tilde{\xi}_r, \forall r = 1, \ldots, m, r \ne i\right)$. We select the alternative a_s where $s = \arg\max_i \hat{p}_i$.

4. Measurement Allocation

The probability of correct selection (*PCS*) is the probability that the alternative identified for selection, a_s, is indeed the most preferred alternative (largest true decision value). The primary goal of this work was to determine, given the assumptions provided in Section 3.1, the number of samples (measurements), n_{ij}, required to maximize the *PCS*

such that $\sum_{i=1}^{m}\sum_{j=1}^{k}n_{ij}$ does not exceed the total experimental budget, B. This sample allocation problem can be expressed by the optimization problem in Equation (3).

$$\max_{n_{ij}} PCS = P\left(a_s \text{ is actually the most preferred alternaitve}\right)$$

$$s.t. \quad \sum_{i=1}^{m}\sum_{j=1}^{k}n_{ij} \le B \tag{3}$$

$$n_{ij} \ge 0 \quad i = 1,\ldots,m, \ j = 1,\ldots,k$$

In the subsections to follow, we derive sample allocation rules by defining the *PCS* and subsequent constraints according to the estimation and selection processes considered.

4.1. Selection of the Best using Maximum Likelihood Estimation

For this case, without loss of generality, we assume that $\xi_1 > \xi_i$ for all $i = 2,\ldots,m$. That is, alternative a_1 is truly the most preferred alternative. Given this assumption and the selection and estimation approaches, we define *PCS*.

$$PCS = P\left(a_s = a_1\right) = P\left(\overline{Y}_1 > \overline{Y}_i, i = 2,\ldots,m\right) = P\left\{\bigcap_{i=2}^{m}\left(\overline{Y}_i - \overline{Y}_1 < 0\right)\right\} \tag{4}$$

In order to compute $\overline{X}_{ij}$, and subsequently $\overline{Y}_i$, it is required that $n_{ij} \ge 1$ for all i,j, so we can restate the sample allocation problem of Equation (3) using Equation (5).

$$\max_{n_{ij}} PCS = P\left\{\bigcap_{i=2}^{m}\left(\overline{Y}_i - \overline{Y}_1 < 0\right)\right\}$$

$$s.t. \quad \sum_{i=1}^{m}\sum_{j=1}^{k}n_{ij} \le B \tag{5}$$

$$n_{ij} \ge 1 \quad i = 1,\ldots,m, \ j = 1,\ldots,k$$

We first consider the $m = 2$ alternatives case. The objective function in Equation (5) thus becomes $\max_{n_{ij}} PCS = P\left(\overline{Y}_2 - \overline{Y}_1 < 0\right)$. Since $\overline{Y}_i \sim N\left(\xi_i, \sum_{j=1}^{k}\frac{\lambda_j^2\sigma_j^2}{n_{ij}}\right)$, then $\overline{Y}_2 - \overline{Y}_1 \sim N\left(\xi_2 - \xi_1, \sum_{i=1}^{2}\sum_{j=1}^{k}\frac{\lambda_j^2\sigma_j^2}{n_{ij}}\right)$. This yields the definition of *PCS* provided by Equation (6).

$$PCS = P\left(\overline{Y}_2 - \overline{Y}_1 < 0\right) = \Phi\left(\frac{\xi_1 - \xi_2}{\sqrt{\sum_{i=1}^{2}\sum_{j=1}^{k}\frac{\lambda_j^2\sigma_j^2}{n_{ij}}}}\right) \tag{6}$$

$\Phi(y)$ in Equation (6) denotes the standard normal cumulative distribution function evaluated at y. Because Φ is a monotonically increasing function and $\xi_1 - \xi_2 > 0$, maximizing *PCS* requires minimizing $\sum_{i=1}^{2}\sum_{j=1}^{k}\frac{\lambda_j^2\sigma_j^2}{n_{ij}}$. Note that this expression is the sum of the mean squared errors (MSE) of the estimators $\overline{Y}_i$ (see [14] for discussion of MSE). The optimization problem in (5), with $m = 2$, is equivalent to:

$$\min_{n_{ij}} z = \sum_{i=1}^{2}\sum_{j=1}^{k}\frac{\lambda_j^2\sigma_j^2}{n_{ij}}$$

$$s.t. \quad \sum_{i=1}^{2}\sum_{j=1}^{k}n_{ij} \le B, \tag{7}$$

$$n_{ij} \ge 1 \quad i = 1,2, \ j = 1,\ldots,k$$

We restated the second constraint of the nonlinear optimization problem in (7) as $-n_{ij} \le -1$, and derived the optimal solution displayed in Equation (8) using the Kuhn-Tucker conditions [17].

$$n_{ab} = \left(\frac{\lambda_b\sigma_b}{\sum_{j=1}^{k}\lambda_j\sigma_j}\right)\frac{B}{2} \quad a = 1,2 \ b = 1,\ldots,k \tag{8}$$

Since the objective function, z, in (7) is the sum of convex functions, then z is too a convex function. As the constraints of this minimization problem are linear, they are also convex. Therefore, the Kuhn-Tucker conditions are necessary and sufficient for the solution displayed in Equation (8) to be an optimal solution.

