UWB Signal Processing: Projection, B-Splines, and Modified Gegenbauer Bases

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***Abstract*—Ultra-wide band (UWB) systems require either rapidly changing or very high sampling rates. Conventional analog-to-digital devices have limited dynamic range. We investigate UWB signal processing via a basis projection method and a basis system designed for UWB signals. The method first windows the signal and then decomposes the signal into a basis via a continuous-time inner product operation, computing the basis coefficients in parallel. The windowing systems are key, and we develop systems that have variable partitioning length, variable roll-off and variable smoothness. These include systems developed to preserve orthogonality of any orthonormal systems between adjacent blocks and *almost orthogonal* windowing systems that are more computable/constructible than the orthogonality preserving systems, built using B-splines. We construct the basis projection method, developing the method with a modified Gegenbauer system designed specifically for UWB signals.**

I. Introduction

An ultra-wide band (UWB) communication system is a large bandwidth system based on the transmission of very short pulses with relatively low energy. These systems operate by running as signaling waveforms, baseband pulses of very short duration, rather than the traditional method using a sinusoidal carrier. The UWB technique has a fine time resolution which makes it a technology appropriate for accurate ranging. The large bandwidth of an UWB system is dominated by its pulse shape and duration. This large system bandwidth relative to the information bandwidth allows UWB systems to operate with a low power spectral density. Such a low power spectral density implies that the UWB signal may be kept near or below the noise floor of detection devices. So, UWB technology has many potential advantages, such as high data rate, low probability of interception and detection, system simplicity, low cost, reduced average power consumption, weak sensitivity to the near-far problem and immunity to interference.

UWB systems present challenges to current methods of signal processing. Despite extensive advances, wideband problems continue to hit barriers in sampling architectures and analog-to-digital conversion (ADC). ADC signal-to-noise and distortion ratio (the effective number of resolution bits) declines with sampling rate due to timing jitter, circuit imperfections, and electronic noise. ADC performance (speed and total integrated noise) can be improved to some extent, e.g., by cooling. However, the energy cost may be significant, and this presents a major hurdle for implementation in miniaturized devices. Digital circuitry has provided dramatically enhanced digital signal processing (DSP) operation speeds, but there has not been a corresponding dramatic energy capacity increase in batteries to operate these circuits. Moore's Law for chips is slowing down, and there is no Moore's Law for batteries or ADCs.

A growing number of applications face this challenge, such as miniature and hand-held devices for communications, robotics, and micro aerial vehicles (MAVs). Very wideband sensor bandwidths are desired for dynamic spectrum access and cognitive radio, radar, and UWB systems. Multichannel and multi-sensor systems compound the issue, such as multiple-input and multiple-output (MIMO), array processing and beamforming, multi-spectral imaging, and vision systems. All of these rely on analog sensing and a digital interface, perhaps with feedback. This motivates mixed-signal circuit designs that tightly couple the analog and digital portions, and operate with parallel reduced bandwidth paths to relax ADC requirements. The goal of such wideband integrated circuit designs is to achieve good tradeoffs in dynamic range, bandwidth, and parallelization, while maintaining low energy consumption [7], [8].

From a signal processing perspective, we can approach this problem by implementing an appropriate signal decomposition in the analog portion that provides parallel outputs for integrated digital conversion and processing [3]. This naturally leads to an architecture with windowed time segmentation and parallel analog basis expansion. The method represents a change of view in sampling, from that of a stationary view of a signal used in classical sampling to an "short-time windowed stationary" view. This viewpoint gives that the time and frequency space "tile" occupied by the signal is processed quickly. The windows give us the tools to partition time-frequency so that the UWB signal can be partitioned uniformly but also quickly and efficiently. With the blocks, the signal can be sampled in parallel [3]. In this paper we view this from the sampling theory perspective, including segmentation and window design, achieving orthogonality between segments, basis expansion and choice of basis, signal filtering, and reconstruction. In particular, we develop the method for modified Gegenbauer bases designed for UWB signals. Mathematical

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definitions and computations for the paper follow those given in Benedetto [1].

