# A stiffness model for control and analysis of a MEMS hexapod nanopositioner 

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#### Abstract

This paper presents a stiffness-based kinematic model for analysis and control of a Micro-ElectroMechanical Systems (MEMS) flexure-based hexapod nanopositioner that was previously built by the National Institute of Standards and Technology (NIST). This nanopositioner is capable of producing high-resolution motions in 6 degrees of freedom (DOF) by actuating three planar X-Y positioning stages. Given a large number of flexure-based nanopositioners, the modeling and controller design has been a challenging task due to their inherent structural complexity and difficulties in measuring 6DOF positioning accuracy. In this paper, we discuss kinematic models for developing an open loop controller and an analytical approach routine for this nanopositioner. These models include a novel model for calculating the nonlinear stiffness of the $\mathrm{X}-\mathrm{Y}$ positioning stage and a stiffness-based Jacobian matrix of the hexapod mechanism for the controller. Compared with Finite Element (FE) simulations, we conclude that the mean error of the $\mathrm{X}-\mathrm{Y}$ stage control model is $1.93 \%$ within a $55 \mu \mathrm{~m}$ range of motion. To validate the control model, the top platform is commanded to trace a circle of diameter $20 \mu \mathrm{~m}$. The result shows a mean error of $3.38 \%$.


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## 1. Introduction

A nanopositioner is a high precision positioning device used in motion control with nanometer precision. Most nanopositioners are made of flexure mechanisms [1-4] that are formed by multiple (often identical) flexure pivots, leaf springs, or their chains that are designed to produce a defined motion upon the application of an appropriate load. These mechanisms have the advantage of no backlash and ultra-high precision. Nanopositioners [5-12] have been widely used in precision engineering and play an important role in emerging nanotechnology and medicine [13-17]. However, the controller modeling and design have been challenging tasks due to their inherent structural complexity and difficulties in measuring 6 degree-of-freedom (DOF) positioning accuracy.

In terms of actuation methods, comb drives [18-20] and thermal actuators [21,22] are commonly used force actuators in Micro-Electro-Mechanical Systems (MEMS). A stiffness-based control model is required for deriving the control model for these actuators, especially when position sensors are not available for feedback control. A lot of prior work by other researchers has been done regarding the stiffness analysis of planar and spatial nanopositioners. Yao et al. [23] calculated the stiffness and derived a kinematic model of a planar micropositioning stage. Ji et al. [24] designed a 6DOF nanopositioner and derived a control model based on the stiffness, the comb drive force, and the integrated capacitive displacement sensor. Yong et al. [25] built and tested a control model of thermal actuation force and motion for a serial kinematic MEMS X-Y stage for multifinger manipulation. Gao et al. [26] did the static analysis of a piezodriven micropositioning stage which adopts the notch flexures. Sun et al. [27] studied the beam stiffness and derived the system spring stiffness of a silicon integrated micro nano-positioning XY-stage with comb drives. Laszczyk et al. [28] designed and

[^0]
## Nomenclature

| [K], [C] | 6-by-6 stiffness and compliance matrices |
| :---: | :---: |
| [ $K^{c}$ ] | the stiffness matrix of the circular notch flexure joint |
| [ $K^{\text {b }}$ ] | the stiffness matrix of a beam flexure with rectangular cross section |
| [ $K^{t}$ ] | the stiffness matrix of the thermal actuation mechanism |
| [ $K^{w}$ ] | the stiffness matrix of a wire flexure |
| [ $K^{X}$ ] | the stiffness matrix of the X stage |
| $\left[K_{j}^{X Y}\right]$ | the stiffness matrix of each $\mathrm{X}-\mathrm{Y}$ stage, $j=1,2,3$ |
| $\hat{T}^{p}$ | the twist representing the motion of the top platform center |
| $\hat{T}_{j}$ | the motion twist of the $\mathrm{X}-\mathrm{Y}$ stage, $j=1,2,3$ |
| $\hat{W}_{j}$ | the actuating wrench on the center of the ends of two wire flexures, $j=1,2,3$ |
| $\Delta^{a}$ | a 6-by-1 vector representing the displacement of three X-Y stage centers by thermal actuation force |
| [Ad] | a 6-by-6 adjoint transformation matrix |
| [R] | a 3-by-3 rotation matrix |
| [ $Z(\theta)$ ] | a 3-by-3 rotation matrix about $z$ axis |

modeled an X-Y microstage for micro-opto-electro-mechanical systems (MOEMS) applications with constant beam stiffness. Shi et al. [29] calculated the workspace of a meso-scale hexapod nanopositioner based on stiffness of flexures and inverse kinematics.

However, these simple models assume a constant stiffness matrix and neglect the nonlinear component in relatively large deflections. Moreover, there is relatively less work done in deriving analytical models of a 6DOF stiffness matrix due to the structural complexity. FE simulation and physical experiments are the two commonly used methods for analyzing the 6DOF stiffness of a nanopositioner. Brouwer et al. [30] modeled and derived the stiffness of a 6DOF manipulator by using software SPACAR. ${ }^{1}$ Yang et al. [31] developed a method to measure in-plane stiffness of a nanopositioner by using atomic force microscopy (AFM) and measured the stiffness of the hexapod nanopositioner [32]. When being compared with the analytical models, these methods are time consuming, costly, and inefficient in the design process.

In this paper, an analytical control model of the $\mathrm{X}-\mathrm{Y}$ positioning stage is derived. It includes the derivation of constant stiffness and nonlinear stiffness by parasitic displacement. Based on the control model of the $\mathrm{X}-\mathrm{Y}$ stage and a 6DOF stiffness matrix of the hexapod mechanism, we derived an open loop control model for the NIST hexapod nanopositioner.

The rest of the paper is organized as follows. Section 2 presents the background and basic approaches used in this paper. In Section 3, we illustrate the derivation of the constant stiffness model of the $\mathrm{X}-\mathrm{Y}$ positioning stage for loading. Section 4 presents the nonlinear stiffness-based actuating model of the $\mathrm{X}-\mathrm{Y}$ positioning stage. Section 5 presents the derivation of the control model of the hexapod nanopositioner. In Section 6, the FE model and an example application of the controller are explained. The errors of the analytical model with the FE model are then calculated and analyzed.

## 2. Background and approaches

In this section, we first illustrate the topology of the hexapod MEMS nanopositioner and then present the basic methodology to be used in compliance and stiffness analysis of general flexure mechanisms.

### 2.1. The hierarchical structure of the NIST hexapod nanopositioner

The hexapod nanopositioner to be studied in this paper was built by the National Institute of Standards and Technology (NIST), shown in Fig. 1(a). The overall footprint of this device is about 12 mm by 10 mm .

The hierarchical structure of the nanopositioner basically follows a top-bottom process. In the top level, the positioner is composed of three main parts: three $\mathrm{X}-\mathrm{Y}$ positioning stages [25], six struts, and one top platform, see Fig. 1(b). Three X-Y positioning stages, which can generate two orthogonal motions, are symmetrically laid out on the base plane. The moving plate of each $\mathrm{X}-\mathrm{Y}$ stage supports two struts, which are firmly attached to the plate at one end and to the top platform at the other end both via the wire flexure joints. This design allows 6DOF motion of the top platform through elastic deformation of these flexures and thus eliminates backlash and increases repeatability. The top platform, which is considered as rigid, is the load-carrying part or the end-effector of the device. Those flexure joints and rigid parts form the bottom level of the positioner. For convenience, we call the combination of the six struts and the top platform as the "hexapod mechanism".

[^1]

Fig. 1. The NIST hexapod nanopositioner. (a) The physical prototype. (b) The schematic view of the hexapod nanopositioner. Each spring symbol represents a general compliant element whose stiffness can be mathematically represented by a six by six stiffness matrix [K].

