EXTREMAL THEOREMS FOR DEGREE SEQUENCE PACKING AND THE 2-COLOR DISCRETE TOMOGRAPHY PROBLEM

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Abstract. A sequence $\pi = (d_1, \ldots, d_n)$ is graphic if there is a simple graph G with vertex set $\{v_1, \ldots, v_n\}$ such that the degree of v_i is d_i. We say that graphic sequences $\pi_1 = (d_1^{(1)}, \ldots, d_n^{(1)})$ and $\pi_2 = (d_1^{(2)}, \ldots, d_n^{(2)})$ *pack* if there exist edge-disjoint n-vertex graphs G_1 and G_2 such that for $j \in \{1,2\}$, $d_{G_j}(v_i) = d_i^{(j)}$ for all $i \in \{1, \ldots, n\}$. Here, we prove several extremal degree sequence packing theorems that parallel central results and open problems from the graph packing literature. Specifically, the main result of this paper implies degree sequence packing analogues to the Bollobás-Eldridge-Catlin graph packing conjecture and the classical graph packing theorem of Sauer and Spencer.

In discrete tomography, a branch of discrete imaging science, the goal is to reconstruct discrete objects using data acquired from low-dimensional projections. Specifically, in the k-color discrete tomography problem the goal is to color the entries of an $m \times n$ matrix using k colors so that each row and column receive a prescribed number of entries of each color. This problem is equivalent to packing the degree sequences of k bipartite graphs with parts of sizes m and n. Here we also prove several Sauer-Spencer-type theorems with applications to the 2-color discrete tomography problem.

Key words. degree sequence, discrete tomography, packing

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1. Introduction.

A sequence of nonnegative integers $\pi = (d_1, d_2, ..., d_n)$ is *graphic* if there is a (simple) graph G of order n having degree sequence π. In this case, G is said to *realize* or be a *realization of* π, and we write $\pi = \pi(G)$. If a sequence π consists of the terms $d_1, \ldots, d_t$ having multiplicities $\mu_1, \ldots, \mu_t$, then we may write $\pi = ({d_1}^{\mu_1}, \ldots, {d_t}^{\mu_t})$.

There are a number of necessary and sufficient conditions for a sequence to be graphic, including the seminal Havel-Hakimi Algorithm [21, 23] and the Erdős-Gallai Criteria [16]. However, a given graphic sequence may have a large family of nonisomorphic realizations, and as such considerable attention has been given to the study of when a graphic sequence has a realization with a given property. Such problems can be divided into two broad classes, described as "forcible" problems and "potential" problems in [30]. Given a graph property $\mathcal{P}$, we say that a graphic sequence π is *forcibly* $\mathcal{P}$*-graphic* if every realization of π has property $\mathcal{P}$, and that π is *potentially* $\mathcal{P}$*-graphic* if at least one realization of π has property $\mathcal{P}$.

Results on forcible degree sequences are often stated as traditional problems in structural or extremal graph theory, where a necessary and/or sufficient condition is given in terms of the degrees of the vertices (or equivalently the number of edges) of a given graph. For instance, minimum degree thresholds for the existence of certain graph structures, such as the threshold for hamiltonicity in Dirac's Theorem [12], can be thought of as forcible theorems. Two older, but exceptionally thorough surveys on

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forcible and potential problems are due to Hakimi and Schmeichel [22] and Rao [31], and a more recent survey on forcible "Chvátal-Type" theorems (in the spirit of [9]) is due to Bauer et al. [3].

A number of degree sequence analogues to classical problems in extremal graph theory appear throughout the literature, including potentially graphic sequence variants of Hadwiger's Conjecture [15, 32], graph Ramsey numbers [6] and the Turán problem (c.f. [17]). In this paper, we consider an extension of the classical graph packing literature to degree sequences. In particular, we prove a potentially $\mathcal{P}$-graphic analogue to a widely-studied graph packing conjecture of Bollobás and Eldridge [4] and, independently, Catlin [8], which implies a graphic sequence version of the Sauer-Spencer graph packing theorem [33]. We conclude by using similar techniques to prove a pair of related results that have applications to discrete imaging science.

1.1. Graph Packing. Two n-vertex graphs G_1 and G_2 *pack* if G_1 is a subgraph of $\overline{G_2}$, or alternatively if G_1 and G_2 can be expressed as edge-disjoint subgraphs of K_n, the complete graph on n vertices. Graph packing has received a great deal of attention in the literature ([26], [35] and [36] are detailed and useful surveys).

In 1978, Sauer and Spencer [33] proved the following classical theorem.

THEOREM 1.1. *Let G_1 and G_2 be graphs of order n with maximum degree Δ_1 and Δ_2 respectively. If*

$$\Delta_1\Delta_2 < \frac{n}{2},$$

then G_1 and G_2 pack.

Likely the most notable open conjecture in graph packing is due to Bollobás and Eldridge [4] and, independently, Catlin [8].