II. Windowing via B-Splines

The first type of system we develop preserves orthogonality of any orthonormal (ON) system between adjacent blocks. The construction here uses any orthonormal basis for $L^2(\mathbb{R})$ and is created by solving a Hermite interpolation problem with constraints. These ON preserving window systems were the motivation for the methods in this paper. They allow us to create a method of time-frequency analysis for a wide class of signals. The second type of system we develop uses the concept of *almost orthogonality* developed by Cotlar, Knapp and Stein [6]. It employs B-spline techniques to create almost orthogonal windowing systems that are more computable/constructible than the orthogonality preserving systems. Preserving orthogonality requires that the (ON) windowing systems $\{\mathbb{W}_k(t)\}$ satisfy $\sum_k [\mathbb{W}_k(t)]^2 \equiv 1$. The almost orthogonal systems require that there exists a δ, $0 \le \delta \le 1/2$ such that for all k

$$1-\delta \le [\mathbb{A}_k(t)]^2 + [\mathbb{A}_{k+1}(t)]^2 \le 1+\delta$$

for $t \in [kT, (k+1)T]$.

A. Orthogonality Preserving Systems

Our first system of signal segmentation uses sine, cosine and linear functions. This was created because it is relatively easy to implement, cuts down on frequency error and preserves orthogonality. Consider a signal block of length $T+2r$ centered at the origin. Let $0 < r \ll T$. Ideally, we would like to make r as small as possible. Define Cap(t) as follows:

$$\begin{cases} 0 & |t| \ge \frac{T}{2}+r\,, \\ 1 & |t| \le \frac{T}{2}-r\,, \\ \sin(\pi/(4r)(t+(T/2+r))) & \frac{-T}{2}-r < t < \frac{-T}{2}+r\,, \\ \cos(\pi/(4r)(t-(T/2-r))) & \frac{T}{2}-r < t < \frac{T}{2}+r\,. \end{cases} \tag{1}$$

Given Cap(t), we form a tiling system $\{\text{Cap}_k(t)\}$ such that $\text{supp}(\text{Cap}_k(t)) \subseteq [kT-r, (k+1)T+r]$ for all k. Note that the Cap window has a continuous roll-off at the endpoints, windows the signal in $[\frac{-T}{2}-r, \frac{T}{2}+r]$ and is identically 1 on $[\frac{-T}{2}+r, \frac{T}{2}-r]$. It has a $1/\omega^2$ decay in frequency space, and also has the property that for all $t \in \mathbb{R}$

$$[\text{Cap}_k(t)]^2 + [\text{Cap}_{k+1}(t)]^2 = 1\,.$$

If we had a signal f with an absolutely convergent Fourier series, then

$$(f\cdot\text{Cap})_k\hat{}[n] = \sum_m f[n-m]\text{Cap}\hat{}[m] = \widehat{f} * \text{Cap}\hat{}[n]\,.$$

The Fourier transform of Cap is a linear combination of $\operatorname{sinc}\omega$ and $\sin\omega$ functions and has an asymptotic $1/\omega^2$ decay.

The theory of splines gives us the tools to generalize this system. The idea is to cut up the time domain into perfectly aligned segments so that there is no loss of information. We also want the systems to be smooth, so as to provide control over decay in frequency, and adaptive, so as to adjust accordingly to changes in frequency band. Finally, we develop our systems so that the orthogonality of bases in adjacent and possible overlapping blocks is preserved.

Definition 1 (ON Window System): An **ON Window System** is a set of functions $\{\mathbb{W}_k(t)\}$ such that for all $k \in \mathbb{Z}$

$$\begin{aligned} &(i.) && \text{supp}(\mathbb{W}_k(t)) \subseteq [kT-r, (k+1)T+r]\,, \\ &(ii.) && \mathbb{W}_k(t) \equiv 1 \text{ for } t \in [kT+r, (k+1)T-r]\,, \\ &(iii.) && \mathbb{W}_k \text{ is symmetric about its midpoint}\,, \\ &(iv.) && \sum [\mathbb{W}_k(t)]^2 \equiv 1\,, \\ &(v.) && \{\widehat{\mathbb{W}_k{}^\circ}[n]\} \in l^1\,. \end{aligned} \tag{2}$$

Conditions $(i.)$ and $(ii.)$ are partition properties, in that they give an exact snapshot of the input function f on $[kT+r, (k+1)T-r]$ with smooth roll-off at the edges. Conditions $(iii.)$ and $(iv.)$ are needed to preserve orthogonality between adjacent blocks. Condition $(v.)$ is needed for the computation of Fourier coefficients. We generate our systems by translations and dilations of a given window $\mathbb{W}_I$, where $\text{supp}(\mathbb{W}_I) = [-T/2-r, T/2+r]$. Our next proposition shows the need for the condition $(v.)$. Let $I = T+2r$ and let $\mathbb{PW}_\Omega$ denote the Paley-Wiener space for bandlimit Ω.