### 2.2. Compliance and stiffness analysis of flexure mechanisms

One important step toward the design and control of flexure mechanisms is compliance analysis or mapping of which the goal is to determine the relationship between deformation and load applied to the device. Here, we apply the screw theory approach to analyze the stiffness and the motion of the hexapod nanopositioner. This method has been well studied by a number of authors, e.g. [33]. For convenience, a brief description of this approach is given below.

We denote the deformation of a flexure mechanism by a general twist $\hat{T}=\left(\theta_{x}, \theta_{y}, \theta_{z} ; \delta_{x}, \delta_{y}, \delta_{z}\right)$ and the load is denoted by a wrench $\hat{W}=\left(F_{x}, F_{y}, F_{z} ; M_{x}, M_{y}, M_{z}\right)$. They are related by,

$$
\hat{W}=[K] \hat{T}, \quad \hat{T}=[C] \hat{W}, \quad[C][K]=[I]
$$

where $[K]$ and $[C]$ are six by six stiffness and compliance matrices, respectively. Depending on how flexure elements are connected, we can have serial flexure chains or parallel flexure chains. Mathematically, the overall compliance matrix of a serial flexure chain is calculated as,

$$
\begin{equation*}
[C]=\sum_{i=1}^{m}\left[A d_{i}\right]\left[C_{i}\right]\left[A d_{i}\right]^{-1} \tag{1}
\end{equation*}
$$

where $m$ is the number of flexure elements and $\left[A d_{i}\right]$ is the so-called 6-by-6 adjoint transformation matrix,

$$
\left[A d_{i}\right]=\left[\begin{array}{cc}
R & 0  \tag{2}\\
D R & R
\end{array}\right]
$$

Here $[R]$ is a 3 -by- 3 rotation matrix. When rotating about the single $\mathrm{x}, \mathrm{y}$, and z axis, $[R]$ can be written as $[X(\cdot)],[Y(\cdot)],[Z(\cdot)]$, respectively. [D] is the skew-symmetric matrix defined by a translational vector $d$. In a similar way, the overall stiffness matrix of a parallel flexure chain is calculated as

$$
\begin{equation*}
[K]=\sum_{i=1}^{m}\left[A d_{i}\right]\left[K_{i}\right]\left[A d_{i}\right]^{-1} . \tag{3}
\end{equation*}
$$

In the remainder of this paper, we use superscripts and subscripts in $[K]$ and $[C]$ for various components of the positioner. For instance, subscripts $j=1,2,3$ are used for each of three $X-Y$ positioning stages and superscripts " $t, b, w, X, X Y$ " are for the thermal actuator, beam flexure, wire flexure, X stage, and $\mathrm{X}-\mathrm{Y}$ stage, respectively.

### 2.3. General analysis procedure

According to the dimensions of the different parts of the mechanism, we divide them into two types: rigid components and compliant components. In our analysis, we are working from the assumption that there is no deformation in the rigid components in order to simplify the analytical model. With regard to compliant components, we assume that the primary deformation that would occur is elastic deformation and the failure of the material occurs when deformation reaches plastic deformation.

In contrast to the hierarchical analysis, the stiffness analysis follows a bottom-up procedure. The kinematic model of the hexapod positioner is shown in Fig. 1(b) in which the spring symbols represent main compliant components such as flexure joints and simplified stage. Mathematically these compliant components are represented by the 6-by- 6 stiffness matrices [ $K$ ]. The global coordinate frame is built at the top center of the top moving platform in the home (undeformed) position.

When the base stages are actuated by the $\mathrm{X}-\mathrm{Y}$ positioning stages to move in x and y directions, the top platform can position and orient in 3D space through elastic deformation of the compliant components. Given the target position of the top platform center, we build this model and calculate the required force as the input. As shown in Fig. 1(b), the bottom $\mathrm{X}-\mathrm{Y}$ positioning stage is a serial connection of two components, X stage and Y stage, whose stiffnesses are noted as $\left[K^{X}\right]$ and $\left[K^{Y}\right]$. The next step in the procedure will be analysis of the hexapod mechanism. As described earlier, the mechanism consists of six struts that are connected in parallel. Each strut is modeled as a serial chain of two wire flexures and one long compliant rod. The compliance matrix of each wire flexure is denoted by $\left[C^{w}\right]$. The matrix $\left[C^{w}\right]$ is calculated in the Appendix A. The compliance matrix of each strut is calculated as $\left[C^{s}\right]$ by formulation (1). By a series of serial and parallel combinations of individual building blocks, we can finally calculate the motion of the top platform, denoted by a twist $\hat{T}^{p}$.

## 3. Stiffness model of the $X-Y$ positioning stage

In this section, we will derive the stiffness of the $\mathrm{X}-\mathrm{Y}$ stages for loading, as shown in Fig. 2. The $\mathrm{X}-\mathrm{Y}$ positioning stage is a kinematic chain of two stages: inside $X$ stage and outside $Y$ stage. Each consists of three main parts: thermal actuator, lever amplifiers, and guiding mechanisms. The coordinate frame is built at the center of the center stage. In what follows, we calculate the stiffness matrix of each of these parts and combine them to obtain the stiffness model of the $\mathrm{X}-\mathrm{Y}$ stages.

### 3.1. Thermal actuator

Thermal actuators are widely used in MEMS mechanisms for producing large forces owing to their scaling capability using multiple parallel beams. The thermal actuator used in this device consists of 15 beams on each side connected in parallel, as shown in Fig. 3. There is an electrical pad on each side for interfacing with a power supply circuit. When the current goes through the thermal actuator beams, the resistance of the beams will increase the temperature of the beams, and this phenomenon is


Fig. 2. The layout of $X-Y$ positioning stage.


Fig. 3. Thermal actuator.
called "joule heating". And then the increased temperature will cause thermal expansion of each beam. Finally, the actuator produces an output force $F$ exerting on the actuation bar, calculated as

$$
\begin{equation*}
F=2 \alpha n \bar{T} E w_{1} h \sin \theta^{t}, \tag{4}
\end{equation*}
$$

where $\alpha, n, \bar{T}, E, w_{1}, h$, and $\theta^{t}$ are physical or material geometric parameters of the thermal actuators defined in Table 1.
Now, we calculate and analyze the stiffness and motion of the $\mathrm{X}-\mathrm{Y}$ stage by following the approach described in Section 2.2. The thermal actuator is composed of 30 long beams with a rectangular cross section of $h$ thick and $w_{1}$ wide. According to the wrench and twist defined in Section 2, the compliance matrix of beams is written as [ $C^{b}$ ] in the Appendix $A$, when the coordinate frame is placed at the center of the free end of a cantilever beam. By substituting parameters in Table 1 and inverting the compliance matrix, we obtain the stiffness matrix of one actuator beam as

$$
\left[K^{t b}\right]=\left[\begin{array}{cccccc}
0 & 0 & 0 & 8.70 \times 10^{-2} & 0 & 0  \tag{5}\\
0 & 0 & -2.16 \times 10^{-2} & 0 & 4.32 \times 10^{-5} & 0 \\
0 & 3.91 \times 10^{-2} & 0 & 0 & 0 & 7.83 \times 10^{-5} \\
3.06 & 0 & 0 & 0 & 0 & 0 \\
0 & 26.1 & 0 & 0 & 0 & 3.91 \times 10^{-2} \\
0 & 0 & 14.4 & 0 & -0.22 & 0
\end{array}\right]
$$

Table 1
Design parameters and values of the $\mathrm{X}-\mathrm{Y}$ positioning stage.