CONJECTURE 1. *Let G_1 and G_2 be n-vertex graphs with maximum degrees $\Delta(G_i) = \Delta_i$ for $i = 1, 2$. If*

$$(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$$

then G_1 and G_2 pack.

If true, Conjecture 1 implies Theorem 1.1. The Bollobás-Eldridge-Catlin conjecture has been settled in several cases, including when $\Delta_1 \leq 2$ by Aigner and Brandt [1] and Alon and Fisher [2]. The case when $\Delta_1 = 3$ was shown by Csaba, Shokoufandeh, and Szemerédi [11] for large n utilizing the regularity lemma. For $\Delta_1, \Delta_2 \geq 300$, Kaul, Kostochka and Yu [25] showed that $(\Delta_1+1)(\Delta_2+1) \leq 0.6n+1$ implies that the two graphs pack, which improves the Sauer-Spencer theorem, and is a partial solution to Conjecture 1. Other partial results were obtained by Corrádi and Hajnal [10] and Hajnal and Szemerédi [20].

1.2. Packing Graphic Sequences. The notion of packing graphic sequences was investigated in [7], where the following key definition appears. If π_1 and π_2 are (not necessarily monotone) graphic sequences, with $\pi_1 = (d_1^{(1)}, \ldots, d_n^{(1)})$ and $\pi_2 = (d_1^{(2)}, \ldots, d_n^{(2)})$, then π_1 and π_2 *pack* if there exist edge-disjoint graphs G_1 and G_2, both with vertex set $\{v_1, \ldots, v_n\}$, such that

$$d_{G_1}(v_i) = d_i^{(1)} \quad \text{and} \quad d_{G_2}(v_i) = d_i^{(2)}.$$

It is critical to note here that the order of the terms in π_1 and π_2 is fixed, so that the statement "π_1 and π_2 pack" is not equivalent to "π_1 and π_2 have realizations that pack". This framework allows for some interesting distinctions between packing graphs and packing graphic sequences. On the other hand, by fixing the ordering of π_1 and π_2, the study of degree sequence packing provides insight into how a pair of graphs with these degree sequences might feasibly pack, if in fact they do.

Given a sequence π, let $\Delta(\pi)$ and $\delta(\pi)$ denote the maximum and minimum terms in π, respectively. Further, given two sequences π_1 and π_2 of the same length, let $\pi_1 + \pi_2$ denote the "vector sum" of π_1 and π_2. One of the main results from [7] is the following.

THEOREM 1.2. *Let π_1 and π_2 be n-term graphic sequences with $\Delta = \Delta(\pi_1 + \pi_2)$ and $\delta = \delta(\pi_1 + \pi_2)$. If*

$$\Delta \leq \sqrt{2\delta n} - (\delta - 1),$$

then π_1 and π_2 pack, except that strict inequality is required when $\delta = 1$. This result is sharp for all n and δ.

As was noted in [7], this theorem can be viewed as an "additive" analogue to the Sauer-Spencer theorem, since $\Delta_1 + \Delta_2 < \sqrt{2n}$ implies that $\Delta_1\Delta_2 < \frac{n}{2}$. We modify and strengthen the techniques introduced in the proof of Theorem 1.2 to obtain our main results here.

1.3. Statement of Main Results. Throughout the statement and proof of the following results, given graphic sequences π_1 and π_2 we let $\Delta_i = \Delta(\pi_i)$ and $\delta_i = \delta(\pi_i)$ for $i \in \{1, 2\}$. Our main result is as follows.

THEOREM 1.3. *Let π_1 and π_2 be graphic sequences with $\Delta_2 \geq \Delta_1$ and $\delta_1 \geq 1$. If*

$$\begin{cases} (\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \delta_1 n + 1 & \text{when } \Delta_2 + 2 \geq \Delta_1 + \delta_1 \\ \frac{(\Delta_2 + 1 + \Delta_1 + \delta_1)^2}{4} \leq \delta_1 n + 1 & \text{when } \Delta_2 + 2 < \Delta_1 + \delta_1, \end{cases}$$

then π_1 and π_2 pack.

Theorem 1.3 holds regardless of the orderings of π_1 and π_2, although these orderings are fixed. Given this, we cannot assume that $\delta(\pi_1) = \delta(\pi_2) = 0$, as it would be possible to order π_1 and π_2 so that the zero terms correspond, which would impact the relative strength of the hypothesis. It seems feasible that the conditions that $\Delta_1 \leq \Delta_2$ and $\delta(\pi_1) \geq 1$ could be replaced by the weaker hypothesis that $\delta(\pi_1 + \pi_2) \geq 1$, although we are unable to obtain such a result at this time.

Theorem 1.3 implies the following analogue to the Bollobás-Eldridge-Catlin conjecture.

