Proceedings from Oceans Engineering for Today's Technology and Tomorrow's Preservation, Vol. III, 1994

Chaotic Behavior of Coastal Currents Due to Random Wind Forcing

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Abstract. Recent analyses revealed that chaotic behavior is possible in a model of currents induced by wind over topography that slopes offshore and varies approximately periodically alongshore. These analyses were based on the assumption that the forcing by wind is harmonic. We examine the more realistic case where the wind field, and therefore the wind forcing, is random. Given a wind field with specified spectral density, and any specified parameters of the bottom topography, we use a generalized Melnikov transform technique to estimate upper bounds for probabilities that chaotic excursions will occur during a specified time interval.

Résumé. Des travaux récents ont révélé la possibilité du comportement chaotique d'un modèle de courants océaniques produits par l'action du vent. Ces travaux reposent sur l'hypothèse simplificatrice que les excitations dues au vent sont harmoniques. Nous examinons le cas plus réaliste où les vitesses du vent et les excitations qui en résultent sont aléatoires. Etant donné un champ des vitesses du vent défini par sa densité spectrale, et un ensemble de paramètres définissant la topographie du fonds de l'océan, nous utilisons une version généralisée de la technique de Melnikov pour estimer des bornes supérieures de probabilités pour que des excursions chaotiques aient lieu pendant un intervalle de temps donné.

I. INTRODUCTION

Recent studies have pointed out the complexity of the time dependence of mesoscale currents in the coastal ocean. As noted in [1], according to observations from moored current meters with limited temporal resolution, most ocean velocity fields contain energy in an apparently continuous band of frequencies below the local inertial frequency. This has been attributed to the wide range of frequencies and scales at which the coastal ocean is forced, and to instabilities and nonlinearities of the flow field that transfer energy between frequencies. An alternative explanation was recently proposed in [1], which showed that an explanation of the observed continuous range of frequencies may lie in the chaotic nature of the flow due to forcing by wind stress. In the context of the theory of deterministic dynamical systems the term "chaotic" was defined about two decades ago. It is applied to systems exhibiting sensitivity to initial conditions, continuous spectral density -- an indication that the chaotic systems' behavior is neither periodic nor quasiperiodic -- and, under certain conditions, irregular exits from regions of phase space associated with the potential wells of the systems' Hamiltonian counterparts.

The system analyzed in [1] represents a simple model for flow over a continental margin. It is based on the following assumptions. The fluid is contained between a channel wall at the coast and the deep ocean. The bottom of the ocean slopes in a direction normal to the coast and has corrugations with harmonic cross-sections parallel to the coast. The amplitudes of the corrugations vary slowly in a direction normal to the coast. The flow consists of a single layer of homogeneous fluid and obeys a quasi-geostrophic potential vorticity equation. The equations of motion of the flow based on these assumptions can be written in the form of a system of three ordinary differential equations, the third of which is slowly varying with time.

An additional assumption used in [1] is that the fluctuating wind stress driving the flow is harmonic. This allows the application to the equations of motion of the deterministic Melnikov approach developed for slowly varying systems with harmonic forcing in [2] and [3]. The necessary condition for chaotic behavior of the harmonically-driven system, given by the Smale-Birkhoff theorem, is that the Melnikov function of the system have simple zeros.

The actual wind field is random, so that the wind-induced stresses that drive the flow are also random, rather than being harmonic. In this paper we study the effect of the randomness of the forcing on the behavior of the flow model proposed in [1]. Our study uses an extension of the Melnikov approach as applied to a class of slowly varying oscillators from the case of harmonic forcing [2] to the case of forcing that approximates as closely as desired a random process with specified spectral density [4]. The extension of the deterministic Melnikov approach to stochastic systems is referred to here as the stochastic Melnikov approach.

In the following sections we present our modification of the model studied in [1], and demonstrate three features of our approach. First, our approach eliminates the need to guess the frequency of a harmonic forcing purportedly equivalent to the actual random forcing. Second, it yields the result that, under random excitation assumed to be Gaussian, the necessary condition for chaos is satisfied for any excitation, however small. Third, for a given intensity of the random excitation, it provides a lower bound of the probability that the flow will not exhibit chaotic jumps (exits) during a specified time interval.