Proposition 1: Let $f \in \mathbb{PW}_\Omega$ and let $\{\mathbb{W}_k(t)\}$ be an ON window system with generating window $\mathbb{W}_I$. Then

$$\frac{1}{I}\int_{-T/2-r}^{T/2-r} [f\cdot\mathbb{W}_I]^\circ(t)\exp(-2\pi int/[I])\,dt = \widehat{f}*\widehat{\mathbb{W}_I}[n]\,. \tag{3}$$

Our general window function $\mathbb{W}_I$ is k-times differentiable, has $\text{supp}(\mathbb{W}_I) = [-T/2-r, T/2+r]$ and has values

$$\mathbb{W}_I = \begin{cases} 0 & |t| \ge T/2+r \\ 1 & |t| \le T/2-r \\ \rho(\pm t) & T/2-r < |t| < T/2+r. \end{cases} \tag{4}$$

We solve for $\rho(t)$ by solving the Hermite interpolation problem

$$\begin{cases} (a.) & \rho(T/2-r) = 1 \\ (b.) & \rho^{(n)}(T/2-r) = 0\,,\ n = 1, 2, \ldots, k \\ (c.) & \rho^{(n)}(T/2+r) = 0\,,\ n = 0, 1, 2, \ldots, k\,, \end{cases}$$

with the conditions that $\rho \in C^k$ and

$$[\rho(t)]^2 + [\rho(-t)]^2 = 1 \text{ for } t \in [\pm(\frac{T}{2}-r), \pm(\frac{T}{2}+r)]\,. \tag{5}$$

The constraint (5) directs us to get solutions expressed in terms of $\sin t$ and $\cos t$. Solving for ρ so that the window is in C^1, we get that $\rho(t)$ equals

$$\begin{cases} \sqrt{\left[1-\frac{1}{2}\left[1-\sin(\frac{\pi}{2r}(\frac{T}{2}-t))\right]^2\right]} & \frac{T}{2}-r \le t \le \frac{T}{2} \\ \frac{1}{\sqrt{2}}\left[1-\sin(\frac{\pi}{2r}(t-\frac{T}{2}))\right] & \frac{T}{2} \le t \le \frac{T}{2}+r\,. \end{cases} \tag{6}$$

With each degree of smoothness, we get an additional degree of decay in frequency.

We designed the ON window systems $\{\mathbb{W}_k(t)\}$ so that they would preserve orthogonality of basis element of overlapping blocks. Because of the partition properties of these systems, we need only check the orthogonality of adjacent overlapping blocks. The best way to think about the construction is to visualize how one would do the extension for a system of sines and cosines. We would extend the odd reflections about the left endpoint and the even reflections about the right. Let $\{\varphi_j(t)\}$ be an ON basis for $L^2[-T/2, T/2]$. Define

$$\widetilde{\varphi_j}(t) = \begin{cases} 0 & |t| \geq T/2 + r \\ \varphi_j(t) & |t| \leq T/2 - r \\ -\varphi_j(-T-t) & -T/2 - r < t < -T/2 \\ \varphi_j(T-t) & T/2 < t < T/2 + r\,. \end{cases} \tag{7}$$

Theorem 1: $\{\Psi_{k,j}\} = \{\mathbb{W}_k \widetilde{\varphi_j}\}$ is an ON basis for $L^2(\mathbb{R})$.
Proof : See [3]. □

B. Almost Orthogonal Systems

We approximate the Cap system with splines. We get windowing systems that nearly preserve orthogonality. Each added degree of smoothness in time adds to the degree of decay in frequency. We use B-splines as our cardinal functions. Let $0 < \alpha \ll \beta$ and consider $\chi_{[-\alpha,\alpha]}$. We want the n-fold convolution of $\chi_{[-\alpha,\alpha]}$ to fit in the interval $[-\beta, \beta]$. Then we choose α so that $0 < n\alpha < \beta$ and let

$$\Psi(t) = \underbrace{\chi_{[-\alpha,\alpha]} * \chi_{[-\alpha,\alpha]} * \cdots * \chi_{[-\alpha,\alpha]}(t)}_{n-times}\,.$$

The β-periodic continuation of this function, $\Psi^\circ(t)$ has the Fourier series expansion

$$\sum_{k\neq 0} \frac{\alpha}{n\beta} \left[\frac{\sin(\pi k\alpha/n\beta)}{2\pi k\alpha/n\beta} \right]^n \exp(\pi i k t/\beta)\,.$$

Cotlar, Knapp and Stein introduced *almost orthogonality* via operator inequalities [6]. The concept allows us to create windowing systems that are more computable/constructible using B-splines.

