Chaotic Transitions in Deterministic and Stochastic Dynamical Systems: Applications of the Melnikov Method in Engineering, Physics, and Neuroscience

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ABSTRACT: The classical Melnikov method provides information on the behavior of deterministic planar systems that may exhibit transitions, i.e. escapes from and captures into preferred regions of phase space. This paper describes and illustrates a unified treatment of deterministic and stochastic systems that extends the applicability of the classical Melnikov method to physically realizable stochastic planar systems with additive, state-dependent, colored, or dichotomous noise. The extended method yields the novel result that motions with transitions are chaotic for either deterministic or stochastic excitation, explains the role in the occurrence of transitions of the system and excitation characteristics, and is a powerful modeling and identification tool.

INTRODUCTION

The Melnikov method is a unified framework for the study of transitions and chaos in a wide class of deterministic and stochastic nonlinear planar dynamical systems with restoring force derived from a multi-well potential. Its applications span a broad

FIG. 1. (a) Potential wells. (b) Phase plane diagram for Duffing-Holmes equation. The orbits that emerge from and converge to saddle point O are the homoclinic orbits. Periodic orbits confined to a well evolve inside a homoclinic orbit around one of the centers C, C'; those that cross the potential barrier evolve outside the homoclinic orbits. The homoclinic orbits are separatrices between these two types of motion.

spectrum of problems in engineering and the applied sciences. In this paper we review fundamental results of Melnikov theory and, to give the reader a sense of its capabilities, present a number of typical applications.

As an example we consider the system

$$\ddot{x} = -V'(x) + \varepsilon(\gamma G(t) - \beta \dot{x}) \qquad (1)$$

where ε is small, $G(t)$ is a sufficiently well behaved forcing function, $\beta > 0$, and $V(x)$ is a double-well potential (Fig. 1a).

We may use the notations $x = x_1, \dot{x} = x_2$. The unperturbed counterpart of Eq. 1 corresponds to the case $\varepsilon = 0$. For this case *two* types of generic motion can occur: (1) motions that evolve around C and C' and never cross the potential barrier; and (2) motions that evolve around the saddle point O and cross that barrier periodically (Fig. 1b). The homoclinic orbits, which approach O as $t \to \pm\infty$, are non-generic and separate the two types of motion. No motion is possible from outside the core defined by the separatrix into the core, and vice-versa.

We now assume that in the perturbed (i.e., forced, dissipative system) (Eq. 1), the forcing is harmonic. In this case, *three* distinct types of steady-state dynamic behavior can occur.

(1) For sufficiently small forcing, depending upon the initial conditions, the system moves periodically about one of system's centers as in Fig. 1b (Fig. 2a).

(2) For sufficiently large forcing the motion crosses the potential barrier periodically as in Fig. 1b (Fig. 2b).
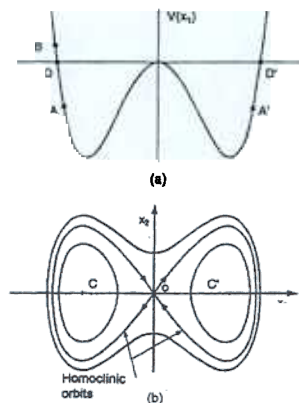
(3) For intermediate excitation amplitudes, and for restricted sets of initial conditions and excitation frequencies, the steady-state motion is *irregular*, even though the system is fully deterministic; hence the term deterministic chaos. The motion evolves about one of the centers, then it undergoes succes-
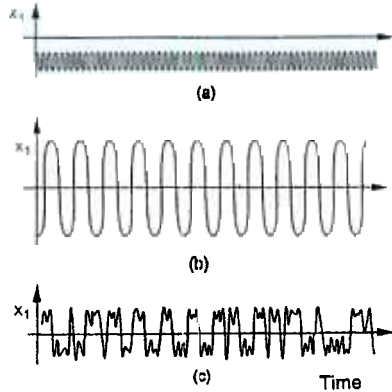

FIG. 2. Types of steady-state dynamic behavior that may be observed in a periodically forced, dissipative planar system with a double-well potential: (a) periodic motion confined to one well; (b) periodic motion visiting both wells; (c) irregular motion with transitions.

sive *transitions*, i.e., it changes successively to motion about another center (Fig. 2c). Transitions in such irregular, deterministic motion are referred to as *chaotic*. A transition away from motion in a potential well is called an *escape*. For a large number of systems it is required that steady-state motions occur within a region called *safe*, that is, i.e., that they not cross a potential barrier. For planar systems subjected to periodic excitation an analytical condition that guarantees the non-occurrence of transitions involves a function - the *Melnikov function* - consisting of a sum of two terms related, through the system's potential, to the dissipation and excitation terms.