II. ANALYTICAL MODEL

A. Equations of Motion. The equations of motion of the flow are

$$\dot{x} = \frac{\partial}{\partial y} H(x,y,z) + \epsilon[-rx + \tau_o + \tau(t)]$$

$$\dot{y} = -\frac{\partial}{\partial x} H(x,y,z) + \epsilon[-ry]$$

$$\dot{z} = \epsilon\{-rz - \tfrac{1}{2}rx^2 + (x-1)[\tau_o + \tau(t)]\} \qquad \text{(1a,b,c)}$$

where x is a basic alongshore speed, y is proportional to the out-of-phase component of a stream function for motion due to the topography, z is the energy-enstrophy of the system, $\epsilon \ll 1$, H(x,y,z) is a Hamiltonian with parameter z given by

$$H(x,y,z) = \tfrac{1}{2}y^2 + zx + \tfrac{1}{2}(\omega_o^2 - z)x^2 - \tfrac{1}{2}x^3 + (1/8)x^4, \qquad (2)$$

$$\omega_o^2 = 1 + \delta^2, \qquad (3)$$

δ is the amplitude of the bottom topography corrugations, ϵr is a friction coefficient related to the eddy viscosity of the ocean flow, and $\epsilon\tau_o$ and $\epsilon\tau(t)$ are, respectively, the steady and fluctuating wind stress at the ocean surface [1].

B. Fluctuating Wind and Surface Wind Stresses. Our modification of the model studied in [1] consists of replacing the harmonic forcing by the random forcing defined in this section.

The random forcing we shall consider is generated by horizontal wind speed fluctuations. The spectral density of the fluctuating horizontal wind speed fluctuations, as developed by Van der Hoven [5], has three main parts. The first part has a peak at a period of about 4 days. The second part, referred to as the spectral gap, has negligible energy and extends over periods of about 5 hrs to 3 min. The third part has a peak at a period of about 1 min and corresponds to fluctuations with relatively small spatial coherence, so that their overall effect on the wind-induced current may be neglected [6]. It can be verified [5] that the part of the spectrum relevant to our problem is closely approximated by the curve

$$S(\omega) = \begin{cases} 0.2823\ell n(\omega) + 1.300 & 0.01 \le \omega \le 0.10 \\ 0.4072\ell n(\omega) + 1.590 & 0.10 \le \omega \le 0.30 \\ -2.71[\ell n(\omega)]^2 + 5 & 0.30 \le \omega \le 3.85 \end{cases} \qquad (4)$$

where $\omega = \Omega/\Omega_{pk}$, Ω is the dimensional frequency and $\Omega_{pk} \approx 2\pi/(4\text{ days})$ is the dimensional frequency corresponding to the spectral peak, which occurs at $\omega = 1$. The units of $S(\omega)$ are m^2/s^2 (Fig. 1). For the spectrum of (4) the standard deviation of the wind speed fluctuations is $\sigma_u \approx 1.33$ m/s.

Surface wind stresses are proportional to the square of the wind speeds [6, pp. 37, 39]. To simplify, we do not account here for the dependence of the proportionality factor on atmospheric stratification. The data used to obtain the Van der Hoven spectrum are consistent with mean speeds of approximately 6 m/s [5], [7], so that the coefficient of variation of the wind speed fluctuations may be assumed to be about 0.20. To a first approximation, one may therefore neglect the square of the wind speed fluctuations in the expansion of the expression $[U+u(t)]^2$, where U and u(t) are the mean and fluctuating wind speed, respectively [6, p. 169], [8]. It may then be assumed that the normalized spectrum of the surface wind stresses is $\Psi_o(\omega) = S(\omega)/\sigma_u^2$ and that, like the distribution of the wind speeds, the distribution of the wind stresses is approximately Gaussian. Finally, we mention that our approximation of the wind stresses assumes that wind directional effects are small, that is, deviations from the prevailing wind direction do not influence our model significantly. A more exact expression for $\Psi_o(\omega)$ can be obtained; however, this is not needed for the purposes of this work.