Definition 2 (Almost ON System): Let $0 < r \ll T$. An **Almost ON System** for adaptive and UWB sampling is a set of functions $\{\mathbb{A}_k(t)\}$ for which there exists δ, $0 \leq \delta < 1/2$, such that

$$\begin{aligned} &(i.) && \operatorname{supp}(\mathbb{A}_k(t)) \subseteq [kT - r, (k+1)T + r]\,, \\ &(ii.) && \mathbb{A}_k(t) \equiv 1 \text{ for } t \in [kT + r, (k+1)T - r]\,, \\ &(iii.) && \mathbb{A}_k((kT + T/2) - t) = \mathbb{A}_k(t - (kT + T/2))\,, \\ &(iv.) && 1 - \delta \leq [\mathbb{A}_k(t))]^2 + [\mathbb{A}_{k+1}(t))]^2 \leq 1 + \delta\,, \\ &(v.) && \{\widehat{\mathbb{A}_k{}^\circ}[n]\} \in l^1\,. \end{aligned}$$

Starting with Cap(t), let $\Delta_{(T,r)} = \frac{T+2r}{m}$. By placing equidistant knot points $-T/2 - r = x_0, -T/2 - r + \Delta_{(T,r)} = x_1, \ldots, T/2 + r = x_m$, we can construct C^{m-1} polynomial splines S_{m+1} approximating Cap(t) in $[(-T/2 - r), (T/2 + r)]$. A theorem of Curry and Schoenberg [15] gives that the set of B-splines $\{B^{(m+1)}_{-(m+1)}, \ldots, B^{(m+1)}_k\}$ forms a basis for S_{m+1}. Therefore, Cap$(t) \approx \sum_{i=-(m+1)}^k a_i B_i^{(m+1)}$. Let

$$\delta = \left\| \sum_{i=-(m+1)}^{k} a_i B_i^{(m+1)} - \text{Cap}(t) \right\|_\infty\,.$$

Then, $\delta < 1/2$, with the largest value for the piecewise linear spline approximation. Moreover, $\delta \to 0$ as m and k increase. Thus we get computable windowing systems that nearly preserve orthogonality. Each added degree of smoothness in time adds to the degree of decay in frequency.

III. Signal Expansions

Given characteristics of the class of input signals, the choice of basis functions used can be tailored to optimal representation of the signal or a desired characteristic in the signal.

Theorem 2 (Projection Formula for ON Windowing):
Let $\{\mathbb{W}_k(t)\}$ be an ON window system, and let $\{\Psi_{k,n}\} = \{\mathbb{W}_k \widetilde{\varphi_n}\}$ be an ON basis that preserves orthogonality between adjacent windows. Let $f \in \mathbb{PW}_\Omega$ and $N = N(T, \Omega)$ be such that $\langle f, \Psi_{k,n}\rangle = 0$ for all $n > N$ and all k. Then, $f(t) \approx f_{\mathcal{P}}(t)$, where

$$f_{\mathcal{P}}(t) = \sum_{k\in\mathbb{Z}} \left[\sum_{n=-N}^{N} \langle f, \Psi_{k,n}\rangle \Psi_{k,n}(t) \right]. \tag{8}$$

This theorem gives a new method for analog to digital conversion. Unlike the Shannon method which examined the function at specific points, then used those individual points to recreate the curve, the projection method breaks the signal into time blocks and then approximates their respective periodic expansions with a Fourier series. This process allows the system to individually evaluate each piece and base its calculation on the needed bandwidth. The individual Fourier series are then summed, recreating a close approximation of the original signal. It is important to note that instead of fixing T, the method allows us to fix any of the three while allowing the other two to fluctuate. From the design point of view, the easiest and most practical parameter to fix is N. For situations in which the bandwidth does not need flexibility, it is possible to fix Ω and T by the equation $N = \lceil T \cdot \Omega \rceil$. However, if greater bandwidth Ω is need, choose shorter time blocks T. The theory can use any ON basis. The basis can be chosen to optimize the analysis of a given class of signals, e.g., modified Gegenbauer (for UWB) or Walsh (for binary).