When $m > 2$, there is no closed-form expression for *PCS* as defined in the objective function of Equation (5). The solution to the $m = 2$ case suggested an approach to overcome this dilemma, so we derived a sample allocation that minimized the sum of the mean squared errors of the estimators $\overline{Y}_i$, for all $i = 1, \ldots, m$ subject to the constraints provided in (5). The solution to this general m alternative problem is provided by Equation (9).

$$n_{ab} = \left(\frac{\lambda_b \sigma_b}{\sum_{j=1}^{k} \lambda_j \sigma_j} \right) \frac{B}{m} \qquad a = 1, \ldots, m, \ b = 1, \ldots, k \qquad (9)$$

Note that in Equations (8) and (9), the sample allocation is dependent only on the second index which represents the attribute. This means that for this single-stage problem, the sample allocations may vary across attribute, but are equivalent across the alternatives.

Because the number of measurements made on any attribute must be an integer value greater than or equal to one, and the total number of measurements must not exceed B, we implement the following rounding rule, which completes our definition of the sample allocation procedure for the selection of the best using maximum likelihood estimation (*MLE allocation rule*).

1. Calculate the n_{ij} according to Equation (9) for alternatives a_i, $i = 1, \ldots, m$ and attributes $j = 1, \ldots, k$.

2. Calculate $n'_{ij} = \lceil n_{ij} \rceil$, where $\lceil \bullet \rceil$ is the ceiling function.

3. Calculate $r_{ij} = n'_{ij} - n_{ij}$.

4. Calculate $O = \sum_{i=1}^{m} \sum_{j=1}^{k} n'_{ij} - B$.

5. Order the $n'_{ij} \neq 1$ in decreasing order of r_{ij}. (For any j, $r_{1j} = \cdots = r_{mj}$; thus, for each j such that $n'_{ij} \neq 1$ the n'_{ij} are ordered in increasing order of i.)

6. Subtract 1 from each of the first O ordered n'_{ij}.

4.2. Selection of the Best using Bayesian Estimation

As we did deriving the MLE allocation rule, we assumed that $\xi_1 > \xi_i$ and defined the PCS using the Bayesian estimation of ξ_i (Equation (2)) and the rule to select a_s where $s = \arg\max_i \tilde{\mu}_i$. With $m = 2$ alternatives this led to the optimization problem in Equation (10).

$$\max_{n_{ij}} PCS = P\left(\tilde{\mu}_2 - \tilde{\mu}_1 < 0 \right) = \Phi \left(\frac{\sum_{j=1}^{k} \lambda_j \frac{\sigma_j^2 \mu_{01j} + n_{1j} \tau_{01j}^2 \mu_{1j}}{\sigma_j^2 + n_{1j} \tau_{01j}^2} - \lambda_j \frac{\sigma_j^2 \mu_{02j} + n_{2j} \tau_{02j}^2 \mu_{2j}}{\sigma_j^2 + n_{2j} \tau_{02j}^2}}{\sqrt{\sum_{i=1}^{2} \sum_{j=1}^{k} \lambda_j^2 \frac{n_{ij} \tau_{0ij}^4 \sigma_j^2}{\left(\sigma_j^2 + n_{ij} \tau_{0ij}^2 \right)^2}}} \right)$$

$$s.t. \quad \sum_{i=1}^{m} \sum_{j=1}^{k} n_{ij} \leq B \qquad (10)$$

$$n_{ij} \geq 0 \quad i = 1, \ldots, m, \ j = 1, \ldots, k$$

Because the unknown attribute values μ_{ij} cannot be separated from the decision variables n_{ij}, we chose to minimized the sum of the mean squared errors of the estimators $\tilde{\xi}_i$ as described in Equation (11) for the general case with m alternatives.