C. Shinozuka Representation of Random Excitation. Since we assume $\tau(t)$ is a Gaussian process with variance σ^2 and one-sided spectral density $\sigma^2\Psi_o(\omega)$, we may approximate the process as

$$\tau(t) = \sigma G(t) \qquad (5)$$

$$G(t) = (1/2\pi)^{1/2}(2/\ell)^{1/2} \sum_{n=1}^{\ell} \cos(\omega_n t + \theta_{no}) \quad (i=1,2,3) \qquad (6)$$

where ℓ is a parameter of the model, $\{\omega_n, \theta_{no}; n=1,..,\ell\}$ are independent random variables, $\{\theta_{no}, n=1,..,\ell\}$ are identically uniformly distributed over the interval $[0,2\pi]$, $\{\omega_n; n=1,..,\ell\}$ are nonnegative with common distribution equal to the spectral density of the process, Ψ_o, and

$$\frac{1}{2\pi}\int_0^{\infty} \Psi_o(\omega)d\omega = 1. \qquad (7)$$

It was shown in [9] that (6) can approximate the Gaussian paths of the random process with spectral density $\Psi_o(\omega)$ as closely as desired provided that the finite value of ℓ is sufficiently large. The approximation by (6) of the process with spectral density $\Psi_o(\omega)$ allows the Melnikov approach to be extended to the case of random excitation [4], [10].

III. GENERALIZED MELNIKOV APPROACH

A. Expression for the Melnikov Process. The dynamics of the unperturbed system ($\epsilon=0$) was studied by Allen et al. (1991), who showed that for $z>z_c=(3/2)\delta^{4/3}+\delta^2-1/2$ the phase plane diagram x-y (z=const) has a saddle point and two elliptic centers with coordinates that satisfy the expressions

$$x^3-3x^2+2(\omega_o^2-z)x+2z=0, \quad y=0, \qquad (8a,b)$$

and contains an asymmetrical homoclinic orbit which asymptotically approaches the saddle point in forward and backward time. The two lobes of the homoclinic orbit separate three distinct regions of phase space, corresponding to three oscillatory regimes: one inside each lobe of the separatrix and a third outside the separatrix. The saddle point corresponds to the intermediate root of (8). The saddle point and the separatrix depend continuously on z and form, respectively, a one-dimensional manifold $\gamma(z)$ of (x,y,z) points and a two-dimensional manifold of (x,y,z) points.

The existence of an oscillatory solution -- a hyperbolic T^{ℓ}-torus -- among the solutions of the invariant manifold M_ϵ of the perturbed system is established by obtaining a solution of the equations

$$\overline{g_3(\gamma(z_o))} = 0, \qquad \overline{d[g_3(\gamma(z_o))]/dz} \neq 0, \qquad (9a,b)$$

where g_3 denotes the expression between large brackets in the r.h.s. of (1c) and the overbars indicate averaging with respect to the angles defining the extended phase space of the system [2], [4]. As shown in [4], (9) yield exactly the same solution as in the harmonic perturbation case studied in [1], that is,

$$z_s=-\tfrac{1}{2}x^2+(x-1)\tau_o/r, \quad \overline{d[g_3(\gamma(z_o))]/dz}\,|_{z=z_s} < 0. \qquad (10a,b)$$

Equation 10a is substituted in (8b). The intermediate root of the resulting equation is denoted by x_s. The coordinates (x_s,y_s,z_s), where y_s is given by (8b), define to first order the hyperbolic T^{ℓ}-torus on the invariant manifold M_ϵ^{ℓ}. From (10b) it follows that the orbit restricted to M_ϵ^{ℓ} is linearly stable [2].

For the case of harmonic forcing studied in [1], the expression for the Melnikov function is

$$m(s,\theta_o) = rC_1+\tau_1 C_2(\omega)\cos(\omega s+\theta_o) \qquad (11)$$

$$C_1=C_1^{\pm}=(x_s-\tau_o/r)[8d\tan^{-1}(V_{m\pm}/(2k_o))-k_o b] \qquad (12)$$

$$C_2(\omega) = C_2^{\pm}(\omega) = -4\pi d\,\frac{\sinh[\omega\cos^{-1}(\alpha_\pm)/k_o]}{\sinh(\omega\pi/k_o)} \qquad (13)$$