The analysis of the error generated by the projection method involves looking at the decay rates of the Fourier coefficients. If we are working with the standard basis, for $f \in C(\mathbb{T}_{2\Phi})$, we can define the modulus of continuity as

$$\mu(\delta) = \sup_{|x-y|\leq\delta} |f(x) - f(y)|\,.$$

This measures the local oscillation of the signal, and we have that

$$|\widehat{f}[n]| \leq \frac{1}{2}\mu(1/n)\,.$$

We say that f satisfies a Hölder condition with exponent α if there exists a constant K such that

$$|f(x+\delta) - f(x)| \leq K\delta^\alpha\,.$$

If f is k-times continuously differentiable and f^k satisfies a Hölder condition with exponent α, then there exists a constant K such that

$$|\widehat{f}[n]| \leq K \frac{1}{n^{k+\alpha}} .$$

The sharp cut-offs $\chi_{[kT,(k+1)T]}$ have a decay of only $\mathcal{O}(1/\omega)$ in frequency. We designed the ON windowing systems so that the windows have decay $\mathcal{O}(1/(\omega)^{k+2})$ in frequency. This makes the error on each block summable.

We assume $\mathbb{W}_k$ is C^k. Therefore, $\widehat{\mathbb{W}_k}(\omega) = \mathcal{O}(1/(\omega)^{k+2})$. We will analyze the error $\mathcal{E}_{k_\mathcal{P}}$ on a given block. Let $M = \|(f \cdot \mathbb{W}_k)\|_{L^2(\mathbb{R})}$. Then $\mathcal{E}_{k_\mathcal{P}}$

$$\begin{aligned} &= \sup\left|(f(t)\cdot \mathbb{W}_k) - \left[\sum_{n=-N}^{N} \langle f, \Psi_{n,k}\rangle \Psi_{n,k}(t)\right]\mathbb{W}_k(t)\right| \\ &= \sup\left[\sum_{|n|>N} \langle f, \Psi_{n,k}\rangle \Psi_{n,k}(t)\right]\mathbb{W}_k(t) \leq \sum_{|n|>N} \frac{M}{n^{k+2}} . \end{aligned}$$

IV. Modified Gegenbauer Systems

The Gegenbauer polynomials are used in an UWB communication system to construct pulses with narrow widths. The Gegenbauer waveform is used to modulate data, and has demonstrated superior performance to classic waveforms, e.g., Gaussian waveforms and the Hermite systems [10], [13].

Using the spirit of [10], [13] (and references therein), we define an ON basis for $L^2[-T/2, T/2]$ using modified Gegenbauer functions. Modified Gegenbauer functions are constructed using Gegenbauer polynomials (see [14, Chapter 18]). The Gegenbauer polynomials are modified so that they zero-out at the endpoints, and normalized to create an ON system. This then allows UWB signals to be expanded in the projection method (8) using the modified Gegenbauer system.

The Gegenbauer polynomials $C_n^\nu : \mathbb{C} \to \mathbb{C}$ are orthogonal over $(-1, 1)$ with orthogonality relation given by [14, Table 18.3.1]

$$\int_{-1}^{1} C_n^\nu(x) C_m^\nu(x) w(x;\nu) dx = h_n^\nu \delta_{n,m}, \tag{9}$$

for $\nu \in (-\frac{1}{2}, \infty) \setminus \{0\}$, where

$$w(x;\nu) := (1-x^2)^{\nu-1/2}, \tag{10}$$

$$h_n^\nu := \frac{2^{1-2\nu}\pi\Gamma(2\nu+n)}{(\nu+n)\Gamma^2(\nu)n!}, \tag{11}$$

the gamma function $\Gamma : \mathbb{C} \setminus -\mathbb{N}_0 \to \mathbb{C}$ is defined in [14, Chapter 5], and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. The Gegenbauer polynomials are defined using the Gauss hypergeometric function [14, (18.5.9)] as