Melnikov's method was extended to quasiperiodically excited systems (Wiggins, 1988) and *stochastic* systems (Frey and Simiu, 1993). The stochastic counterpart of the Melnikov function is a *Melnikov process*. Through the modulus of the *Melnikov transfer function* -- a function of frequency that depends upon the system's potential -- the Melnikov method provides information on the degree to which the frequency components of the excitation promote transitions effectively.

In Section 2 we review basic results on Melnikov functions and processes, and chaotic dynamics results pertaining to the escape problem. The remaining sections are devoted to applications.

2. MELNIKOV FUNCTIONS AND PROCESSES. NECESSARY CONDITION FOR THE OCCURRENCE OF EXITS

We consider the phase space x_1, x_2, θ, where $\theta = \omega t$, T is a constant, the axis $O\theta$, denoted by Γ_0, is normal to the plane x_1, x_2, and O is the saddle point of the unperturbed system (Fig. 3).
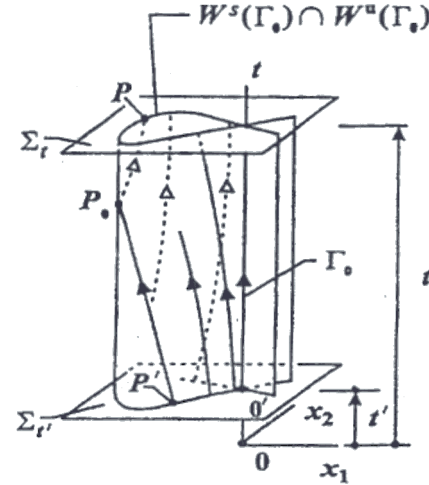

FIG 3. Stable and unstable manifolds for unperturbed system.

The stable (unstable) manifold $W^s(\Gamma_0)$ $(W^u(\Gamma_0))$ of the hyperbolic orbit Γ_0 is defined as the set of points such that the orbits passing through them approach Γ_0 as $t \rightarrow \infty$ $(t \rightarrow -\infty)$. A cross-section of the stable and unstable manifolds with any plane θ =const has the coordinates of the system's homoclinic orbits. The stable and unstable manifolds coincide.

It follows from the persistence theorem that, for sufficiently small ε the perturbed system has a hyperbolic orbit Γ_ε contained in a close neighborhood of Γ_0. The stable (unstable) manifold is defined as the set of points such that the orbits passing through those points approach Γ_ε as $t \rightarrow \infty$ $(t \rightarrow -\infty)$.

For the perturbed system the stable and unstable manifolds no longer coincide. To first order the distance between them is proportional to the *Melnikov function*, which can be shown to have the expression

$$M(t) = \beta \ c + \gamma \ m(t) \qquad (2)$$

$$c = -\int_{-\infty}^{\infty} h^2(\tau) d\tau,$$

$$m(t) = \int_{-\infty}^{\infty} h(\tau) G(\tau - t) d\tau.$$

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where $h(\tau)$, the ordinate of the homoclinic orbit in the x_1, x_2 plane can be obtained by integrating Eq. 1 with initial conditions at the saddle point. It can further be shown that: (1) if $M(t)$ has simple zeros the stable and unstable manifolds intersect transversely, and (2) transverse intersections are a necessary condition for chaotic behavior and escapes. A cross-section through intersecting stable and unstable manifolds, which yields a so-called homoclinic tangle, is depicted in Fig. 4.
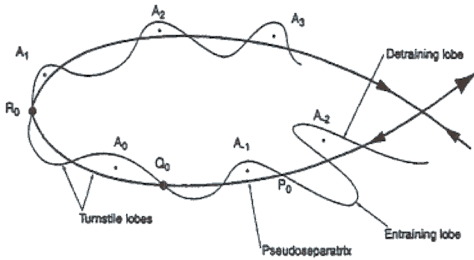

FIG. 4. Plane cross-section through intersecting stable and unstable manifolds. Chaotic transport takes point A_{-2}, located within the core bounded by the pseudoseparatrix, through A_{-1} and A_0 to point A_1, located outside the core.