| Symbol | Design parameter | Values |
| :--- | :--- | :--- |
| $h$ | Out of plane thickness of the MEMS device | $30 \mu \mathrm{~m}$ |
| $e_{1}$ | Vertical distance between thermal actuator bar and notch | $667.4 \mu \mathrm{~m}$ |
| $e_{2}$ | Horizontal distance between thermal actuator bar and notch | $300 \mu \mathrm{~m}$ |
| $e_{3}$ | Short horizontal distance between two notches | $100 \mu \mathrm{~m}$ |
| $e_{4}$ | Long horizontal distance between two notches | $1000 \mu \mathrm{~m}$ |
| $e_{5}$ | Horizontal distance between thermal actuator bar and top notch | $1400 \mu \mathrm{~m}$ |
| $e_{6}$ | Vertical distance between notch and X stage center | $415 \mu \mathrm{~m}$ |
| $e_{7}$ | Horizontal distance between notch and Y stage center | $1958.5 \mu \mathrm{~m}$ |
| $e_{8}$ | Vertical distance between X stage center and Y stage center | $468.5 \mu \mathrm{~m}$ |
| $\theta^{t}$ | Angle of thermal actuator beam | 0.068 rad |
| $l^{b t}$ | Length of thermal actuator beam | $1000 \mu \mathrm{~m}$ |
| $w_{1}$ | Width of thermal actuator beam | $22.3 \mu \mathrm{~m}$ |
| $w_{2}$ | Distance between two thermal actuator beams | $23 \mu \mathrm{~m}$ |
| $w_{3}$ | Width of thermal actuator beam along actuator bar | $22.3 \mu \mathrm{~m}$ |
| $r$ | Radius of the circular flexure hinges | $40 \mu \mathrm{~m}$ |
| $w_{4}$ | Flexure hinge neck width | $7 \mu \mathrm{~m}$ |
| $w_{5}$ | Amplifier beam width | $100 \mu \mathrm{~m}$ |
| $n$ | Number of thermal actuator beams on each side | 15 |
| $E$ | Young's module | 130 GPa |
| $Y$ | Yield strength | 7 GPa |
| $v$ | Poisson's ratio | 0.28 |
| $\alpha$ | Coefficient of thermal expansion | $3 \times 10^{-9} \mathrm{~T}+3 \times 10^{-6}\left({ }^{\circ} \mathrm{C}-1\right.$ |
| $\bar{T})^{\mathrm{a}}$ | Actuator average temperature | $<550^{\circ} \mathrm{C}$ |

[^2]where each element has a specific physical meaning and unit. For instance, the element in row 6 and column 5 of the matrix, -0.22 , gives the required moment in unit $N \mu \mathrm{~m}$ about the $z$ axis to produce a $1 \mu \mathrm{~m}$ translation in $y$ direction.

The coordinate transformations of each beam relative to the frame shown in Fig. 3, are given by

$$
\begin{aligned}
& {\left[R_{1}^{i}\right]=\left[Z\left(\frac{\pi}{2}+\theta^{t}\right)\right] \quad \mathbf{d}_{1}^{i}=\left(-\left(w_{2}+\frac{w_{3}}{2}\right) i,-50,0\right),} \\
& {\left[R_{2}^{i}\right]=\left[Z\left(\frac{3 \pi}{2}-\theta^{t}\right)\right] \quad \mathbf{d}_{2}^{i}=\left(-\left(w_{2}+\frac{w_{3}}{2}\right) i, 50,0\right),}
\end{aligned}
$$

where $i=1, \ldots, 15$. The subscript 1 and 2 represent the two sides of the actuator. After applying coordinate transformation for thirty beams which are parallel connected to the center actuator bar, the stiffness of the thermal actuator is calculated as,

$$
\left[K^{t}\right]=\sum_{i=1}^{15}\left(\left[A d_{1}^{i}\right]\left[K^{t b}\right]\left[A d_{1}^{i}\right]^{-1}+\left[A d_{2}^{i}\right]\left[K^{t b}\right]\left[A d_{2}^{i}\right]^{-1}\right)=\left[\begin{array}{cccccc}
0 & 0 & 0 & 1.33 \times 10^{-2} & 0 & 0  \tag{6}\\
0 & 0 & -498.0 & 0 & 2.60 & 0 \\
0 & 0.368 & 0 & 0 & 0 & 2.35 \times 10^{-3} \\
827.0 & 0 & 0 & 0 & 0 & 0 \\
0 & 174.0 & 0 & 0 & 0 & 0.368 \\
0 & 0 & 1.23 \times 10^{5} & 0 & -498 & 0
\end{array}\right] .
$$

In this matrix, the maximum value of $1.23 \times 10^{5} \mathrm{~N} \mu \mathrm{~m} /$ rad represents the moment about the $z$ axis, $M_{z}$, required to produce a rotation of $\theta_{z}=1 \mathrm{rad}$. This value indicates that the device has a very high in-plane stiffness. The value of $2.35 \times 10^{-3} \mathrm{~N} / \mu \mathrm{m}$ indicates that it is relatively easier for the mechanism to translate out of the plane, i.e., it has a low out-of-plane stiffness. This result motivates the study of 6DOF stiffness of the X-Y stage.

### 3.2. Compliance of the circular notch flexure

All of the notch flexures in our device have an identical geometry with a half circle of radius $r=40 \mu \mathrm{~m}$, minimum neck thickness $w_{4}=7 \mu \mathrm{~m}$, as shown in Fig. 4(a). Although several analytical models for calculating the stiffness of notch flexures $[34,35]$ exist, the error is too large ( $>5 \%[36]$ ). Thus, we choose to use the FE method to obtain the 6-by-6 compliance matrix of the notch flexure.

As shown in the FE model in Fig. 4(a), one end of the flexure is fixed and a coordinate frame is built at the free end of the flexure. The facets of both ends are considered to be constrained in two planes with kinematic constraints of 6 DOFs. This is equivalent to attaching two rigid plates at both ends so that there is no stress concentration along the edges of the ends. With these defined boundary conditions, a bending moment $M_{z}$ is applied to the free end. Let us denote the bending angle of the free end by $\theta_{z}$.

We run FE simulations in Abaqus. A linear function is used to fit the simulation data $\theta_{z}$ vs. $M_{z}$, shown in Fig. 4(b). The slope $0.178 \mathrm{rad} / \mathrm{N} \mu \mathrm{m}$ gives the rotational compliance $\theta_{z} / M_{z}$. We also obtain the translational compliance in the $y$ direction by a moment along the $z$ direction as $\delta_{y} / M_{z}=7.11 \mu \mathrm{~m} / N \mu \mathrm{~m}$. Therefore, we obtain the last column of the compliance matrix shown in Eq. (7).


Fig. 4. (a) FE model of notch flexures. (b) The von Mises stress and bending angle $\theta_{z}$ vs. the bending moment $M_{z}$.

Similarly, we can separately apply a force and moment along $x, y$, and $z$ directions and we obtain the other compliant elements. Eventually, the following compliance matrix is obtained,

$$
\left[C^{c}\right]=\left[\begin{array}{cccccc}
0 & 0 & 0 & 9.12 \times 10^{-2} & 0 & 0  \tag{7}\\
0 & 0 & -0.802 & 0 & 2.01 \times 10^{-2} & 0 \\
0 & 7.10 & 0 & 0 & 0 & 0.178 \\
1.51 & 0 & 0 & 0 & 0 & 0 \\
0 & 301.0 & 0 & 0 & 0 & 7.11 \\
0 & 0 & 42.6 & 0 & -0.801 & 0
\end{array}\right] .
$$

### 3.3. Displacement of the amplifier and guiding beams

The lever amplifier is used to produce a larger translational displacement at the sacrifice of output force. The displacement amplification ratio of the lever mechanism is calculated as $e_{4} / e_{3}=10$, where $e_{4}$ and $e_{3}$ are shown in Fig. 5(a) and defined in Table 1.

As shown in Fig. 2, the lever amplifier mechanism has two symmetrical parts and each part consists of three notch flexure joints and a middle Z-shaped amplifier beam. When the thermal actuation force is exerted, the translational and rotational displacements in the opposite direction are canceled due to the geometric symmetry. Thus, only translational displacement in the same direction remains to move the connected stage. Note that the stroke of the amplifier is constrained by the allowable rotational angle of the notch flexure. Although the stiffness of single crystal silicon is quite low, the FE simulations show a maximum rotation angle of $5^{\circ}$ with a safety factor of 3.5 .