COROLLARY 1.4. *Let π_1 and π_2 be graphic sequences with $\Delta_2 \geq \Delta_1$ and $\delta_1 \geq 1$. If*

$$(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1,$$

then π_1 and π_2 pack. This result is best possible.

Much as the Bollobás-Eldridge-Catlin conjecture implies the Sauer-Spencer theorem, we also obtain the following.

COROLLARY 1.5. *Let π_1 and π_2 be graphic sequences with $\Delta_2 \geq \Delta_1$ and $\delta_1 \geq 1$. If $\Delta_1\Delta_2 < \frac{n}{2}$, then π_1 and π_2 pack. This result is best possible.*

1.4. Sharpness. In [24], Kaul and Kostochka characterized the sharpness examples for Theorem 1.1. Specifically, graphs G_1 and G_2 satisfying $\Delta_1\Delta_2 = \frac{n}{2}$ pack, unless n is even, G_1 is a matching of size $\frac{n}{2}$, and either $\frac{n}{2}$ is odd and $G_2 = K_{\frac{n}{2},\frac{n}{2}}$, or G_2 is any graph that contains $K_{\frac{n}{2}+1}$ as a component.

In a similar manner, to see that Corollaries 1.4 and 1.5 are sharp, let n be even and consider $\pi_1 = (1^n)$ and $\pi_2 = (\frac{n}{2}^{\frac{n+2}{2}}, 0^{\frac{n-2}{2}})$. These sequences are uniquely realized as a perfect matching and $K_{\frac{n}{2}+1} \cup \left(\frac{n}{2} - 1\right) K_1$, which do not pack, regardless of the orderings of π_1 and π_2. The proof of the following theorem is inherent in the proofs of Corollaries 1.4 and 1.5, so we omit the proof in the interest of concision.

THEOREM 1.6. *Theorem 1.3 is strictly stronger than Corollary 1.4 unless* $\delta_1 = 1$. *Further, Corollary 1.4 is strictly stronger than Corollary 1.5 unless* $\Delta_1 = \delta_1 = 1$.

A *k-factor* of a graph G is a spanning k-regular subgraph of G. Kundu's k-factor Theorem [28], proved independently by Lovász for $k = 1$ [29], states that a graphic sequence $\pi = (d_1, \dots, d_n)$ has a realization containing a k-factor if and only if $\pi' = (d_1 - k, \dots, d_n - k)$ is also graphic. Together with Theorem 1.6, this allows us to partially characterize the sharpness of Corollary 1.4 and completely characterize the sharpness of Corollary 1.5. The latter characterization is analogous to the characterization for graph packing from [24].

THEOREM 1.7. *Let π_1 and π_2 be graphic sequences with $\Delta_2 \geq \Delta_1$ and $\delta_1 \geq 1$.*

(a) If $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 2$ and $\delta_1 \neq 1$, then π_1 and π_2 pack.

(b) If $\Delta_1 \Delta_2 = \frac{n}{2}$, then π_1 and π_2 pack unless $\Delta_1 = 1$ and $\pi_1 + \pi_2$ is not graphic.

2. Proofs of Theorem 1.3 and Corollaries 1.4 and 1.5. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be graphs. We say a vertex pair (x, y) is a *bad pair for* (G_1, G_2) or a (G_1, G_2)*-bad pair* if $xy \in E_1 \cap E_2$. Let $b(G_1, G_2)$ denote the number of (G_1, G_2)-bad pairs. We begin by proving Theorem 1.3.

Proof. [Proof of Theorem 1.3] Let π_1 and π_2 be graphic sequences that do not pack. Choose $G_1 = G(\pi_1)$ and $G_2 = G(\pi_2)$ to have the fewest bad pairs among all realizations of π_1 and π_2 and let $G = G_1 \cup G_2$. For a given (G_1, G_2)-bad pair (x, y) we define $I(x, y) = V - (N_G(x) \cup N_G(y))$. Among all choices of G_1 and G_2 that minimize $b(G_1, G_2)$, choose G_1, G_2 and a bad pair (x, y) such that the size of $I = I(x, y)$ is maximum. For $i \in \{1, 2\}$, let $Q_i(y)$ be $N_{G_i}(y) - N_G[x]$ and define $Q_i(x)$ similarly. If either $Q_1(x)$ or $Q_1(y)$ is nonempty, assume without loss of generality that $|Q_1(x)| \leq |Q_1(y)|$. Otherwise, if both $Q_1(x)$ and $Q_1(y)$ are empty, then assume without loss of generality that $|Q_2(x)| \leq |Q_2(y)|$.

Throughout the proof we will make use of the following sets. First, let $\overline{Y} = V(G) - N_G[y]$. Define A to be a subset of $N_{G_1}(\overline{Y})$ such that every vertex of A has at least two neighbors in G_1 in $\overline{Y}$. Finally, let $B = N_{G_1}(\overline{Y}) - A$ and $R = A \cup \{v \in N_G[y] : A \subseteq N_G(v)\}$.