It can be shown that the homoclinic tangle makes possible the occurrence of chaotic motions and escapes of the type shown in Fig. 2c.

If $G(t)$ is quasiperiodic (i.e., a sum of harmonic terms with amplitudes a_i, generally incommensurate frequencies ω_i, and phase angles ε_i ($i=1,2,..n$), it follows from Eqs. 2

$$m(t) = \sum_{i=1}^{n} a_i |\alpha(\omega_i)| \cos[\omega_i t + \theta_i - \psi(\omega_i)] \quad (3)$$

in which the transfer function $\alpha(\omega_i)$ is the Fourier transform of $h(\tau)$, and $\psi(\omega_i)$ is the argument of $\alpha(\omega_i)$ (Wiggins, 1988).

A physically realizable stochastic excitation can be closely approximated by a sum of harmonic terms with random parameters. A realization of the stochastic process corresponds to a fixed set of parameters. Therefore, instead of a Melnikov function, a system with a stochastic excitation possesses an ensemble of Melnikov functions, i.e., a *Melnikov process*. If the system's excitation $\gamma G(t)$ is Gaussian with unit variance, the Melnikov process has expectation $-\beta c$, and spectral density

$$\Psi_M(\omega) = \gamma^2 |\alpha(\omega_i)|^2 \Psi_0(\omega), \quad (4)$$
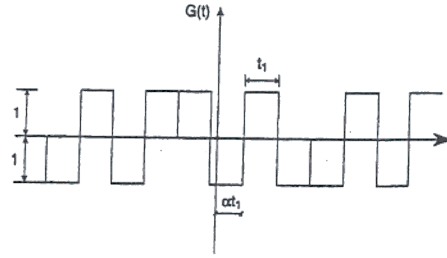

FIG. 5. Realization of a square-wave dichotomous noise process.

where $\Psi_0(\omega)$ is the spectral density of $G(t)$. Eq. 4 shows that, depending upon its shape, $\alpha(\omega_i)$ may reduce or increase the contribution of the excitation's various frequency components to the Melnikov process. This observation is useful in practical applications, notably the open-loop control of escapes.

3. CONDITION FOR NONOCCURRENCE OF ESCAPES IN SYSTEMS EXCITED BY DICHOTOMOUS NOISE

The dichotomous noise $G(t)$ we consider is depicted in Fig. 5. No escapes can occur if $M(t)$ has no simple zeros. For example, since $|G(t)| \leq 1$, for the double-well potential $V(x) = x^4/4 - x^2/2$, which has the shape shown in Fig. 1a, the condition that Eq. 2 not have simple zeros (that escapes cannot occur) yields $\gamma/\beta < 0.47$ (Sivathanu, Hagwood, and Simiu, 1995). To our knowledge this is the only method for obtaining a criterion guaranteeing the nonoccurrence of escapes due to dichotomous noise.

4. VESSEL CAPSIZING

The empirical restoring force in the equation of rolling motion of a vessel is derived from an M-shaped potential, rather than a W-shaped potential as in Fig. 1a. The spectrum of the Melnikov process induced by the wave force has the form of Eq. 4; the spectrum of the wave force is equal to the spectrum of the waves times the square of the modulus of an empirical transfer function.

Using the phase space flux factor -- a functional of the Melnikov function, -- Hsieh, Troesch, and Shaw (1994) estimated capsizing probabilities of a vessel for various time intervals and significant wave heights.

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5. OPEN-LOOP CONTROL OF ESCAPES

Consider Eq. 1, where $G(t)=G_f(t)-kG_c(t-\tau)$, $\epsilon\gamma\ G_f(t)$ is a stationary excitation process, $\epsilon\gamma\ kG_c(t-\tau)$ is a stationary control process whose addition to the system will reduce the escape rate, τ is a time lag, and $0<k<1$. For the trivial choice $G_c(t)/G_f(t-\tau)$ and small enough τ the escape rate will be smaller in the controlled than in the uncontrolled system. For $\tau=0$ the ratio between the average powers of the control and excitation force is $q=k^2$. Melnikov theory can be used to obtain open-loop control forces that achieve reductions comparable to those due to a trivial control force, but with a smaller ratio q, i.e., with less energy. It follows from Eq. 4 that this can be accomplished if the frequency content of the control force is concentrated in the frequency interval where $|\alpha(\omega_i)|$ is highest. For details and results, see Simiu and Franaszek (1997) and Basios et al. (1999).