The guiding mechanism also has two symmetrical parts and each consists of two notch flexure joints and a middle Z-shaped guiding beam. The purpose of the guiding mechanism is to increase the out-of-plane stiffness. The parallel design of the guiding mechanism is also used to ensure the X stage and the Y stage move in pure translation upon exerting the thermal actuation force.

Since the amplifier and guiding beams have the same structures, we use the same model, shown in Fig. 5(a), to simplify the stiffness calculation. In this model, the amplifier beam and the guiding beam are both considered as a combination of three deformable gray beams and blue rigid segments. Point $P_{k}(k=1, \ldots, 4)$ is located at the cross section center on one end of the deformable notch flexure or the gray beam at the home position. We serially connect the beams according to the positions of $P_{k}$ and derive the stiffness at the home position as $\left[K^{a l}\right]$. As shown in Fig. 5(b), the amplifier beam reaches a new position as the $\mathrm{X}-\mathrm{Y}$ stage is moving. In the model, we first assume the amplifier beam does a rigid body motion and rotates around $P_{0}$. Then, we denote the new position of point $P_{k}(k=1, \ldots, 4)$ by $P_{k^{\prime}}$ and derive the stiffness at the new position as $\left[K^{a n}\right]$.


Fig. 5. Model of lever amplifier. (a) Amplifier beam at original position. (b) Amplifier beam at deformed position.

### 3.4. Stiffness of $X-Y$ positioning stage

In this section, we derive the stiffness of the $\mathrm{X}-\mathrm{Y}$ positioning stage which is subject to a loading on the stage center. As shown in Fig. 2, the X stage and Y stage have the same topology. Therefore, we will only show the derivation of the stiffness matrix of the X stage, $\left[K^{X}\right]$. The stiffness of the Y stage, $\left[K^{Y}\right]$, can be derived similarly.

Fig. 6(a) shows the CAD drawing of the $X$ stage and we build the coordinate frame in the stage center. A schematic view of the topology of the $X$ stage is shown in Fig. 6(b), where $\left[K^{t}\right]$ is the stiffness of thermal actuator and $\left[K^{c}\right]$ is the stiffness of notch flexures. In order to derive the stiffness of the X stage, we developed a model that divides the stage into four building blocks shown as the dotted line in Fig. 6(c). These four blocks are connected to the center stage in parallel. Due to symmetry, blocks 1 and 2 have an identical topology and dimensions. Furthermore, block 3 and 4 are also identical. The derivation of the stiffness matrix of the $\mathrm{X}-\mathrm{Y}$ positioning stage is given below:

1. Calculate the stiffness in blocks 1 and 2 as $\left[K^{b 1}\right]=\left[K^{b 2}\right]$. Both blocks 1 and 2 are formed by three notch flexures, a half thermal actuator, and one amplifier beam. The half thermal actuator $\left[K^{t}\right] / 2$ is firstly serially connected to a notch flexure and secondly connected to another notch flexure in parallel. Thirdly, they are serially connected to the amplifier beam $\left[\mathrm{K}^{a l}\right]$ and another notch flexure. The other end of this flexure is attached to the center stage which is considered as a rigid body.
2. Calculate the stiffness in blocks 3 and 4 as $\left[K^{b 3}\right]=\left[K^{b 4}\right]$. Both blocks 3 and 4 are a serial chain of two notch flexures and a guiding beam $\left[K^{a l}\right]$. Following the formulation of serial flexure chains given in Eq. (1), we calculate $\left[K^{b 3}\right]$ and $\left[K^{b 4}\right]$.
3. Calculate the stiffness of each independent $X$ and $Y$ positioning stage as $\left[K^{X}\right]$ and $\left[K^{Y}\right]$. As shown in Fig. 6(c), blocks 1-4 are connected to the center stage in parallel. Therefore the stiffness of the X positioning stage is calculated as

$$
\left[K^{X}\right]=\sum_{i=1}^{4}\left[K^{b i}\right]
$$

where we assume that appropriate coordinate transformations have been applied. Following a similar procedure, we can obtain the stiffness matrix of the $Y$ stage, $\left[K^{Y}\right]$. Their compliance matrices are calculated as the inverse of their stiffness matrix, i.e.,

$$
\left[C^{X}\right]=\left[K^{X}\right]^{-1}, \quad\left[C^{Y}\right]=\left[K^{Y}\right]^{-1} .
$$

4. Calculate the compliance matrix of each of the three $X-Y$ positioning stages $\left[C_{j}^{X Y}\right]$. The $X$ stage is serially connected with a $Y$ stage to form an $\mathrm{X}-\mathrm{Y}$ positioning stage shown in Fig. 1(b). The compliance matrix of each $\mathrm{X}-\mathrm{Y}$ positioning stage is calculated as

$$
\begin{equation*}
\left[C_{j}^{X Y}\right]=\left[A d_{j}^{X}\right]\left[C^{X}\right]\left[A d_{j}^{X}\right]^{-1}+\left[A d_{j}^{Y}\right]\left[C^{Y}\right]\left[A d_{j}^{Y}\right]^{-1}, j=1, \ldots, 3, \tag{8}
\end{equation*}
$$

where $\left[A d_{j}^{X}\right]$ and $\left[A d_{j}^{Y}\right]$ are the 6-by-6 adjoint matrix representing the coordinate transformations of the X or Y stage to each X Y stage center, respectively, according to the layout of the $\mathrm{X}-\mathrm{Y}$ stages in Fig. 1(b).


Fig. 6. The models of the $X$ positioning stage. Except the amplifying and guiding beams, the shaded blue segments are considered rigid bodies while spring symbols represent compliant segments. (a) The CAD drawing of the $X$ positioning stage. (b) The schematic view of the $X$ stage showing various components. (c) The schematic view of the model for calculating the stiffness.

## 4. Actuation force of $X-Y$ positioning stage

In this section, we derive the model for actuating the $\mathrm{X}-\mathrm{Y}$ positioning stage which is composed of two components: linear component $F^{l}$ and nonlinear component $F^{n}$.

### 4.1. Linear force analysis

In order to apply the formulation of serial and parallel flexure chains, we divide the center stage into two identical pieces, each connecting with block 3 or 4 shown in Fig. 6(c). This leads us to a topology with blocks 5, 6, and 7 shown in Fig. 7. Block 5 is the thermal actuator. Block 6 and 7 are identical.

We follow a similar process and derive the stiffness for the thermal actuation force with the following steps.

1. Calculate $\left[K^{b 34}\right]$, the parallel stiffness of blocks 3 and 4 in Fig. 6(c).
2. Calculate $\left[K^{b 6}\right]=\left[K^{b 7}\right]$, the stiffness of blocks 6 and 7 . Block 6 is mainly a serial chain of the amplifier beam $\left[K^{a l}\right]$ in block 1 and half of the stiffness of blocks 3 and $4,\left[K^{b 34}\right] / 2$. Block 7 is mainly a serial chain of the amplifier beam $\left[K^{a l}\right]$ in block 2 and $\left[K^{b 34}\right]$. 3. Let $\left[K^{b 5}\right]=\left[K^{t}\right]$ as the block 5. Recall $\left[K^{t}\right]$ is given in Eq. (6).
3. Calculate the entire stiffness of the positioning stage as a parallel chain of blocks 5,6 , and 7 , i.e.,

$$
\left[K^{l}\right]=\sum_{i=5}^{7}\left[K^{b i}\right],
$$

where we assume appropriate coordinate transformations have been applied. The terminal component of this parallel connection is the thermal actuator bar, on which thermal actuation force is exerted.