$$C_n^\nu(x) := \frac{(2\nu)_n}{n!} {}_2F_1\left(\begin{array}{c} -n, 2\nu+n \\ \nu+\frac{1}{2} \end{array}; \frac{1-x}{2}\right),$$

where the Pochhammer symbol $(\cdot)_n : \mathbb{C} \to \mathbb{C}$ for $n \in \mathbb{N}_0$ is defined by $(a)_n := (a)(a+1)\cdots(a+n-1)$, and the Gauss hypergeometric function is defined in [14, Chapter 15]. They have a Rodrigues-type formula [14, Table 18.5.1]

$$C_n^\nu(x) := \frac{(-1)^n (2\nu)_n}{2^n(\nu+\frac{1}{2})_n n!} \frac{1}{w(x;\nu)} \frac{d^n}{dx^n} w(x;\nu+n),$$

and can also be computed using three-term recurrence relations [14, Table 18.9.1], or using trigonometric [11, p. 220] series expressions. Note that Gegenbauer polynomials can be given in terms of the more general Jacobi polynomials symmetric in parameters with [14, (18.7.1)]

$$C_n^\nu(x) = \frac{(2\nu)_n}{(\nu+\frac{1}{2})_n} P_n^{(\nu-1/2,\nu-1/2)}(x).$$

Consider the modified Gegenbauer function $\mathcal{C}_n^\nu : [-T/2, T/2] \times (0, \infty) \to \mathbb{R}$ defined by

$$\mathcal{C}_n^\nu(t;T) := \sqrt{\frac{2\, w\left(\frac{2t}{T};\nu\right)}{T h_n^\nu}}\, C_n^\nu\left(\frac{2t}{T}\right).$$

It is easy to see from (9) that these functions form an ON basis for $L^2[-T/2, T/2]$ with $\nu \in (\frac{1}{2}, \infty)$, namely

$$\int_{-T/2}^{T/2} \mathcal{C}_n^\nu(t;T)\mathcal{C}_m^\nu(t;T) dt = \delta_{m,n}.$$

Note that we exclude the parameters $\nu \in (-1/2, 1/2]$ in order to keep the endpoints $\pm L/2$ in the domain of integration. By using (10), (11) one has

$$\begin{aligned} \mathcal{C}_n^\nu(t;T) = \frac{2^{2\nu-1/2}\Gamma(\nu)}{T^\nu} &\sqrt{\frac{(n+\nu)n!}{\pi\Gamma(2\nu+n)}} \\ &\times \left(\left(\tfrac{T}{2}\right)^2 - t^2\right)^{\nu/2-1/4} C_n^\nu\left(\frac{2t}{T}\right). \end{aligned}$$

We will compute $\mathcal{E}_{k_\mathcal{P}}$ in terms of the modified Gegenbauer system. The decay of the system is key to the improved performance of this system for UWB signals [10], [13].

V. Biorthogonal and Frame Constructions

The collection $\{\Psi_{k,j}\} = \{\mathbb{W}_k \widetilde{\varphi_j}(t)\}$ forms an ON basis for $L^2(\mathbb{R})$. In this section, we develop the biorthogonal basis to $\{\Psi_{k,j}\}$. We can write down the folding operation used to create the ON windows as an operator [9]. Recall that an operator U is *unitary* if its transpose is its inverse, i.e., $U^* = U^{-1}$. Folding is a unitary operator. Fix a point α in $\mathbb{R}$, and define the reflection, or mirror in α as $\mathcal{M}_\alpha f(t) = f(2\alpha - t)$. Let $\chi_\alpha^l = \chi_{(-\infty,\alpha]}$ and $\chi_\alpha^r = \chi_{(\alpha,\infty)}$ be the left and right cutoff functions, and let ρ_α^u and ρ_α^d be the up and down ramp functions. In terms of these operators, $\mathcal{M}_0\mathbb{W}_I(t) = \mathbb{W}_I(t)$ and $\mathcal{M}_0\rho^u(t) = \rho^d(t)$.