We prove Theorem 1.3 by counting the number of edges in G_1 between R and $V(G) - R$ to reach a contradiction. In order to gain the desired count, we first show particular edge structures in I, $\overline{Y}$, and $N_{G_1}(\overline{Y})$. We then show that A is not empty and further that R is a vertex cover of G_1.

We proceed by proving a sequence of claims, the first of which follows immediately from the straightforward fact that $4xy \leq (x + y)^2$ for all real x and y.

CLAIM 1. $(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \frac{1}{4}(\Delta_2 + 1 + \Delta_1 + \delta_1)^2$.

CLAIM 2. *If u and v are vertices in G such that xu and yv are not in $E(G)$, then uv is not in $E(G)$.*

Proof. Assume otherwise, and without loss of generality let uv be an edge of G_1. We may then exchange the edges xy and uv with the non-edges xu and yv in G_1 to create another realization of π_1. Since xu and yv are not in G, this reduces the number of bad pairs, a contradiction. □

Claim 2 immediately implies that I is an independent set in G.

CLAIM 3. $\overline{Y} \neq \emptyset$.

Proof. Toward contradiction, suppose that $N_G[y] = V(G)$. Thus $|N_G[y]| = n$, and therefore $\Delta_1 + \Delta_2 \geq n$. By assumption, $(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \delta_1 n + 1$, which implies that

$$(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \delta_1(\Delta_1 + \Delta_2) + 1.$$

Expanding and rearranging, this yields

$$0 \leq \Delta_1(\delta_1 - 1 - \Delta_2) - \delta_1 + 1.$$

However, $\delta_1 - 1 - \Delta_2 < 0$ and $-\delta_1 + 1 \leq 0$, a contradiction. Consequently, $N_G[y] \neq V(G)$. $\square$

Claim 4. *$\overline{Y}$ is independent in G_1.*

Proof. Otherwise, suppose there are vertices u and v in $\overline{Y}$ that form an edge in G_1. By Claim 2, both u and v must be adjacent to x. If there is some vertex $z \in Q_1(y)$, then removing the edges uv, xy and yz from G_1 and adding the non-edges yu, yv and xz to G_1 would create another realization of G_1' of π_1 such that $b(G_1', G_2) < b(G_1, G_2)$. If $Q_1(y)$ is empty, then since $|Q_1(x)| \leq |Q_1(y)|$, we have that $Q_1(x)$ is also empty, and therefore since we have assumed $|Q_2(x)| \leq |Q_2(y)|$ and $u \in Q_2(x)$, there is some z in $Q_2(y)$. We then exchange the edges yz and xu in G_2 and the edges uv and xy in G_1, for the non-edges yu and xz in G_2 and the non-edges xu and yv in G_1 to again create realizations of π_1 and π_2 with fewer than $b(G_1, G_2)$ bad pairs. Thus, $\overline{Y}$ is independent in G_1. $\square$

Claim 5. *$N_{G_1}(\overline{Y}) \cup \{x, y\}$ is a clique in G.*

Proof. Let $u \in \overline{Y}$ and $w \in N_{G_1}(u)$. By Claim 4, $w \notin \overline{Y}$ and therefore $w \in N_G[y]$. If $w \neq x$, then since $uy \notin E(G)$, by Claim 2, $wx \in E(G)$. Thus, $N_{G_1}(u) \subseteq N_G[x] \cap N_G(y)$.

Consequently, suppose $w, w' \in N_{G_1}(\overline{Y})$ are such that $ww' \notin E(G)$. Let $u \in N_{G_1}(w) \cap \overline{Y}$ and $u' \in N_{G_1}(w') \cap \overline{Y}$ (u and u' need not be distinct). Note that without loss of generality $x \neq w$ since $xw' \in E(G)$. If $u \in I$, then replacing the edges uw, $u'w'$ and xy in G_1 with the non-edges xu, yu' and ww' contradicts the minimality of $b(G_1, G_2)$. Thus $u \notin I$, and likewise $u' \notin I$.

Next, assume there is some $z \in Q_1(y)$. By Claim 2, $uz \notin E(G)$. Remove the edges wu, $w'u'$ and yz from G_1 and add the edges ww', yu' and zu to create a realization G_1' of π_1 with $b(G_1', G_2) = b(G_1, G_2)$. However, neither x nor y are adjacent to vertices in $\{z\} \cup I(x, y)$, which contradicts the maximality of I.