By taking the row 1 and column 4 element of the matrix [ $K^{l}$ ], we can calculate the linear component of the thermal actuation force as

$$
F^{l}\left(\delta_{x}^{c}\right)=\delta_{x}^{c}\left[\begin{array}{llllll}
1 & 0 & 0 & 0 & 0 & 0
\end{array}\right]\left[K^{l}\right]\left[\begin{array}{l}
0  \tag{9}\\
0 \\
0 \\
1 \\
0 \\
0
\end{array}\right]
$$

where $\delta_{x}^{c}$ is the translational displacement of the stage center in the $x$ direction according to the coordinate frame of Fig. 6(a).

### 4.2. Nonlinear force analysis

In the linear force analysis, the actuation force is a linear function of the displacement $\delta_{x}^{c}$. As shown in the error analysis (to be discussed later), this linear force may generate a significant error. Furthermore, in linear analysis, blocks 6 and 7 are two


Fig. 7. The schematic view of the model for calculating the linear actuation force.


Fig. 8. The schematic view of the model for calculating the nonlinear actuation force. (a) The schematic view of the $X$ stage. (b) The schematic view of the kinematic model of the half $X$ stage. (c) The schematic view of the compliant model of the half $X$ stage.
independent components. Under external loading, these two blocks may separate or overlap with each other. To correct this, we now propose a nonlinear force model. This model takes into account the nonlinear kinematics of the X and Y stages.

In Fig. 8, the red lines represent the new position deformed from the previous position (shaded in blue). The intersection of two center lines is the home position of the stage center. In Fig. 8(a), the blue position illustrates the home position of the center stage, amplifier, and guiding beams. $P_{c}$ is the position of stage center in the blue position, while $P_{c}{ }^{\prime}$ is the stage center in the new red position. Thus, stage center $P_{c}$ coincides with the intersection of the stage centerlines. The red position shows the deformed mechanism by the thermal actuation force. $P_{c}$ and $P_{c}{ }^{\prime}$ are located along the horizontal center line. There is no vertical movement since the thermal actuation force is along the horizontal centerline and the structures are symmetrical to this line.

In Fig. 8(b), we treat the notch flexures as revolute joints and the other parts as rigid components denoted with lines. Essentially, we obtain a parallelogram four bar linkage which only allows translation in $x$ and $y$ direction. We cut the stage center along the horizontal center line so that the center stage is divided into the top and bottom sections. The free end of the center stage introduces an extra DOF in vertical translation, which is originally constrained by the symmetrical structure about the horizontal center line. The blue position represents the home position as in Fig. 8(a). In the red position, the amplifier beam and guiding beam rotate around $P_{0}$. Given the main horizontal displacement $\delta_{\chi}^{c}$, we can derive the corresponding rotation angle of the amplifier and guiding beams as $\phi=\arcsin \left(\delta_{x}^{c} / e_{4}\right)$. The parasitic vertical displacement can be calculated as

$$
\begin{equation*}
\delta_{y}^{c}=e_{4}(1-\cos \phi)=e_{4}-\sqrt{e_{4}^{2}-\left(\delta_{x}^{c}\right)^{2}} \tag{10}
\end{equation*}
$$

In Fig. 8(c), we use the compliant model to substitute the kinematic model of the red position in Fig. 8(b). The topology of block 8 and 9 are similar to block 1 and 3 in Fig. 6. We first substitute $\left[K^{a l}\right]$ in block 1 and 3 as $\left[K^{a n}\right]$ and calculate the stiffness at the new position as $\left[K^{b 8}\right]$ and $\left[K^{b 9}\right]$. Secondly, we calculate the stiffness of the mechanism at $P_{c}$ as $\left[K^{n}\right]$, where block 8 and 9 are connected in parallel to the center stage, shown in Fig. 8(c). Finally, we calculate the required wrench $\hat{R}$ to vertically move $P_{c}$ to $P_{c}{ }^{\prime}$. Note this brings the top and bottom pieces aligned vertically, i.e., $\delta_{y}^{c}=0$.

The first element of $\hat{R}$ denotes the required force $F_{x}$ in the $x$ direction. By the principle of virtual work, the required force on the thermal actuator bar equals to $F_{x} e_{4} / e_{3}$, where $e_{4} / e_{3}=10$ is the amplifier ratio. Therefore, we can derive the nonlinear component of actuation force $F^{n}$ as

$$
F^{n}\left(\delta_{x}^{c}\right)=2 \frac{e_{4}}{e_{3}}\left[\begin{array}{llllll}
1 & 0 & 0 & 0 & 0 & 0
\end{array}\right]\left[K^{n}\right]\left[\begin{array}{c}
0  \tag{11}\\
0 \\
0 \\
0 \\
e_{4}-\sqrt{e_{4}^{2}-\left(\delta_{x}^{c}\right)^{2}} \\
0
\end{array}\right]
$$

At last, we calculated the required thermal actuation force as the sum of the linear force $F^{I}\left(\delta_{x}^{c}\right)$ and the nonlinear component $F^{n}\left(\delta_{x}^{c}\right)$, i.e.,

$$
\begin{equation*}
F\left(\delta_{x}^{c}\right)=F^{l}\left(\delta_{x}^{c}\right)+F^{n}\left(\delta_{x}^{c}\right) \tag{12}
\end{equation*}
$$

According to the different geometrical parameters of $X$ and $Y$ position, $F\left(\delta_{x}^{c}\right)$ in Eq. (12) can be denoted as $F^{X}$ and $F^{Y}$.

## 5. Derivation of the control model of the hexapod mechanism

In this section, we will discuss the calculation of the stiffness of the hexapod mechanism. Based on the derived stiffness and the control model of the X-Y stage, we calculate the Jacobian matrix and finally derive the control model of the hexapod.

### 5.1. Geometric description

For convenience, we define the following parameters for describing the geometry of the kinematic model. As shown in Fig. 9(a), the struts have a total length $L$ and diameter $D$, and have a short flexure joint of length $l$ and diameter $d$ at each end ( $L \gg l$ ). As shown in Fig. 9(b), the distance between the neighboring intersecting points of the struts at the top platform is $c_{3}$. The distance between the non-neighbor intersecting points of the struts at the top platform is $c_{4}$. For the base stages, the distance between the neighboring intersecting points of the struts is $c_{1}$ The distance between the non-neighboring intersecting points of the struts at the base is $c_{2}$.

Fig. 9(b) shows the geometrical relationship of the twelve points. We denote the position of the six points at the top platform and the six at the base stages by $\mathbf{A}_{i}$ and by $\mathbf{B}_{i}$ respectively. The points on the moving platform can be described in the global coordinate frame as

$$
\mathbf{A}_{i}^{0}=\left[Z\left(\alpha_{i}\right)\right]\left\{\begin{array}{c}
r_{a}  \tag{13}\\
0 \\
-t
\end{array}\right\}, \quad \mathbf{B}_{i}^{0}=\left[Z\left(\beta_{i}\right)\right]\left\{\begin{array}{c}
r_{b} \\
0 \\
-H
\end{array}\right\}, \quad i=1, \ldots, 6,
$$

where $[Z(\cdot)]$ is the 3-by-3 rotation matrix about the $z$ axis. $r_{a}$ and $r_{b}$ are the radii of the strut attachment points (bottom plates in home position). $H$ is the height of the hexapod mechanism at the original undeformed position and it is derived as the unknown by solving equation $\left(a_{1}^{0}-b_{1}^{0}\right)^{T}\left(a_{1}^{0}-b_{1}^{0}\right)-L^{2}=0 . t$ is the thickness of the top platform. Angles $\alpha_{i}$ and $\beta_{i}$ are tabulated in Table 2. All struts are made of tungsten material with Young's modulus of elasticity $E=411 \mathrm{GPa}$, yield stress $\sigma_{Y}=550 \mathrm{MPa}$, and Poisson's ratio $\nu=0.28$.

a


Fig. 9. Geometrical description of the hexapod mechanism. (a) The strut. (b) The layout of the top platform and the bottom stages.