Definition 3 (Folding Operation): The **folding operation** about a point α is given by

$$\mathcal{F}_\alpha = \chi_\alpha^l(1+\mathcal{M}_\alpha)\rho_\alpha^u + \chi_\alpha^r(1-\mathcal{M}_\alpha)\rho_\alpha^d . \tag{12}$$

Lemma 1: The folding operator is unitary if and only if the sin-cos condition holds, i.e.,

$$(\rho_\alpha^u)^2 + (\rho_\alpha^d)^2 = 1 . \tag{13}$$

The collection $\{\Psi_{k,j}\} = \{\mathbb{W}_k\widetilde{\varphi_j}(t)\}$ forms an ON basis for $L^2(\mathbb{R})$. Therefore, it has a biorthogonal Riesz basis $\{\Psi^*_{k,j}\}$. The basis is uniquely determined by the biorthogonality relationship

$$\langle\{\Psi_{m,n}\},\{\Psi^*_{k,j}\}\rangle = \delta_{m,k}\cdot\delta_{n,j}.$$

Theorem 3 (Biorthogonal Basis): The basis $\{\Psi_{k,j}\} = \{\mathbb{W}_k\widetilde{\varphi_j}(t)\}$ has a unique biorthogonal basis $\{\Psi^*_{k,j}\}$, with uniqueness given by the biorthogonality relationship $\langle\{\Psi_{m,n}\},\{\Psi^*_{k,j}\}\rangle = \delta_{m,k}\cdot\delta_{n,j}$. The basis $\{\Psi^*_{k,j}\}$ is given by $\{\Psi_{k,j}\} = \{\widetilde{\mathbb{W}_k\widetilde{\varphi_j}(t)}\}$, where $\widetilde{\mathbb{W}_k}$ is the translation of the window

$$\widetilde{\mathbb{W}_I} = \begin{cases} 0 & |t| \geq T/2+r\,, \\ 1 & |t| \leq T/2-r\,, \\ \frac{\rho_0^u(t)}{{\rho_0^u}^2+{\rho_0^d}^2} & -T/2-r < t < -T/2+r\,, \\ \frac{\rho_0^d(t)}{{\rho_0^u}^2+{\rho_0^d}^2} & T/2-r < t < T/2+r\,. \end{cases} \quad (14)$$

The windowing systems above allow us to develop *Signal Adaptive Frame Theory*. The idea is as follows. If we work with an ON windowing system $\{\mathbb{W}_k(t)\}$, let $\{\Psi_{k,j}\}$ be an ON basis that preserves orthogonality between adjacent windows. Let $f\in\mathbb{PW}_\Omega$ and $N = N(T,\Omega)$ be such that $\langle f\cdot\mathbb{W}_k,\Psi_{k,n}\rangle = 0$ for all $n > N$ and all k. Then

$$f(t) = \sum_{k\in\mathbb{Z}}\left[\sum_{n\in\mathbb{Z}}\langle f\cdot\mathbb{W}_k,\Psi_{k,n}\rangle\Psi_{k,n}(t)\right]. \quad (15)$$

This also gives

$$\|f\|^2 = \sum_{k\in\mathbb{Z}}\left[\sum_{n\in\mathbb{Z}}|\langle f\cdot\mathbb{W}_k,\Psi_{k,n}\rangle|^2\right]. \quad (16)$$

Given that $\{\Psi_{k,j}\} = \{\mathbb{W}_k\widetilde{\varphi_j}(t)\}$ is an ON basis for $L^2(\mathbb{R})$, we have a representation of a given function f in L^2. The set $\{\Psi_{k,j}\} = \{\mathbb{W}_k\widetilde{\varphi_j}(t)\}$ is an exact normalized tight frame for L^2. The restriction that these basis elements present is computability. They become increasingly difficult to compute as the smoothness in time/decay in frequency increases.

A way around this is to use B-spline theory and frame theory. The ideas behind this connection go back to the curvelet work of Candès and Donoho. The paper of Borup and Neilsen [2] gives a nice overview of this connection, and we will refer to that paper for the background from which we develop our approach. The set $\{\mathbb{B}_k(t)\}$ form an *admissible* cover, in that they form a partition of unity and have overlap with only their immediate neighbors.