$$\min_{n_{ij}} z = \sum_{i=1}^{m} \sum_{j=1}^{k} \frac{\lambda_j^2 \sigma_j^2 \tau_{0ij}^2}{\sigma_j^2 + n_{ij} \tau_{0ij}^2}$$

$$s.t. \quad \sum_{i=1}^{m} \sum_{j=1}^{k} n_{ij} \le B \tag{11}$$

$$n_{ij} \ge 0 \quad i = 1, \ldots, m, \quad j = 1, \ldots, k$$

The optimal solution to this general *m* alternative problem, found using the Kuhn-Tucker conditions, is displayed in Equation (12).

$$n_{ab} = \left(\frac{\lambda_b \sigma_b}{\sum_{j=1}^{k} \lambda_j \sigma_j} \right) \left(\frac{B}{m} - \sum_{j=1}^{k} \frac{\lambda_j \sigma_j \sigma_b}{\lambda_b \tau_{0ab}^2} + \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{k} \frac{\sigma_j^2}{\tau_{0ij}^2} \right) \quad a = 1, \ldots, m, \ b = 1, \ldots, k \tag{12}$$

When there is little prior knowledge of the true attribute values, μ_{ij}, that is, the prior distribution for the μ_{ij} are diffuse and the variances of the prior distributions, τ_{0ij}^2, are very large, then the limiting posterior distribution for μ_{ij} can be stated as displayed in Equation (13):

$$p\left(\mu_{ij} \mid x_{ij1}, \ldots, x_{ijn_{ij}} ; i, j = 1, 2 \right) \xrightarrow{\rightarrow} N\left(\overline{X}_{ij}, \frac{\sigma_j^2}{n_{ij}} \right) \quad \text{as } \tau_{0ij}^2 \to \infty \tag{13}$$

Further, as $\tau_{0ij}^2 \to \infty$, the allocation solution for the selection of the best using Bayesian estimation displayed in Equation (12) converges to the solution found using maximum likelihood estimation provided in Equation (9).

We finalize our definition of the sample allocation procedure for the selection of the best using Bayesian estimation (*Bayes allocation rule*) by providing a rounding rule that assures that the number of measurements made on any attribute is a non-negative integer and the total number of measurements does not exceed *B*. The rounding rule is identical to the *MLE allocation rule* rounding rule with one exception: Step 5 is replaced by the following:

5. Order the n'_{ij} in decreasing order of r_{ij} (For any *j*, $r_{1j} = \cdots = r_{mj}$; thus, for each *j*, the n'_{ij} are ordered in increasing order of *i*.)

4.3. Multinomial Selection using Bayesian Estimation

Under the Bayesian estimation model, we also implemented a multinomial selection procedure (described in Section 3.2.2). Under this procedure, again assuming $\xi_1 > \xi_i$ for all $i = 2, \ldots, m$, the *PCS* is defined as in Equation (14).

$$PCS = P(a_s = a_1) = P(\hat{p}_1 > \hat{p}_i, i = 2, \ldots, m) \tag{14}$$

With $m = 2$ alternatives, $\hat{p}_1 > \hat{p}_2$ is true if and only if $\hat{p}_1 > 0.5$, thus $PCS = P(\hat{p}_1 > 0.5)$. In the limit, $\hat{p}_1$ approaches $p_1 = P(\tilde{\xi}_1 > \tilde{\xi}_2)$, which, provided the distribution of $\tilde{\xi}_i$ in Equation (2), is computed as Equation (15).

$$p_1 = P\left(\tilde{\xi}_1 > \tilde{\xi}_2 \right) = P\left(\tilde{\xi}_2 - \tilde{\xi}_1 < 0 \right) = \Phi\left(\frac{\tilde{\mu}_1 - \tilde{\mu}_2}{\sqrt{\sum_{i=1}^{2} \sum_{j=1}^{k} \frac{\lambda_j^2 \sigma_j^2 \tau_{0ij}^2}{\sigma_j^2 + n_{ij} \tau_{0ij}^2}}} \right) \tag{15}$$

From Equation (15) it is seen that $p_1 > 0.5$ if and only if $\tilde{\mu}_1 - \tilde{\mu}_2 > 0$. So, we restate the *PCS* as Equation (16).

$$PCS = P(\hat{p}_1 > 0.5) = P(\tilde{\mu}_2 - \tilde{\mu}_1 < 0) \tag{16}$$

Because the *PCS* definition of Equation (16) is the same as the *PCS* definition from the selection of the best procedure using Bayesian estimation in Section 4.2 (Equation (10)), the allocation rule for this multinomial selection procedure is also the allocation rule provided by Equation (12). We suspect that this relation between the selection of the best and the multinomial selection procedures holds only for symmetric posterior distributions of the decision values, such as the normal distribution considered in this work.