It remains to consider the case where $Q_1(y) = \emptyset$. Similar to the proof of Claim 4, since $Q_1(y)$ is empty, $Q_1(x)$ is empty, therefore $u, u' \in Q_2(x)$ and there must be a vertex z in $Q_2(y)$. Also note that since $u, u' \in Q_2(x)$ the edges xu and xu' are in G_2. Exchanging the edges wu, $w'u'$ and xy in G_1 with ux and the non-edges $u'y$ and ww' creates another realization G_1' of π_1 such that (u, x) is a (G_1', G_2)-bad pair and $b(G_1', G_2) = b(G_1, G_2)$. However, by Claim 2 u is not adjacent to vertices in $\{z\} \cup I(x, y)$, and x is not adjacent to vertices in $\{z\} \cup I(x, y)$. Therefore $I(u, x) > I(x, y)$. Hence, $N_{G_1}(\overline{Y}) \cup \{x, y\}$ is a clique in G. $\square$

Claim 6. *$A \neq \emptyset$.*

Proof. For sake of contradiction, suppose A is empty, and therefore $N_{G_1}(\overline{Y}) = B$. Since $\overline{Y}$ is independent in G_1 we have that $\delta_1 |\overline{Y}| \leq |B|$. Thus,

$$n = |\overline{Y}| + |N_G[y]| \leq \frac{|B|}{\delta_1} + \Delta_1 + \Delta_2.$$

We proceed by showing that $|B| \leq \Delta_1 + \Delta_2 - 2$, which establishes the desired contradiction. By the definition of $\overline{Y}$, y is not adjacent to vertices in $\overline{Y}$, and therefore $y \notin B$. If $x \notin B$, then $|B| \leq |N_G(y)| - |\{x\}| \leq \Delta_1 + \Delta_2 - 2$. If $x \in B$, then since x has a neighbor in $\overline{Y}$, $Q_1(x) \neq \emptyset$. By assumption $|Q_1(x)| \leq |Q_1(y)|$, thus there is some vertex z in $N_G[y]$ not adjacent to x. Now we have that $|B| \leq |N_G[y] - \{y, z\}| \leq \Delta_1 + \Delta_2 - 2$. Inserting this upper bound of $|B|$ into the above inequality we have that

$$\delta_1 n + 1 \leq (\delta_1 + 1)(\Delta_1 + \Delta_2) - 1.$$

By Claim 1,

$$(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \frac{(\Delta_2 + 1 + \Delta_1 + \delta_1)^2}{4}.$$

so that the hypothesis of the theorem yields

$$(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq (\delta_1 + 1)(\Delta_1 + \Delta_2) - 1,$$

which implies

$$\Delta_2(\Delta_1 - 1) < \delta_1(\Delta_1 - 1),$$

so that either $0 < 0$, if $\Delta_1 = 1$, or $\delta_1 > \Delta_2$, a contradiction. □

By Claim 6, $A \neq \emptyset$ and by Claim 5, $N_{G_1}(\overline{Y}) \cup \{x, y\}$ is a clique in G and therefore $N_{G_1}(\overline{Y}) \cup \{x, y\} \subseteq R$.

Claim 7. *Every edge of G_1 is incident with R.*

Proof. Towards contradiction let zz' be an edge of G_1 not incident with R. By Claim 4 we know that z and z' must be in $N_G[y] - R$, so there exist vertices w and w' (not necessarily distinct) in A which are not adjacent to z and z' (respectively). Also, we have distinct vertices u and u' in $\overline{Y}$ such that wu and $w'u'$ are edges in G_1.

We can remove the edges zz', uw and $u'w'$ from G_1 and add the non-edges wz, $w'z'$ and uu' to form a realization G_1' of π_1. It is possible that, via this edge-exchange, (u, u') is a bad pair of (G_1', G_2), implying that $b(G_1', G_2) = b(G_1, G_2) + 1$. However, the sets $Q_1(x)$, $Q_2(x)$, $Q_1(y)$ and $Q_2(y)$ are not affected by these exchanges. Now, $\overline{Y}$ is no longer independent in G_1 and (x, y) is still a bad pair. As in the proof of Claim 4 we now exchange edges to obtain a realization G_1'' of π_1 such that (x, y) and (u, u') are no longer bad pairs and no other bad pairs are created. Thus, $b(G_1'', G_2) < b(G_1, G_2)$, a contradiction.

Therefore R is a vertex cover of G_1, as desired. □

We conclude the proof by finding lower and upper bounds on the number of edges in G_1 between R and $V - R$, which we denote by $e_1(R, V - R)$. The necessary lower bound follows easily from the assertion that $V - R$ is independent in G_1

$$\delta_1(n - |R|) \leq e_1(R, V - R).$$

While $\Delta_1|R|$ is a straightforward upper bound for $e_1(R, V - R)$, we require a stronger bound to obtain the desired result.