Table 2
Geometric dimensions of the hexapod mechanism.

$$
\begin{aligned}
& l=0.08 \mathrm{~mm}, d=0.015 \mathrm{~mm}, D=0.075 \mathrm{~mm}, t=0.432 \mathrm{~mm}, H=4.127 \mathrm{~mm}, L=5.4 \mathrm{~mm} \\
& c_{1}=0.4 \mathrm{~mm}, c_{2}=7.2 \mathrm{~mm}, c_{3}=0.4 \mathrm{~mm}, c_{4}=2.5 \mathrm{~mm}, r_{a}=1.572 \mathrm{~mm}, r_{b}=4.277 \mathrm{~mm} \\
& \alpha_{1}=187.3^{\circ}, \alpha_{2}=172.7^{\circ}, \alpha_{3}=67.3^{\circ}, \alpha_{4}=52.7^{\circ}, \alpha_{5}=-52.7^{\circ}, \alpha_{6}=-67.3^{\circ} \\
& \beta_{1}=237.3^{\circ}, \beta_{2}=122.7^{\circ}, \beta_{3}=117.3^{\circ}, \beta_{4}=2.7^{\circ}, \beta_{5}=-2.7^{\circ}, \beta_{6}=-117.3^{\circ}
\end{aligned}
$$

### 5.2. Stiffness of the hexapod mechanism

Let us first calculate the stiffness of the hexapod mechanism. As shown in Fig. 9, the strut is modeled as a serial chain of a cylindrical rod (diameter $D$ ) and two wire flexure joints (diameter $d$ ) at both ends. The compliance matrix of the rod and the wire flexures are derived from the matrix $\left[C^{w}\right]$ in the Appendix A by substituting the parameters in Table 2.

By following the formulation for serial chains, we obtain the compliance matrix of the strut as [ $C^{S}$ ] and we show the stiffness [ $K^{\varsigma}$ ] in Fig. 10. In blocks 2 and 3 of Fig. 10, two $\left[K^{s}\right]$ are connected in parallel and then serially connected to an X-Y positioning stage. As shown in Fig. 11, the coordinate transformation of no. 4 strut to the center of the top platform is calculated by three steps, where $\gamma_{1}=\arcsin \left(\left(c_{4}-c_{1}\right) / 2 L\right)$ and $\gamma_{2}=\arccos \left(H / L \cos \gamma_{1}\right)$. We build the coordinate frame at the center of the top platform and calculate the compliance by parallel connecting blocks 2 and 3 as $\left[C_{1}^{T}\right]$. In block 1 , two $\left[K^{S}\right]$ are parallel connected without an $\mathrm{X}-\mathrm{Y}$ stage. By serially connecting $\left[C_{1}^{T}\right]$ with block 1 and locating the coordinate frame at the $x_{1} y_{1}$ in Fig. 10, we can calculate the compliance [ $C_{1}^{\mathrm{Hex}}$ ].

### 5.3. The control models of the hexapod mechanism

When separating the connection between the bottom $\mathrm{X}-\mathrm{Y}$ stage and the end of the wire flexure shown in Fig. 10, we use the free body diagram approach to analyze the hexapod. We denote the actuating wrench on the center of the two ends by $\hat{W}_{j}$ and the displacement by twist $\hat{T}_{j}$. As the movement of the end of the wire equals the movement of the bottom center stage, we have

$$
\begin{equation*}
\hat{T}_{j}=\hat{T}_{j}^{t}+\hat{T}_{j}^{\prime} \tag{14}
\end{equation*}
$$



Fig. 10. The schematic view and free body diagram of the hexapod nanopositioner.


Fig. 11. Coordinate transformation for stiffness calculation.
where $\hat{T}_{j}^{t}$ is the motion twist caused by thermal actuation force and $T_{j}^{\prime}$ is the displacement caused by the reacting wrench on the center stage. With Eq. (8) and previously derived $\left[C_{j}^{H e x}\right]$ to derive the following equations

$$
\begin{equation*}
\hat{T}_{j}=\left[C_{j}^{H e x}\right] \hat{W}_{j}, \hat{T}_{j}^{\prime}=-\left[C_{j}^{X Y}\right] \hat{W}_{j}, \hat{T}_{j}^{t}=\left[M_{j}\right] \Delta^{a}, \tag{15}
\end{equation*}
$$

where

$$
\left[M_{1}\right]=\left[\begin{array}{llllll}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right], \quad\left[M_{2}\right]=\left[\begin{array}{llllll}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right], \quad\left[M_{3}\right]=\left[\begin{array}{llllll}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] .
$$

$\Delta^{a}$ is a 6-by- 1 displacement vector of the actuating stages, written as

$$
\Delta^{a}=\left[\begin{array}{l}
\delta_{1 x} \\
\delta_{1 y} \\
\delta_{2 x} \\
\delta_{2 y} \\
\delta_{3 x} \\
\delta_{3 y}
\end{array}\right] .
$$

Substituting Eq. (15) into Eq. (14) yields the wrench exerted on the center of two wire flexures for a given $\Delta^{a}$ as

$$
\begin{equation*}
\hat{W}_{j}=\left(\left[C_{j}^{H e x}\right]+\left[C_{j}^{X Y}\right]\right)^{-1}\left[M_{j}\right] \Delta^{a} . \tag{16}
\end{equation*}
$$

By the coordinate transformation matrix $\left[A d_{j}\right]$, we translate the wrench from the local coordinate frame $x_{j} y_{j}$ in Fig. 10 to the global coordinate frame $x y$ at the center of top platform. Applying the law of superposition, i.e., adding the displacement caused by the wrenches on each of the three bottom stages, we can derive the movement of the top platform center as

$$
\begin{equation*}
\hat{T}^{p}=\sum_{j=1}^{3}\left[C_{j}^{T}\right]\left[A d_{j}\right] \hat{W}_{j} . \tag{17}
\end{equation*}
$$

When we substitute Eq. (16) into Eq. (17), we can derive the required actuating displacement for a target position of the center of top platform as

$$
\begin{equation*}
\Delta^{a}=\left(\sum_{j=1}^{3}\left[C_{j}^{T}\right]\left[A d_{j}\right]\left(\left[C_{j}^{H e x}\right]+\left[C_{j}^{X Y}\right]\right)^{-1}\left[M_{j}\right]\right)^{-1} \hat{T}^{p} . \tag{18}
\end{equation*}
$$

The terms in the parentheses of the above equation can be considered as a Jacobian matrix that relates the input actuator displacement and the output displacement of the top platform. If we have positional sensors for the thermal actuator, we may use Eq. (18) for our kinematic controller.

However when position sensors are not available, we would need an open loop control model that depends on the thermal force model described earlier. To derive this model, if we substitute the value from Eq. (18) into $\delta_{x}^{c}$ in Eq. (12), we can derive the equation for actuation forces,

$$
\hat{W}^{a}=\left[\begin{array}{c}
F_{1}^{X}  \tag{19}\\
F_{1}^{Y} \\
F_{2}^{X} \\
F_{2}^{Y} \\
F_{3}^{X} \\
F_{3}^{Y}
\end{array}\right]=\left[\begin{array}{c}
F^{X}\left(\delta_{1 x}\right) \\
F^{Y}\left(\delta_{1 y}\right) \\
F^{X}\left(\delta_{2 x}\right) \\
F^{Y}\left(\delta_{2 y}\right) \\
F^{X}\left(\delta_{3 x}\right) \\
F^{Y}\left(\delta_{3 y}\right)
\end{array}\right], \begin{gathered}
F^{X}\left(\delta_{1 x}\right) \leq 0 \\
F^{Y}\left(\delta_{1 y}\right) \leq 0 \\
F^{X}\left(\delta_{2 x}\right) \leq 0 \\
F^{Y}\left(\delta_{2 y}\right) \geq 0 \\
F^{X}\left(\delta_{3 x}\right) \geq 0 \\
F^{Y}\left(\delta_{3 y}\right) \geq 0
\end{gathered}
$$

where the inequality equations indicate the direction of thermal expansion. Note that the force control model in Eq. (19) is a nonlinear function of the displacement of X and Y positioning stages.