6. STOCHASTIC RESONANCE

Stochastic resonance is the phenomenon wherein, for a system excited by a low-frequency harmonic in the presence of noise, the output signal-to-noise ratio can be enhanced by *adding* noise or a harmonic excitation. From Melnikov theory it follows that the spectral density of the added noise or the frequency of the added harmonic excitation must correspond to the largest ordinates of $|\alpha(\omega)|$. The enhancement is due to the chaotic nature of the motion, similar to the motion of Fig. 2c, induced in the system by the excitation, the original noise, and the added noise or harmonic. If the escape rate is approximately equal to the excitation frequency, energy from the broadband spectrum inherent in the chaotic motion is transferred by a synchronization-like mechanism at that frequency, thus enhancing the signal-to-noise ratio (Franaszek, Simiu, 1996) – see Fig. 6.

7. SNAP-THROUGH OF A TRANSVERSELY EXCITED BUCKLED COLUMN

Holmes and Marsden (1981) applied Melnikov theory to a buckled column with continuous mass subjected to a continuously distributed transverse excitation with harmonic time dependence. This dynamical system can be shown to have restoring forces derived from a potential similar to Fig. 1a. An extension of the theory to the case of excitation by dichotomous noise yields a criterion for the nonoccurrence of snap-through similar to the criterion of Section 3 – see Simiu and Franszek (1996). An evolution in time of the buckled column shape is depicted in Fig. 7.

8. WIND-INDUCED ALONG-SHORE CURRENTS OVER OCEAN FLOOR WITH VARIABLE TOPOGRAPHY

Observations yielded by moored current meters suggest that wind-induced along-shore mesoscale velocity fields over a sloping, corrugated ocean floor (Fig. 8) contain energy in a continuous range of low frequencies, a possible indication of chaotic behavior. Allen et al. (1991) developed a simplified model of the flow to which the Melnikov approach is applicable. For wind speed fluctuations conforming to accepted meteorological models, Simiu (1996) obtained criteria for the probability that the flow motion can be chaotic during given time intervals.
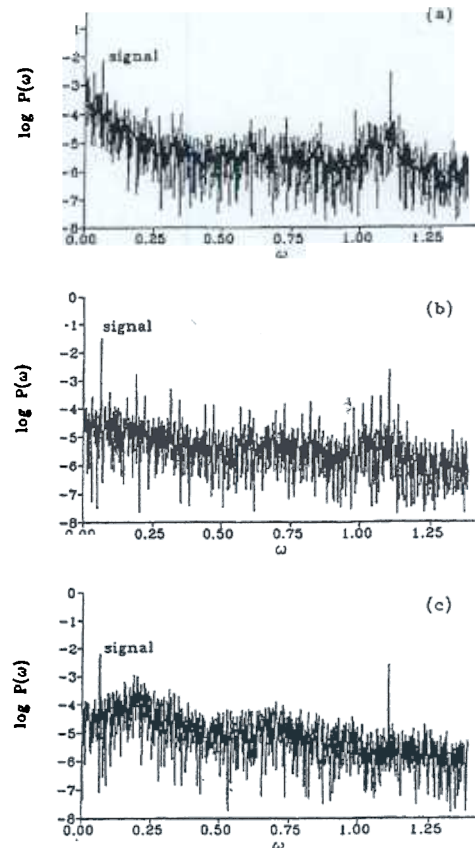

FIG. 6. Power spectra of motion induced in a bistable system by harmonic and noise. The motion is chaotic in all cases. In case (b) the noise is stronger than in case (a), and the escape rate is approximately equal to the signal frequency; a synchronization-like phenomenon occurs that transfers broadband energy associated with the chaotic motion to the signal frequency and thus increases the signal-to-noise ratio. For even larger noise, as in case (c), that phenomenon does not occur and the signal-to-noise ratio decreases.
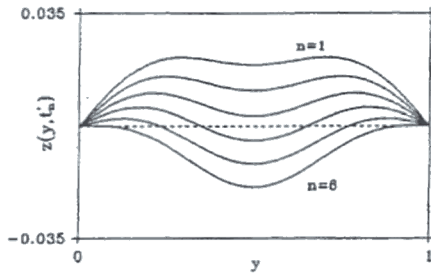
FIG. 7. Evolution in time of buckled column shape undergoing snap-through induced by continuous stochastic transverse excitation.