5. Numerical Experiments

In Section 4 we derived the *MLE allocation rule* and the *Bayes allocation rule*. Here we briefly describe a computational experiment that compared the performance of these rules to a uniform allocation that provides an equal sample allocation to each of the attributes across all alternatives (a common allocation approach, consistent with the principle of balance in the traditional design of experiments discipline).

We generated 500 concave efficient frontiers (decision cases), each consisting of $m = 5$ alternatives described by $k = 2$ attributes. The true values of the attributes were randomly assigned from the domain [100, 200], subject to the constraints necessary for nondominance and concavity. For each decision case, the standard deviation of the measurement error for each attribute was randomly assigned from a uniform distribution with parameters $min = 1$ and $max = 30$. We considered 19 different decision models defined by $(\lambda_1, \lambda_2) = (0.05, 0.95), \ldots, (0.95, 0.05)$ and a sample budget of $B = 50$. For the allocations provided by the two allocation rules and a uniform allocation, we simulated 10 000 measurement experiments and calculated the frequency of correct selection (*fcs*) using the selection of the best selection procedure under both MLE and Bayesian estimation. The prior distribution for μ_{ij} used in the Bayesian estimations, $N(150, 35^2)$, was equivalent for all attributes for all decision cases. We calculated the relative *fcs* (or *rel fcs*) as the ratio of the *fcs* provided by the rule-based allocation to the maximum *fcs* obtained across all sample allocations that provided equivalent allocations across alternatives. Within the confines of this problem, the *rel fcs* measure allows us to quantify how much better the selection could have been if a different sample allocation were chosen. See [18] for further details.

6. Results

We took the perspective that the estimation and selection procedures were provided by the decision-maker and the only remaining decision was how to allocate the overall budget, *B*, amongst the alternatives and attributes. Thus we were not concerned with the overall *PCS*, but rather the performance of the allocation relative to the optimal allocation (i.e., *rel fcs*) under the given selection, estimation and budget constraints. The left panel of Figure 3 displays the performance of the uniform allocation and the MLE allocation rule (using MLE estimation procedures) as a function of decision model (λ_1 value). Similar results are displayed in the right panel for the uniform and Bayes allocation rules, but under Bayesian estimation. The performance of a rule, for each decision weight, was defined to be the average *rel fcs* of its sample allocation across the 500 test cases. The uncertainties in the average *rel fcs* were expressed as 95 % confidence intervals using the normality assumptions provided by the Central Limit Theorem.
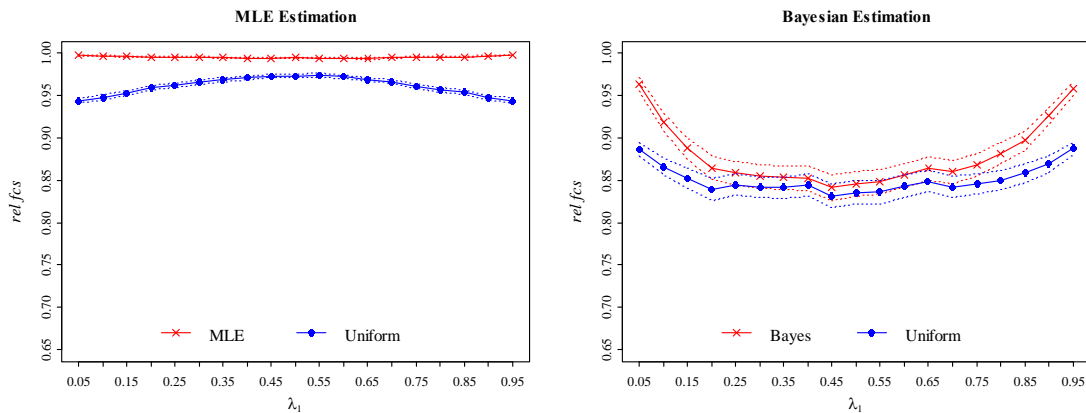

Figure 3: Relative frequency of correct selection averaged across all decision cases for each λ_1 value. The dotted lines represent the 95 % confidence intervals.

From the left panel of Figure 3, we observed that the MLE allocation rule provided a *rel fcs* near 1 across all decision models under the given selection, estimation and budget constraints. We also note that this allocation rule outperforms the common uniform allocation across all decision models. The Bayes allocation rule provided performance results (right panel) that were statistically indistinguishable from the uniform allocation for

$0.25 \leq \lambda_1 \leq 0.75$. This result may be driven by the general and nonspecific prior distribution provided for all attributes and alternatives for all decision cases. While the *rel fcs* results under the Bayesian approach are often smaller than those proved by the MLE approach, this does not mean that the absolute *fcs* was smaller. The absolute *fcs* values were seen to be statistically indistinguishable with the Bayesian approach outperforming the MLE approach when λ_1 was near 0 and 1.