$$k_o = (z_s-\omega_o^2+3x_s-(3/2)x_s^2)^{1/2} > 0 \qquad (14)$$

$$d=k^2+(x_s-1)^2, \quad \alpha_\pm = \frac{bV_{m\pm}}{8k_2-bV_{m\pm}}, \qquad (15)$$

$$b=4(x_s-1), \quad V_{m\pm}=\tfrac{1}{2}[b\pm(b^2+16k^2)^{1/2}]. \qquad (16)$$

$C_2(\omega)$ may be interpreted as a linear transfer function. It is also referred to as scaling factor [11]. From the linearity of the Melnikov function with respect to the perturbative terms [12], it follows that, if the forcing is given by (5), (6), the expression for the Melnikov process induced by the random forcing is

$$m(s,\theta_{1o},\theta_{2o},..,\theta_{\ell o})=$$

$$rC_1+[(1/2\pi)(2/\ell)]^{1/2}\sigma\sum_{n=1}^{\ell}C_2(\omega_n)\cos[\omega_n t+(\omega_n s+\theta_{no})] \qquad (17)$$

where C_1 and C_2 are given by (12) and (13). The expectation and variance of the Melnikov process are [10, p. 326]:

$$E[m(s,\theta_{1o},\theta_{2o},..,\theta_{\ell o})]=rC_1. \qquad (18)$$

$$Var[m(s,\theta_{1o},\theta_{2o},..,\theta_{\ell o})] = \frac{\sigma^2}{2\pi}\int_0^{\infty} C_2^2(\omega)\Psi_o(\omega)d\omega. \qquad (19)$$

B. Mean Upcrossing Rate of Melnikov Function as an Index of Chaotic Behavior. For a given realization of the forcing process, the necessary condition for the occurrence of chaos is that the corresponding realization of the Melnikov process have simple zeros. Since, in the limit of large ℓ, the Melnikov process with expectation (18) and variance (19) is Gaussian, the probability that a realization of the process will have simple zeros is one -- no matter how small the noise -- provided that the time interval is infinitely long [10]. However, the probability that the Melnikov function has simple zeros during a finite time interval T is $p_T < 1$. Let us denote the mean and the standard deviation of the Melnikov function by E[m] and σ_m. If the ratio $k = E[m]/\sigma_m$ is sufficiently large (say, $k > 1.5$; i.e., if the excitation is sufficiently small in relation to the damping), then p_T can be closely approximated by using: (a) the Kac-Rice formula [13] for the rate of upcrossing of a threshold k by a standardized Gaussian process with one-sided spectral density $\Psi_m(\omega)$, E(k), and (b) the assumption that the upcrossing is a rare event described by a Poisson process, so that

$$p_T = 1 - \exp[-E(k)T] \tag{20}$$

$$E(k) = \nu\exp(-k^2/2) \tag{21}$$

$$\nu = (1/2\pi)\{[\int_0^\infty \omega^2\Psi_m(\omega)d\omega]/[\int_0^\infty \Psi_m(\omega)d\omega]\}^{1/2} \tag{22}$$

Note that the restriction to relatively large k does not apply to (21).

We denote by t_{ex} the time spent by the system in a region of phase space associated with a potential well before it exits from that region. In the limit of weak perturbations the mean value of t_{ex} must be at least as large as $1/E(k)$ (there can be no exit from that region as long as the stable and unstable manifolds do not intersect). Therefore, in that limit, $1/E(k)$ is a lower bound for the mean exit time, $1 - p_T$ is a lower bound for the probability that $t_{ex} > T$, and p_T is an upper bound for the probability that $t_{ex} < T$.

IV. EXAMPLE

We consider the case $\delta = 0.3003$, $\tau_o/r = 3.236$, also studied in [1]. From (8a) and (10a), $x_s = 1.236$, $z_s = 0$. From (8), the unperturbed system has the fixed points {0,0}, {1.236,0} and {1.764,0}. From (12) to (16), $C_1^+ = 2.524$, $C_{1-} = -7.076$, and