Suppose $|R| \leq \Delta_2 + 1$. Since $\{x, y\} \subseteq R$, in G_1 both x and y have at most $\Delta_1 - 1$ neighbors in $V - R$. The remaining vertices of R each have at most Δ_1 neighbors in $V - R$. Thus, $e_1(R, V - R)$ is bounded above by

$$(|R| - 2)\Delta_1 + 2(\Delta_1 - 1) = |R|\Delta_1 - 2.$$

Combining the upper and lower bounds on $e_1(R, V-R)$ yields

$$\delta_1 n + 1 < |R|(\Delta_1 + \delta_1).$$

By our assumption on $|R|$ we have the following contradiction,

$$\delta_1 n + 1 < (\Delta_2 + 1)(\Delta_1 + \delta_1).$$

Now assume that $|R| = \Delta_2 + 1 + t$, where t is a positive integer. Notice that $|N_G[y]| \leq \Delta_1 + \Delta_2$ implies that

$$|N_G[y] - R| \leq \Delta_1 + \Delta_2 - (\Delta_2 + 1 + t) = \Delta_1 - t - 1.$$

As y has no neighbors in $\overline{Y}$, y has at most $\Delta_1 - t - 1$ neighbors of G_1 in $V - R$. If there is another vertex $w \in R - N_{G_1}(\overline{Y})$, then w also has no neighbors in $\overline{Y}$ and thus has at most $\Delta_1 - t - 1$ neighbors of G_1 in $V - R$. If $R - N_{G_1}(\overline{Y}) = \{y\}$, then R is a clique. In this case, x has at most $\Delta_1 + \Delta_2 - |R|$ neighbors of G_1 in $V - R$. As $\Delta_1 + \Delta_2 - |R| = \Delta_1 - t - 1$, we have that there are at least two vertices in R with at most $\Delta_1 - t - 1$ neighbors of G_1 in $V - R$.

Each of the remaining vertices of R have at most $\Delta_1 - t$ neighbors of G_1 in $V - R$. In particular, if $v \in B$, then v has one neighbor of G_1 to $\overline{Y}$ and at most $\Delta_1 - t - 1$ neighbors of G_1 to $N_G[y] - R$. If $v \in A$, then v is adjacent to every vertex of A, and therefore has at most $\Delta_1 + \Delta_2 - |R| + 1$ neighbors of G_1 to $V - R$, which is $\Delta_1 - t$.

Therefore, we have that

$$e_1(R, V-R) \leq 2(\Delta_1 - t - 1) + (|R| - 2)(\Delta_1 - t) = |R|(\Delta_1 - t) - 2.$$

Combining this with the lower bound of $e_1(R, V-R)$, we have

$$\delta_1 n + 1 < |R|(\Delta_1 + \delta_1 - t).$$

Since $\Delta_2 + 1 + t = |R|$, we expand the right side to obtain

$$\delta_1 n + 1 < (\Delta_2 + 1)(\Delta_1 + \delta_1) - t(\Delta_2 + 1 - (\Delta_1 + \delta_1)) - t^2.$$

If $\Delta_2 + 2 \geq \Delta_1 + \delta_1$, then we contradict our claim that $(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \delta_1 n + 1$. Otherwise, $\Delta_2 + 2 < \Delta_1 + \delta_1$. In this case, the right side is maximized when $t = \frac{1}{2}(-\Delta_2 - 1 + \Delta_1 + \delta_1)$, which yields

$$\delta_1 n + 1 < \frac{(\Delta_2 + 1 + \Delta_1 + \delta_1)^2}{4}.$$

This contradiction completes the proof. □

We next prove Corollary 1.4.

Proof. Assume that $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$. Then

$$\begin{aligned}(\Delta_2 + 1)(\Delta_1 + \delta_1) &= (\Delta_2 + 1)(\Delta_1 + 1 + \delta_1 - 1) \\ &= (\Delta_2 + 1)(\Delta_1 + 1) + (\Delta_2 + 1)(\delta_1 - 1) \\ &\leq \delta_1 n + 1,\end{aligned}$$

where the last inequality follows from the hypothesis and the fact that $\delta_1 \geq 1$. Thus the result follows when $\Delta_2 + 2 \geq \Delta_1 + \delta_1$.

Suppose then that $\Delta_2 + 2 < \Delta_1 + \delta_1$, which implies $\delta_1 \geq 2$ and also that

$$\begin{aligned} \frac{(\Delta_2 + 1 + \Delta_1 + \delta_1)^2}{4} &< \frac{[2(\Delta_1 + \delta_1)]^2}{4} \\ &= (\Delta_1 + \delta_1)^2. \end{aligned}$$

Note that $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ implies that $\Delta_1 \leq \sqrt{n}$, while $\delta_1 \geq 2$ implies that $\sqrt{\delta_1 n} - \delta_1 > \sqrt{n}$. Hence $(\Delta_1 + \delta_1)^2 \leq \delta_1 n$, and the result follows. $\square$

Finally, we give the straightforward proof that Corollary 1.4 implies Corollary 1.5.