7. Conclusions and Future Work

The optimal allocation rule depends upon the estimation and selection procedures used. The results of our computational experiments show that, in the cases considered, the *MLE allocation* rule performs at a near optimal level, while the derived *Bayes allocation* rule performed only slightly better than the common uniform allocation. These results emphasize that decision modeling and experimental design should be done jointly rather than independently (which, unfortunately, is currently not uncommon). Such a cooperative approach can improve the overall selection results of the project. This work extends previous work [18] and adds to our knowledge of how to allocate samples across attributes and alternatives in a multiple attribute selection problem. We plan to study the impact of the Bayesian prior distribution on the performance of the Bayes allocation rule. We will also consider another important area of experimental testing: Bernoulli trials in pass-fail testing. Although this work focused on a single-stage experiment, future work will consider multiple stage and sequential experiments.

References

1. Keeney, R.L., and Raiffa, H., 1993, Decisions with Multiple Objectives - Preferences and Value Tradeoffs, 2nd Edition, Cambridge University Press, New York.
2. Kirkwood, C.W., 1997, Strategic Decision Making: Multiobjective Decision Analysis with Spreadsheets, Duxbury Press, Belmont, CA.
3. Dyer, J. S., 2005, "MAUT - Multiattribute Utility Theory," appears in Multiple Criteria Decision Analysis - State of the Art Surveys, Figueira, J., Greco, S., and Ehrgott, M. (eds.), Springer, New York, 265-295.
4. Box, G.E., Hunter, J.S., and Hunter, W.G., 2005, Statistics for Experimenters, 2nd Edition, John Wiley & Sons Inc., Hoboken, NJ.
5. Montgomery, D.C., 2013, Design and Analysis of Experiments, 8th Edition, John Wiley & Sons, New York.
6. Gupta, S.S., and Panchapakesan, S., 1979, Multiple Decision Procedures, John Wiley & Sons, New York.
7. Bechhofer, R.E., Santer, T.J., and Goldsman, D.M., 1995, Design and Analysis of Experiments for Statistical Selection, Screening, and Multiple Comparisons, John Wiley and Sons, Inc., New York.
8. Kim, S.-H., and Nelson, B.L., 2006, "Selecting the Best System," appears in Handbook in Operations Research and Management Science, Henderson, S.G., and Nelson, B.L., Elsevier, Oxford, 13, 501-534.
9. Butler, J., Morrice, D.J., and Mullarkey, P.W., 2001, "A Multiple Attribute Utility Theory Approach to Ranking and Selection," Management Science, 47(6), 800-816.
10. Miller, J.O., Nelson, B.L., and Reilly, C.H., 1998, "Efficient Multinomial Selection in Simulation," Naval Research Logistics, 45, 459-482.
11. Tollefson, E., Goldsman, D.M., Kleywegt, A., and Tovey, C., 2014, "Optimal Selection of the Most Probable Multinomial Alternative," Sequential Analysis, 33, 491-508.
12. Chen, C.-H., and Lee, L.H., 2011, Stochastic Simulation Optimization: An Optimal Computing Budget Allocation, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ.
13. Joint Committee for Guides in Metrology, 2008, Evaluation of Measurement Data — Guide to the Expression of Uncertainty in Measurement, Joint Committee for Guides in Metrology, Sevres Cedex, France.
14. Casella, G., and Berger, R.L., 2002, Statistical Inference, 2nd Edition, Duxbury, Pacific Grove, CA.
15. Gelman, A., Carlin, J.B., Stern, H.S., and Rubin, D.B., 2004, Bayesian Data Analysis, 2nd Edition, Chapman & Hall/CRC, New York.
16. Goldsman, D.M., 1984, "On Selecting the Best of K Systems: An Expository Survey of Indifference-zone Multinomial Procedures," Proceedings of the 1984 Winter Simulation Conference, November 28-30, Dallas, TX, 107-112.
17. Winston, W.L., 2004, Operations Research, 4th Edition, Thompson, Belmont, CA.
18. Leber, D.D., and Herrmann, J.W., 2014, "Resource Allocation for Selection Decisions with Measurement Uncertainty," Proceedings of the 2014 Industrial and Systems Engineering Research Conference, May 31-June 3, Montreal, Canada.