$$C_2^+(\omega) = -4.8\sinh(2.064\omega)/\sinh(5.500\omega) \tag{23}$$

$$C_2^-(\omega) = -4.8\sinh(3.436\omega)/\sinh(5.500\omega) \tag{24}$$

For harmonic forcing with period $\omega = 1$ (a case examined by Allen et al., and assumed therein to correspond to a dimensional time T=4 days), $C_2^+ = -0.152$, $C_2^- = -0.609$, and the necessary condition for exits from the region corresponding to the interior of the left well is satisfied for $\sigma/r > 8.220$, where σ is the standard deviation of the harmonic forcing. The inequality $\sigma/r > 11.74$ obtains for the right well. If $\sigma/r < 8.22$ there can be no exits from either the left region or, a fortiori, from the right region. Note that, for $\omega = 1$, the excitation needed to satisfy the necessary condition for the occurrence of chaos (which is also the necessary condition for the occurrence of exits associated with chaotic behavior) is stronger for the right well [which is the smaller of the two wells -- see [1] and weaker for the larger well.

We now consider the case of random forcing with spectrum $\sigma^2\Psi_o(\omega)$. Figures 2 and 3 show, respectively, the square of the transfer function, $[C_2^-(\omega)]^2$, and the normalized spectral density of the Melnikov function, $\Psi_{mo-}(\omega) = [C_2^-(\omega)]^2\Psi_o(\omega)$. Equation 22 applied to $\Psi_{mo-}(\omega)$ and $\Psi_{mo+}(\omega)$ yields $\nu^- = 0.0702$ and $\nu^+ = 0.0534$, respectively. The transfer functions are seen to suppress or considerably reduce the spectral components of the wind stress with frequencies $\omega > 1.75$ or so, and to amplify low frequency components. The standard deviations of the Melnikov process are $\sigma_{m+} = 0.359\sigma$ and $\sigma_{m-} = 0.778\sigma$, respectively, so $k_- = |E[m_-]/\sigma_{m-}| = 9.1r/\sigma$ and $k_+ = E[m_+]/\sigma_{m+} = 7.03r/\sigma$. We assume $\sigma/r < 8.22$, say $\sigma/r = 4$, so $k_- = 2.275$, $k_+ = 1.76$. From (21), $E(k_-) = 0.0053$ and $E(k_+) = 0.0113$. We have $1/E(k_-) = 188.6$ (corresponding to $188.6 \times 4/(2\pi) = 119.7$ days) and $1/E(k_+) = 88.5$ (56.3 days), so $1/E(k_-) > 1/E(k_+)$, as one would intuitively expect, since the left well is larger than the right well. For the left well, for T = 1 month (i.e., 47.1 nondimensional time units), $p_{T-=1mo} = 0.22$ (21); for one year $p_{T-=1yr} = 0.95$. For the right well $p_{T+=1mo} = 0.41$, $p_{T+=1yr} = 0.998$.

Suppose that a decision would hinge on whether, given the wind spectrum (4), the probability of non-occurrence of chaotic jumps in the current would be at least 0.5 during one month. The lower bounds $1 - p_{T-=1mo} = 1 - 0.22 = 0.78$ and $1 - p_{T+=1mo} = 0.59$ would provide a conservative basis for such a decision.

For ilustrations of chaotic behavior for our system, see [1].

V. SUMMARY AND CONCLUSIONS

We considered a model of offshore currents driven by wind speed fluctuations, studied in [1] for the periodic fluctuations case. We examined the behavior of this model under forcing by randomly fluctuating wind. The following results were obtained:

(1) Necessary conditions for chaos induced by stochastic

excitations depend on the product of the forcing spectrum and the square of the transfer function in the expression for the stochastic Melnikov transform.

(2) The development of a stochastic Melnikov process allows the estimation, via the mean time between zero upcrossings of the Melnikov process, of: (a) lower bounds for the mean time of exit (i.e., the mean time between chaotic jumps) from regions of phase space associated with potential wells; and, provided that the excitation is sufficiently small in relation to the damping, (b) upper bounds p_T for the probabilities p_{ex} of occurrence of exits from those regions during a specified time interval T (lower bounds $1-p_T$ for the probabilities $1-p_{ex}$ that no exits will occur during T).

The possibility of chaos was assessed on the basis of the Melnikov approach for two cases. In the first case [1] it was assumed that the excitations induced by wind are harmonic and that the entire energy of the wind stresses is concentrated at the frequency corresponding to the peak of the wind stress spectrum. Under these assumptions the deterministic Melnikov approach is applicable. This approach led to the conclusion that, for a certain parameter range, no exits are possible. Other choices of the harmonic forcing frequency can be made, but no guidance is available on how well those choices will reflect the effects of the actual, stochastic forcing unless the problem is examined by using a stochastic, rather than a deterministic approach.