Proof. For $\Delta_1 > 1$, since $\Delta_1 \geq \Delta_2$, we have that $\Delta_1 + \Delta_2 \leq 2\Delta_2$ and $2\Delta_2 \leq \Delta_1\Delta_2$, thus

$$\Delta_1\Delta_2 + \Delta_1 + \Delta_2 \leq 2\Delta_1\Delta_2.$$

By assumption, $2\Delta_1\Delta_2 < n$, therefore $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$. If, instead, $\Delta_1 = 1$, then $2\Delta_2 < n$ or $2\Delta_2 + 1 \leq n$, and since $(\Delta_1 + 1)(\Delta_2 + 1) - 1 = 2\Delta_2 + 1$, we have that $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ as desired. $\square$

3. Discrete Tomography. *Tomography* is the process of imaging through sectioning, for example constructing a three dimensional image from a series of 2-dimensional cross-sections or projections. Of interest here is *discrete tomography*, which uses low-dimensional projections to reconstruct discrete objects, such as the atomic structure of crystalline lattices and other polyatomic structures.

3.1. The k-color Tomography Problem. Numerous papers (c.f. [13, 14, 18, 19]) study the *k-color Tomography Problem*, in which the goal is to color the entries of an $m \times n$ matrix using k colors so that each row and column receives a prescribed number of entries of each color. The colors represent different types of atoms appearing in a crystal, and the number of times an atom appears in a given row or column is generally obtained using high resolution transmission electron microscopes [27, 34]. This is precisely the problem of packing the degree sequences of k bipartite graphs with partite sets of size m and n.

3.2. Sauer-Spencer-Type Theorems. A sequence $\pi = (a_1, \ldots, a_r; b_1, \ldots, b_s)$ is *bigraphic* if there is a bipartite graph G such that $\pi = \pi(G)$ with partite sets X and Y, and the degrees of the vertices in X and Y are $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$, respectively. Two bigraphic sequences, $\pi_1 = (a_1^{(1)}, \ldots, a_r^{(1)}; b_1^{(1)}, \ldots, b_s^{(1)})$ and $\pi_2 = (a_1^{(2)}, \ldots, a_r^{(2)}; b_1^{(2)}, \ldots, b_s^{(2)})$ pack if there exist edge-disjoint bipartite graphs G_1 and G_2, both with partite sets $X = \{x_1, \ldots, x_r\}$ and $Y = \{y_1, \ldots, y_s\}$, such that for $j \in \{1, 2\}$,

$$d_{G_j}(x_i) = a_i^{(j)}$$

for $1 \leq i \leq r$, and

$$d_{G_j}(y_i) = b_i^{(j)}$$

for $1 \leq i \leq s$.

The following is a tomographic analogue to Corollary 1.5.

Theorem 3.1. *Let π_1 and π_2 be bigraphic sequences with parts of sizes r and s, and $\Delta_i = \Delta(\pi_i)$ and $\delta_i = \delta(\pi_i)$ for $i \in \{1, 2\}$, such that $\Delta_1 \leq \Delta_2$ and $\delta_1 \geq 1$. If*

$$\Delta_1\Delta_2 \leq \frac{(r + s)}{4}$$

then π_1 and π_2 pack.

The other main result of this section, which takes δ_1 into account, improves on Theorem 3.1 when $\delta_1 \geq 3$.

THEOREM 3.2. *Let π_1 and π_2 be bigraphic sequences with parts of sizes r and s, and $\Delta_i = \Delta(\pi_i)$ and $\delta_i = \delta(\pi_i)$ for $i \in \{1, 2\}$, such that $\Delta_1 \leq \Delta_2$ and $\delta_1 \geq 1$. If*

$$\Delta_1\Delta_2 \leq \delta_1 \frac{(r+s)}{8}$$

then π_1 and π_2 pack.

As before, we say a vertex pair (x, y) is a *bad pair for* (G_1, G_2) or a (G_1, G_2)-*bad pair* if $xy \in E(G_1) \cap E(G_2)$.

Let π_1 and π_2 be bigraphic sequences that do not pack, choose $G_1 = G(\pi_1)$ and $G_2 = G(\pi_2)$ to have the fewest bad pairs among all realizations of π_1 and π_2 and let $G = G_1 \cup G_2$. Fix a (G_1, G_2)-bad pair (x, y) and let X and Y be the partite sets of G, where $x \in X$ and $y \in Y$. Let $I_X = X - N_G(y)$ and $I_Y = Y - N_G(x)$. We now have the following lemmas, the first of which is analogous to Claim 2.

LEMMA 3.3. *The set $I_X \cup I_Y$ is independent.*

Proof. Suppose otherwise, so in particular let $z \in I_X$ and $z' \in I_Y$ such that $zz' \in E(G)$. Exchanging the edges xy and zz' with the non-edges zy and $z'x$ decreases the number of (G_1, G_2)-bad pairs, contradicting the choice of G_1 and G_2. □

LEMMA 3.4. *The subgraph of G induced by $N_{G_1}(I_Y) \cup N_{G_1}(I_X) \cup \{x, y\}$ is a complete bipartite graph.*

Proof. First, note that by Lemma 3.3 and the definition of I_Y, x is adjacent to every vertex in $N_{G_1}(I_X)$ and likewise, y is adjacent to every vertex in $N_{G_1}(I_Y)$. Suppose then that there is some $w \in N_{G_1}(I_Y)$ and $w' \in N_{G_1}(I_X)$ such that ww' is not an edge in G. Now we have that there is some $z' \in I_Y$ and $z \in I_X$ such that wz' and $w'z$ are edges in G_1. Exchanging the edges $w'z, wz'$ and xy (all in G_1) with the non-edges ww', xz' and yz decreases the number of bad pairs in G, a contradiction. □

We are now ready to prove Theorems 3.1 and 3.2.