## 6. Error analysis and case studies

In this section, FE models of the hexapod and the $\mathrm{X}-\mathrm{Y}$ positioning stage are built to analyze the errors of the control models. We study two cases on how to use the control model of the hexapod nanopositioner.

### 6.1. Finite element modeling

Firstly, we use Abaqus to build a FE model of the hexapod nanopositioner in Fig. 12(a), which is meshed into 400600 tetrahedral elements in four components: the hexapod mechanism with 72823 elements and three X-Y positioning stages. Each X-Y positioning stage has 109259 elements. As shown in Fig. 12(b), the flexures joints are meshed with more elements. The local size of the elements of the wire flexures and notch flexure is $5 \mu \mathrm{~m}$ while the global size of the elements of the $\mathrm{X}-\mathrm{Y}$ positioning stage and the hexapod mechanism are $100 \mu \mathrm{~m}$ and $260 \mu \mathrm{~m}$, respectively. Based on Eq. (19), a set of actuation force data is calculated as the concentrated forces in the Abaqus model.

We set the Abaqus model with static steps and apply the full-Newton solution technique while activating nonlinear geometry. Ten 2.4 GHz processors are used in parallelizing calculation to finish the analysis. It takes more than 14 min for running a simulation. In the test of the $\mathrm{X}-\mathrm{Y}$ positioning stage control model, we disassemble the $\mathrm{X}-\mathrm{Y}$ positioning stage from the hexapod FE model with the same settings and boundary conditions. By comparing the analytical model with the FE model, we can calculate the error by

$$
\begin{equation*}
\text { Error }=\left|\frac{\text { analytical value }-\mathrm{FE} \text { value }}{\mathrm{FE} \text { value }}\right| \times 100 \% \tag{20}
\end{equation*}
$$



Fig. 12. The Finite Element model of the hexapod nanopositioner. (a) Modeling of the hexapod nanopositioner. (b) The modeling of the flexures.


Fig. 13. Linear and nonlinear actuation force and error percentage (comparison with FEA) vs. command displacement of the stage.


Fig. 14. The hexapod nanopositioner is actuated to trace a circle with $20 \mu \mathrm{~m}$ diameter. (a) Comparison of FE simulation with the analytical solution. (b) Plots of six thermal actuation forces. (c) Errors of translations along $x$ and $y$ axes. (d) Rotations of the top platform by FE simulations.

### 6.2. Error analysis of the model for the $X-Y$ positioning stage

In the analysis of the actuating model for the $\mathrm{X}-\mathrm{Y}$ stage, we compare the linear model given in Eq. (9) and the nonlinear model in Eq. (12). After we apply the calculated force in the FE simulation, we record the data of the displacement and calculate the errors by Eq. (20). In Fig. (13), the red lines represent the derived force by the linear actuating model and the nonlinear model according to the left vertical axis with unit $N$. The blue lines represent the errors in percentage comparing with FE simulations. From Fig. (13), we can draw the following conclusions.

1. Under small deflections, the nonlinear model is very close to the linear model. The stiffness of the nonlinear model increases when the stages undergo a large deflection.
2. As the target displacement increases, the forces of the two models differ significantly. When the displacement equals to $55 \mu \mathrm{~m}$, the force calculated by the linear model is 0.104 N , while it is 0.19 N for the nonlinear model. This represents almost a 90 \% difference, hence justifies the nonlinear model.
3. Since the nonlinear model consists of the linear model and a nonlinear component, the error of the nonlinear model is also a combination of their errors. The error $5.33 \%$ at the beginning of the blue lines shows that the majority of error is due to the linear model, which overestimates the displacement. The linear model began to underestimate the displacement around $13 \mu \mathrm{~m}$ and the error increases quickly. As the displacement increases, the nonlinear component becomes more dominant and this cancels the increasing error caused by the linear model. As a result, the total error of the nonlinear control model is going down.
4. The mean error is $1.93 \%$ for the nonlinear control model within a $55 \mu \mathrm{~m}$ range of motion, which is significantly lower than 16.19 \% for the linear model.

### 6.3. Case studies

Case 1: tracing a circle with a $\mathbf{2 0} \mu \mathrm{m}$ diameter. As an example of the control model, the hexapod is commanded to trace a circle with a $20 \mu \mathrm{~m}$ diameter in the plane $\delta_{z}=20 \mu \mathrm{~m}$ with $\theta_{x}, \theta_{y}, \theta_{z}=0$ as shown in Fig. 14(a). Fig. 14(b) shows the calculated thermal actuation forces for each position by Eq. (19). We also need to check if the calculated forces are within the range of inequality constraint equations of (19). Fig. 14(c) and (d) show the hexapod model errors of drawing the circle. Based on these tests, we can draw the following conclusions.

1. In Fig. 14(c), the difference of the absolute value between the analytical model and the FE model becomes larger as $\delta_{x}$ and $\delta_{y}$ increase. However, the percentage errors decrease as $\delta_{x}$ and $\delta_{y}$ increase. Thus, the maximum percentage errors of $\delta_{x}$ and $\delta_{y}$ are located at the positions near $\delta_{x}=0$ and $\delta_{y}=0$, respectively. The mean percentage errors of the translational displacements in $x, y$, and $z$ directions are $3.94 \%, 3.65 \%$, and $2.55 \%$, respectively. Thus, the mean error of the translational displacements in three directions is $3.38 \%$.
2. Fig. 14 (d) shows the parasitic rotation angles along three axes. The absolute mean values are respectively $5.28 \times 10^{-6} \mathrm{rad}$, $1.395 \times 10^{-5} \mathrm{rad}$, and $4.23 \times 10^{-6} \mathrm{rad}$. The max values are $1.45 \times 10^{-5} \mathrm{rad}, 3.09 \times 10^{-5} \mathrm{rad}$, and $1.11 \times 10^{-5} \mathrm{rad}$. Thus, we can conclude that the parasitic rotation angles are very small.

Case 2: Specified single and coupled motion of the platform. In order to verify the control model, the hexapod is proposed to realize the multi DOFs motions of the top platform. As shown in Table 3, the hexapod is first commanded to perform $30 \mu \mathrm{~m}$ translation in the $z$ direction which is treated as a base for further movement. Then, the hexapod is commanded to perform 2DOF

Table 3
The hexapod nanopositioner is actuated to reach coupled DOFs.