For each window $\mathbb{B}_k(t)$, let $\phi_{n,k}(t)$ be the shifted $\exp[\pi i t T/n]$ centered in the window. Then define $\Phi_{k,n} = \mathbb{B}_k(t)\phi_{k,n}(t)$. Given $f\in L^2$, we can write

$$f(t) \approx \sum_{k\in\mathbb{Z}}\left[\sum_{n\in\mathbb{Z}}\langle f\cdot\mathbb{B}_k,\Phi_{k,n}\rangle\Phi_{k,n}(t)\right]. \quad (17)$$

For this system we can compute

$$A\|f\| \leq \sum_{k\in\mathbb{Z}}\left[\sum_{n\in\mathbb{Z}}|\langle f\cdot\mathbb{B}_k,\Phi_{k,n}\rangle|^2\right] \leq B\|f\|\,.$$

The signal will be underrepresented on some blocks, over-represented on others. This is a function of how much of the signal is concentrated in the overlap regions. The frame bounds will be tightened for the almost orthogonal windowing systems. The fact that the almost ON windows $\{\mathbb{A}_k(t)\}$ approximate the ON windowing system will result in approximating the expansion of the signal contained in the overlapping region in an ON basis. The closer the approximation, the better the frame bounds. Developing these signal adaptive frames, their bounds and the associated frame operators will be a major point of emphasis in future work. We conjecture the following:

$$\mathcal{A}_{1-\delta}\|f\|^2 \leq \sum_{k\in\mathbb{Z}}\left[\sum_{n\in\mathbb{Z}}|\langle f\cdot\mathbb{A}_k,\Psi_{k,n}\rangle|^2\right] \leq \mathcal{A}_{1+\delta}\|f\|^2\,,$$

and that this goes to normalized tight frame as $\delta\to 0$.

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References

[1] J. J. Benedetto, *Harmonic Analysis and Applications*, CRC Press, Boca Raton, FL, 1997.
[2] L. Borup and M. Neilsen, "Frame Decomposition of Decomposition Spaces" *Jour. Fourier Anal. Appl.* **13** (1), 39–70, 2007.
[3] S. D. Casey, "Adaptive Signal Processing," to appear in *Excursions in Harmonic Analysis*, Springer-Birkhäuser Applied and Numerical Harmonic Analysis Series, 29 pp., 2015.
[4] S. D. Casey, S. Hoyos, and B. M. Sadler, "Adaptive and ultra-wideband sampling via signal segmentation and projection," to appear in *Proc. IEEE*, 24 pp., 2015.
[5] R. Coifman and Y. Meyer, "Remarques sur l'analyse de Fourier a fenetre." *CR Acad. Sci. Paris* **312**, 259–261, 1991.
[6] Grafakos, L.: Classical and Modern Fourier Analysis. Pearson Education, Upper Saddle River, NJ (2004)
[7] S. Hoyos and B. M. Sadler, "Ultra wideband analog-to-digital conversion via signal expansion," *IEEE Transactions on Vehicular Technology*, **54** (5), 1609–1622, 2005.
[8] S. Hoyos and B. M. Sadler, "UWB mixed-signal transform-domain direct-sequence receiver," *IEEE Transactions on Wireless Communications*, **6** (8) 3038–3046, 2007.
[9] B. Jawerth and W. Sweldens, "Biorthogonal smooth local trigonometric bases," *Jour. Fourier Anal. Appl.*, **2** (2), 109–133, (1995)
[10] A. Lamari, F. Elbahhar, A. Rivenq, J.M. Rouvaen, and M. Heddebaut, "Performance Evaluation of a Multi-band UWB Location and Communication System Based on Modified Gegenbauer Functions," *Wireless Personal Communications*, **48** (6), 361–384, 2009.
[11] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and theorems for the special functions of mathematical physics (Third enlarged edition)*, Die Grundlehren der mathematischen Wissenschaften, Band 52, Springer-Verlag New York, Inc., New York 1966.
[12] H. S. Malvar, "Biorthogonal and nonuniform lapped transforms for transform coding with reduced blocking and ringing artifacts," *IEEE Trans. Signal Process.*, **46** (4) 1043-1053, 1998.
[13] A. Okassa M'foubat, C. Tatkeu, and F. Elbahhar, "UWB System Based on Modified Gegenbauer Function in the MISO Channel," *Wireless Engineering and Technology*, **4**, 117–123, 2013.
[14] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST handbook of mathematical functions*, Cambridge University Press, Cambridge, 2010.
[15] I. J. Schoenberg, *Cardinal Spline Interpolation* (CBMS–NSF Conference Series in Applied Mathematics, 12), SIAM, Philadelphia, PA, 1973.
[16] R. Young, *An Introduction to Nonharmonic Fourier Series* Academic Press, New York, 1980.