Proof. [Proof of Theorem 3.1] Observe first that each vertex in $N_{G_1}(I_X)$ (respectively $N_{G_1}(I_Y)$) can have at most Δ_1 neighbors in I_X (resp. I_Y) so that

$$|I_X| + |I_Y| \leq \Delta_1(|N_{G_1}(I_X)| + |N_{G_1}(I_Y)|).$$

We further have that

$$n - (|N_G(x)| + |N_G(y)|) \leq |I_X| + |I_Y|,$$

and

$$|N_{G_1}(I_Y)| + |N_{G_1}(I_X)| \leq |N_G(x)| + |N_G(y)| - 2 \leq 2(\Delta_1 + \Delta_2) - 4.$$

Taken together, these yield that

$$n - (2(\Delta_1 + \Delta_2) - 2) \leq \Delta_1(2(\Delta_1 + \Delta_2) - 4),$$

so

$$\frac{n}{2} \leq (\Delta_1 + 1)(\Delta_1 + \Delta_2) - 2\Delta_1 - 1.$$

As $\Delta_1\Delta_2 \leq \dfrac{n}{4}$, it follows that

$$2\Delta_1\Delta_2 \leq \Delta_1^2 + \Delta_1\Delta_2 + \Delta_1 + \Delta_2 - 2\Delta_1 - 1,$$

so that

$$\Delta_1\Delta_2 - \Delta_2 \leq \Delta_1^2 - \Delta_1 - 1.$$

However, then

$$\Delta_2 \leq \Delta_1 - \frac{1}{\Delta_1 - 1},$$

a contradiction, since $\Delta_1 \leq \Delta_2$. □

Proof. [Proof of Theorem 3.2] By Lemma 3.3, I_X and I_Y are independent, so every vertex in $I_X \cup I_Y$ must have at least δ_1 neighbors in $N_{G_1}(I_Y) \cup N_{G_1}(I_X)$. Also, as in Theorem 3.1, each vertex in $N_{G_1}(I_Y) \cup N_{G_1}(I_X)$ has at most Δ_1 neighbors in $I_X \cup I_Y$. Therefore,

$$\delta_1(|I_X| + |I_Y|) \leq \Delta_1(|N_{G_1}(I_Y)| + |N_{G_1}(I_X)|)$$

so that

$$|I_X| + |I_Y| \leq \frac{\Delta_1}{\delta_1}\left(|N_{G_1}(I_Y)| + |N_{G_1}(I_X)|\right).$$

Again, we have that

$$|N_{G_1}(I_Y)| + |N_{G_1}(I_X)| \leq |N_G(x)| + |N_G(y)| - 2 \leq 2(\Delta_1 + \Delta_2 - 2).$$

Let $r + s = n$, so that

$$|I_X| + |I_Y| = n - \left(|N_G(x)| + |N_G(y)|\right).$$

Combining the above equations yields

$$n - 2(\Delta_1 + \Delta_2) + 2 \leq 2\frac{\Delta_1}{\delta_1}(\Delta_1 + \Delta_2 - 2).$$

By isolating Δ_2,

$$\frac{\delta_1 n}{2(\Delta_1 + \delta_1)} - \Delta_1 + \frac{2\Delta_1 + \delta_1}{\Delta_1 + \delta_1} \leq \Delta_2.$$

Notice that $\Delta_1 + \delta_1 \leq 2\Delta_1$, so we have

$$\frac{\delta_1 n}{4\Delta_1} \leq \Delta_2 + \Delta_1 - \frac{2\Delta_1 + \delta_1}{\Delta_1 + \delta_1}.$$

By assumption, $\Delta_2 \leq \dfrac{\delta_1 n}{8\Delta_1}$, so

$$2\Delta_2 \leq 2\left(\frac{\delta_1 n}{8\Delta_1}\right) \leq \Delta_2 + \Delta_1 - \frac{2\Delta_1 + \delta_1}{\Delta_1 + \delta_1},$$

which implies

$$\Delta_2 \leq \Delta_1 - \frac{2\Delta_1 + \delta_1}{\Delta_1 + \delta_1}.$$

Since $\dfrac{2\Delta_1 + \delta_1}{\Delta_1 + \delta_1} > 0$, and $\Delta_2 \geq \Delta_1$, we arrive at a contradiction, completing the proof. □

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