|  | $\delta_{x}$ | $\delta_{y}$ | $\delta_{z}$ | $\theta_{x}(\mathrm{rad})$ | $\theta_{y}(\mathrm{rad})$ | $\theta_{z}(\mathrm{rad})$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\hat{T}_{\text {FE }}$ | $-1.62 \mathrm{E}-2$ | 6.31E-4 | $3.07 \mathrm{E}+1$ | $4.00 \mathrm{E}-7$ | $-2.23 \mathrm{E}-5$ | $-8.23 \mathrm{E}-7$ |
| $\hat{T}_{A}$ | 0 | 0 | $3.0 \mathrm{E}+1$ | 0 | 0 | 0 |
| $\hat{T}_{\text {FE }}$ | $-1.04 \mathrm{E}+1$ | 6.83E-4 | $3.08 \mathrm{E}+1$ | $2.20 \mathrm{E}-7$ | $-4.65 \mathrm{E}-5$ | $-7.20 \mathrm{E}-7$ |
| $\hat{T}_{A}$ | $-1.0 \mathrm{E}+1$ | 0 | $3.0 \mathrm{E}+1$ | 0 | 0 | 0 |
| $\hat{T}_{\text {FE }}$ | $-1.03 \mathrm{E}-2$ | $-1.04 \mathrm{E}+1$ | $3.07 \mathrm{E}+1$ | $9.13 \mathrm{E}-6$ | $-1.89 \mathrm{E}-5$ | 7.44E-6 |
| $\hat{T}_{A}$ | 0 | $-1.0 \mathrm{E}+1$ | $3.0 \mathrm{E}+1$ | 0 | 0 | 0 |
| $\hat{T}_{\text {FE }}$ | 4.10E-2 | $-6.64 \mathrm{E}-2$ | $3.07 \mathrm{E}+1$ | $-4.10 \mathrm{E}-3$ | $-2.19 \mathrm{E}-5$ | $1.43 \mathrm{E}-5$ |
| $\hat{T}_{A}$ | 0 | 0 | $3.0 \mathrm{E}+1$ | $-4.0 \mathrm{E}-3$ | 0 | 0 |
| $\hat{T}_{\text {FE }}$ | $-8.43 \mathrm{E}-2$ | $9.53 \mathrm{E}-4$ | $3.07 \mathrm{E}+1$ | $9.85 \mathrm{E}-7$ | $-4.20 \mathrm{E}-3$ | $-7.33 \mathrm{E}-7$ |
| $\hat{T}_{A}$ | 0 | 0 | $3.0 \mathrm{E}+1$ | 0 | $-4.0 \mathrm{E}-3$ | 0 |
| $\hat{T}_{\text {FE }}$ | $4.41 \mathrm{E}-2$ | $1.37 \mathrm{E}-1$ | $3.07 \mathrm{E}+1$ | $1.64 \mathrm{E}-6$ | $-2.50 \mathrm{E}-5$ | $-4.19 \mathrm{E}-3$ |
| $\hat{T}_{A}$ | 0 | 0 | $3.0 \mathrm{E}+1$ | 0 | 0 | $-4.00 \mathrm{E}-3$ |
| Error (\%) | 4.12 | 4.25 | 2.39 | 2.34 | 4.69 | 4.56 |

movements, which consist of the base movement coupled with $-10 \mu \mathrm{~m}$ of translation or -0.004 rad of rotation. Based on these tests, we can draw the following conclusions.

1. Some single DOF displacements cannot be achieved and the device must be actuated for coupled motion with a nonzero $\delta_{z}$. Again this is because the thermal actuators can only move in one direction.
2. Table 3 shows the twist values in coupled DOFs movements and the corresponding errors in single DOF movements. The results of the analytical model are very close to the FE model. The maximum error is $4.69 \%$ for coupled motion $\theta_{y}=-0.004 \mathrm{rad}$ with $\delta_{z}=30 \mu \mathrm{~m}$. The maximum parasitic value of the movement is $0.137 \mu \mathrm{~m}$ in translation and $4.65 \times 10^{-5}$ rad in rotation.
3. Due to the inequality constraints in Eq. (19), the workspace is limited. More specifically, the workspace will be limited by the stress of the wire flexures and the workspace of the $\mathrm{X}-\mathrm{Y}$ positioning stage, which depends on the maximum allowed temperature and stress distribution. Clearly this motivates the future work of optimizing workspace of this device.

## 7. Conclusions

A method for calculating the in-plane nonlinear stiffness is proposed for building the actuating equation of the $\mathrm{X}-\mathrm{Y}$ positioning stage. This method can be applied to the stiffness derivation and analysis of various in-plane parallel mechanisms. The analytical models for calculating the 6DOF stiffness matrix of the $\mathrm{X}-\mathrm{Y}$ positioning stage and the hexapod nanopositioner are derived. The derivation process can be applied to the stiffness analysis of other compliant mechanisms in three dimensions. Based on the Jacobian matrix derived from constant stiffness of the hexapod mechanism and the nonlinear control model of the X-Y stage, an open loop control model of the hexapod nanopositioner is derived. An example of drawing a circle is presented to show the application of the control model. FE simulations of the $\mathrm{X}-\mathrm{Y}$ positioning stage and the hexapod nanopositioner are built for analyzing the errors of the analytical model. Future work will be to derive the workspace of the nanopositioner and to reduce the modeling error by kinematic calibration via physical experiments.

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## Appendix A. Compliant matrix of blade flexures

The compliance matrix of a blade flexure shown in Supplementary Fig. 1 about its free end can be written as

$$
\left[C^{b}\right]=\frac{l}{E I_{z}}\left[\begin{array}{cccccc}
0 & 0 & 0 & \frac{1}{\chi \beta} & 0 & 0  \tag{21}\\
0 & 0 & -\frac{l \kappa}{2} & 0 & \kappa & 0 \\
0 & \frac{l}{2} & 0 & 0 & 0 & 1 \\
\frac{l^{2} \eta}{12} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{l^{2}}{3} & 0 & 0 & 0 & \frac{l}{2} \\
0 & 0 & \frac{l^{2} \kappa}{3} & 0 & -\frac{l \kappa}{2} & 0
\end{array}\right]
$$

where

$$
\begin{equation*}
\kappa=\frac{I_{z}}{I_{y}}=\frac{t^{2}}{w^{2}}, \quad \beta=\frac{J}{I_{z}}, \quad \eta=\frac{t^{2}}{l^{2}}, \quad \chi=\frac{G}{E}=\frac{1}{2(1+\nu)}, \tag{22}
\end{equation*}
$$

are non-dimensional constants determined by geometries and material properties. $I_{y}, I_{z}$ are the area moments of inertia about $y$ and $z$ axis. $J$ is the polar moment inertia about $x$ axis. $E$ and $G$ are Young's modulus and shear modulus, respectively, and $v$ is the poisson's ratio. $\beta$ is the ratio of torsion constant over moment of inertia. For a rectangular cross section, $\beta$ is defined by

$$
\begin{equation*}
\beta=12\left(\frac{1}{3}-0.21 \frac{t}{w}\left(1-\frac{1}{12}\left(\frac{t}{w}\right)^{4}\right)\right) . \tag{23}
\end{equation*}
$$

## Compliant matrix of circular wire flexures

The compliance and stiffness matrices of a wire flexure with a circular cross section shown in Supplementary Fig. 2 has the same form as that of a wire flexure with a rectangular cross section. The compliance matrix given by Eq. (21) can be simplified by substituting cross sectional area, $A=\pi d^{2} / 4$, area moments of inertia about axes $y$ and $z, I_{y}=I_{z}=\pi d^{4} / 64$ and second moment of area about axis $x, J=2 I_{z}$, written as,

$$
\left[C^{W}\right]=\frac{l}{E I_{z}}\left[\begin{array}{cccccc}
0 & 0 & 0 & \frac{1}{2 \chi} & 0 & 0  \tag{24}\\
0 & 0 & -\frac{l}{2} & 0 & 1 & 0 \\
0 & \frac{l}{2} & 0 & 0 & 0 & 1 \\
\frac{l^{2} \eta}{16} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{l^{2}}{3} & 0 & 0 & 0 & \frac{l}{2} \\
0 & 0 & \frac{l^{2}}{3} & 0 & -\frac{l}{2} & 0
\end{array}\right],
$$

where $\eta=d^{2} / l^{2}$. The stiffness matrix of a wire flexure is the inverse of its compliance matrix, i.e., $\left[K^{w}\right]=\left[C^{w}\right]^{-1}$.
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[^1]:    ${ }^{1}$ Certain commercial equipment, instruments, or materials are identified in this paper in order to specify the experimental procedure adequately. Such identification is not intended to imply recommendation or endorsement by the National Institute of Standards and Technology, nor is it intended to imply that the materials or equipment identified are necessarily the best available for the purpose.

[^2]:    ${ }^{\text {a }}$ Data source [25]