9. THE AUDITORY NERVE SYSTEM AS A CHAOTIC DYNAMICAL SYSTEM

Figure 9 shows a typical time history of the response of a nerve fiber to harmonic excitation in the presence of weak noise. Experimental results reported in the 1960's are consistent with a model of the nerve fiber response, including irregular "firings," as a planar dynamical system to which the Melnikov approach is applicable (Franaszek and Simiu, 1998). Similar models can be used to represent and analyze encephalograms and other plots of physiological or physical phenomena exhibiting irregular transitions.
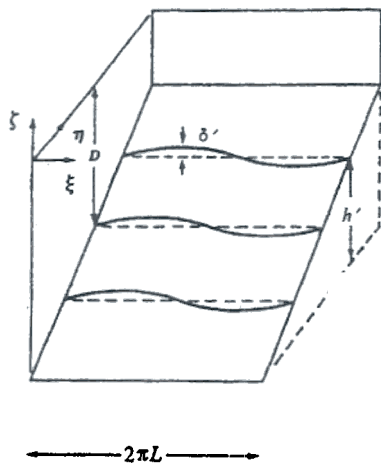

FIG. 8. Model of sloping continental shelf showing one wavelength of along-shore periodic corrugation (after Allen, Samelson, and Newberger (1991).
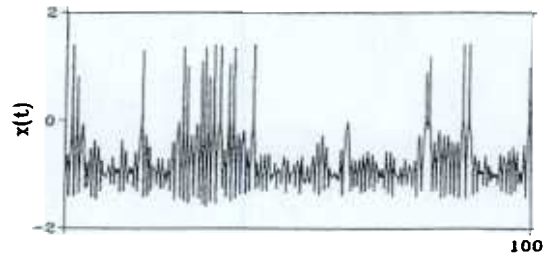

FIG. 9. Time history of motion induced in auditory nerve fiber by harmonic excitation in the presence of weak noise.

REFERENCES

Allen, J.S., Samelson, R.M. and Newberger, P.A., 1991, "Chaos in a model of forced quasi-geostrophic flow over topography: an application of Melnikov's method," *J. Fluid Mech.* **226** 511-547.

Basios, V., Bountis, T. , and Nicolis, G., 1999, "Controlling the onset of homoclinic chaos due to parametric noise," *Physics Lett.* **A 251** 250-258.

Franaszek, M. and Simiu, E, 1996a, "Stochastic resonance: A chaotic dynamics approach," *Physical Rev.* **E 54** 1288-1304.

Franaszek, M. and Simiu, E., 1996b, "Noise-induced Snap-Mass: A Chaotic Dynamics Approach," *Int. J. Nonlinear Mech.*, **31** 861-869.

Franaszek, M. and Simiu, E., 1998, "Auditory nerve fiber modeling: A stochastic Melnikov approach," *Phys. Rev.* **E57** 5870-5876.

Frey, M. and Simiu, E., 1993, "Noise-induced chaos and phase space flux," *Physica* **D 63** 321-340.

Holmes, P. and Marsden, J., 1981, "A Partial Differential Equation with Infinitely Many Periodic Orbits: Chaotic Oscillations of a Forced Beam," *Arch. Ration. Mech. Anal.* **76** 135-166.

Hsieh, S.R., Troesch, A.W., and Shaw, S.W., 1994, "A nonlinear probabilistic method for predicting vessel capsizing in random beam seas," *Proc. R. Soc. London* **A 446** 195-211.

Simiu, E., 1996, "Melnikov Process for Stochastically Perturbed Slowly Varying Oscillators: Application to a Model of Wind-driven Coastal Currents," *J. Appl. Mech.* **63** 429-436.

Simiu, E., 2002, *Chaotic Transitions in Deterministic and Stochastic Dynamical Systems.* Princeton University Press, Princeton and Oxford.

Simiu, E. and Franaszek, M., 1997, "Melnikov-based Open-loop Control of Escape for a Class of Nonlinear Systems," *ASME J. Dynamical Systems, Measurement, and Control* **119** 590-594.

Sivathanu, Y., Hagwood, C., and Simiu, E., 1995, "Exits in Multistable Systems Excited by Coin-toss Square Wave Dichotomous Noise: A Chaotic Dynamics Approach," *Physical Rev.* **E 52** 4669-4675.

Wiggins, S, 1988, *Global Bifurcations and Chaos: Analytical Methods*, Springer-Verlag, New York.

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