In the second case the wind stress fluctuations were assumed to be Gaussian with a Van der Hoven spectral density. This assumption guarantees that, no matter how small the forcing, exits can occur, although the mean exit times may be quite long. Lower bounds for the probability that exits will not occur during specified time intervals were obtained, which could provide useful conservative estimates on the probability of non-occurrence of chaotic transport during a specified time interval.

The deterministic Melnikov function provides only a necessary condition for the occurrence of chaos, that is, the fact that the Melnikov function has simple zeros does not mean by itself that chaos will actually occur. In fact, chaos will generally occur only for considerably larger excitations than the minimum excitation for which the Melnikov function has simple zeros.

Similarly, the fact that, with probability p_T, the stochastic Melnikov process will have simple zeros during a time interval T does not mean that chaotic transport will occur during T with probability p_T. In fact, the excitation to which there corresponds a probability p_T of chaotic transport during T is considerably larger than the excitation that causes simple zeros of the Melnikov process to occur during T with probability p_T. Put differently, for a given excitation, $1-p_T$ is a weak lower bound of the probability $1-p_{ex}$ that chaotic transport will not occur during the time interval T, that is, $1-p_T$ provides a conservative estimate of that probability.

ACKNOWLEDGMENT

Partial support of the Office of Naval Research, Ocean Engineering Division, is acknowledged with thanks. The writer also wishes to acknowledge useful discussions with M.R. Frey of the Department of Mathematics, Bucknell University.

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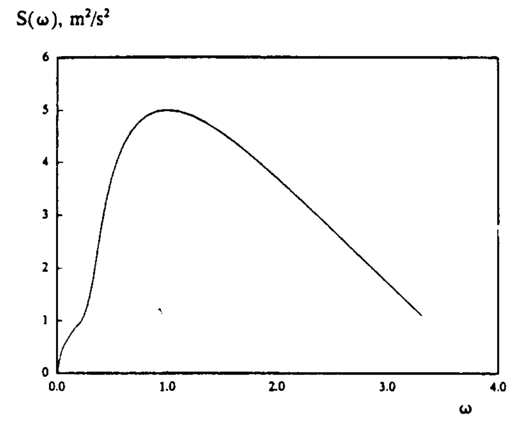

Fig. 1. Spectral density of wind speed fluctuations, $S(\omega)$.

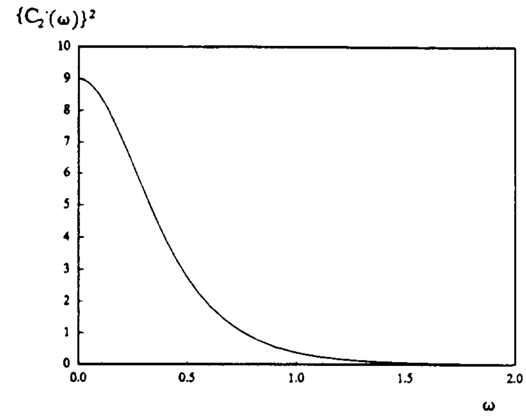

Fig. 2. Square of transfer function, $\{C_2^-(\omega)\}^2$.

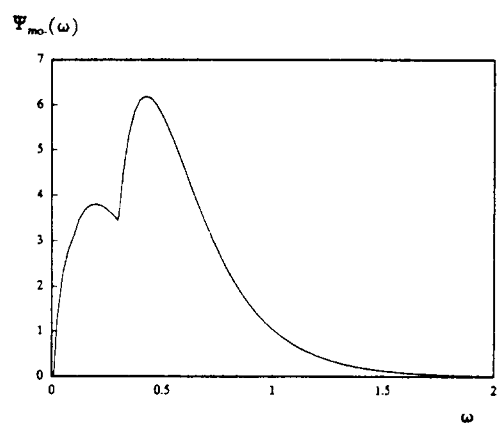

Fig. 3. Spectral density of the Melnikov process, $\Psi_{m\omega^-}(\omega)$.