The Decoupled Potential Integral Equation for Time-Harmonic Electromagnetic Scattering

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Abstract

We present a new formulation for the problem of electromagnetic scattering from perfect electric conductors. While our representation for the electric and magnetic fields is based on the standard vector and scalar potentials $\boldsymbol{A}, \phi$ in the Lorenz gauge, we establish boundary conditions on the potentials themselves, rather than on the field quantities. This permits the development of a well-conditioned second kind Fredholm integral equation which has no spurious resonances, avoids low frequency breakdown, and is insensitive to the genus of the scatterer. The equations for the vector and scalar potentials are decoupled. That is, the unknown scalar potential defining the scattered field, ϕ^{scat}, is determined entirely by the incident scalar potential ϕ^{inc}. Likewise, the unknown vector potential defining the scattered field, $\boldsymbol{A}^{\text{scat}}$, is determined entirely by the incident vector potential $\boldsymbol{A}^{\text{inc}}$. This decoupled formulation is valid not only in the static limit but for arbitrary $\omega \geq 0$.

Keywords. Charge-current formulations, electromagnetic theory, electromagnetic (EM) scattering, low-frequency breakdown, Maxwell equations.

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1 Introduction

In this paper, we consider the problem of exterior scattering of time-harmonic electromagnetic waves by perfect electric conductors. For a fixed frequency ω, we assume that the electric and magnetic fields take the form

$$\begin{aligned} \mathcal{E}(\boldsymbol{x},t) &= \Re\{\boldsymbol{E}(\boldsymbol{x})e^{-i\omega t}\}, \\ \mathcal{H}(\boldsymbol{x},t) &= \Re\{\boldsymbol{H}(\boldsymbol{x})e^{-i\omega t}\}, \end{aligned} \tag{1}$$

so that Maxwell's equations are

$$\begin{aligned} \nabla \times \boldsymbol{E}(\boldsymbol{x}) &= i\omega\mu \boldsymbol{H}(\boldsymbol{x}), \\ \nabla \times \boldsymbol{H}(\boldsymbol{x}) &= -i\omega\epsilon \boldsymbol{E}(\boldsymbol{x}). \end{aligned} \tag{2}$$

Following standard practice, we write the *total* electric and magnetic fields as a sum of the (known) incident and (unknown) scattered fields:

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}^{\rm inc} + \boldsymbol{E}^{\rm scat}, \\ \boldsymbol{H} &= \boldsymbol{H}^{\rm inc} + \boldsymbol{H}^{\rm scat}. \end{aligned} \tag{3}$$

The scattered field in the exterior must satisfy the Sommerfeld-Silver-Müller radiation condition:

$$\boldsymbol{H}^{\rm scat}(\boldsymbol{x}) \times \tfrac{\boldsymbol{x}}{|\boldsymbol{x}|} - \sqrt{\tfrac{\mu}{\epsilon}}\boldsymbol{E}^{\rm scat}(\boldsymbol{x}) = o\Big(\tfrac{1}{|\boldsymbol{x}|}\Big), \quad |\boldsymbol{x}| \to \infty\,. \tag{4}$$

It is well-known that when the scatterer, denoted by D, is a perfect conductor, the conditions to be enforced on its boundary are [1, 2]

$$\boldsymbol{n} \times \boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{0}|_{\partial D} \;\Rightarrow\; \boldsymbol{n} \times \boldsymbol{E}^{\rm scat}(\boldsymbol{x}) = -\boldsymbol{n} \times \boldsymbol{E}^{\rm inc}(\boldsymbol{x})|_{\partial D}, \tag{5}$$

$$\boldsymbol{n} \cdot \boldsymbol{H}(\boldsymbol{x}) = 0|_{\partial D} \;\Rightarrow\; \boldsymbol{n} \cdot \boldsymbol{H}^{\rm scat}(\boldsymbol{x}) = -\boldsymbol{n} \cdot \boldsymbol{H}^{\rm inc}(\boldsymbol{x})|_{\partial D}, \tag{6}$$

where $\boldsymbol{n}$ is the outward unit normal to the boundary ∂D of the scattered. It is also well-known that

$$\boldsymbol{n} \cdot \boldsymbol{E}(\boldsymbol{x}) = \frac{\rho}{\epsilon}|_{\partial D}, \tag{7}$$

$$\boldsymbol{n} \times \boldsymbol{H}(\boldsymbol{x}) = \boldsymbol{J}|_{\partial D}, \tag{8}$$

where $\boldsymbol{J}$ and ρ are the induced current density and charge on the surface ∂D. In order to satisfy the Maxwell equations, $\boldsymbol{J}$ and ρ must satisfy the continuity condition $\nabla_s \cdot \boldsymbol{J} = i\omega\rho$, where $\nabla_s \cdot \boldsymbol{J}$ denotes the surface divergence of the tangential current density. It is also well-known that the exterior problem for $\boldsymbol{E}^{\rm scat}$ has a unique solution for $\omega > 0$ when boundary conditions are prescribed on its tangential components (see, for example, [3]):

$$\boldsymbol{n} \times \boldsymbol{E}^{\rm scat}(\boldsymbol{x}) = \mathbf{f}(\mathbf{x})|_{\partial D}\,, \tag{9}$$

for an arbitrary tangential vector field $\mathbf{f}$. On a perfect conductor, $\mathbf{f}(\mathbf{x}) = -\boldsymbol{n} \times \boldsymbol{E}^{\rm inc}(\boldsymbol{x})$ to enforce (5).

1.1 The vector and scalar potential

Scattered electromagnetic fields are typically represented in terms of the induced surface current $\boldsymbol{J}$ and charge ρ using the vector and scalar potentials in the Lorenz gauge:

$$\boldsymbol{E}^{\text{scat}} = i\omega \boldsymbol{A}^{\text{scat}} - \nabla \phi^{\text{scat}}, \tag{10}$$

$$\boldsymbol{H}^{\text{scat}} = \frac{1}{\mu} \nabla \times \boldsymbol{A}^{\text{scat}}, \tag{11}$$

where

$$\boldsymbol{A}^{\text{scat}}[\boldsymbol{J}](\boldsymbol{x}) = \mu S_k[\boldsymbol{J}](\boldsymbol{x}) \equiv \mu \int_{\partial D} g_k(\boldsymbol{x}-\boldsymbol{y}) \boldsymbol{J}(\boldsymbol{y}) dA_{\boldsymbol{y}},$$

$$\phi^{\text{scat}}[\rho](\boldsymbol{x}) = \frac{1}{\epsilon} S_k[\rho](\boldsymbol{x}) \equiv \frac{1}{\epsilon} \int_{\partial D} g_k(\boldsymbol{x}-\boldsymbol{y}) \rho(\boldsymbol{y}) dA_{\boldsymbol{y}}, \tag{12}$$

with

$$g_k(\boldsymbol{x}) = \frac{e^{ik|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|}$$

and $k = \omega\sqrt{\epsilon\mu}$. The Lorenz gauge is defined by the relation

$$\nabla \cdot \boldsymbol{A}^{\text{scat}} = i\omega\mu\epsilon\phi^{\text{scat}}\,. \tag{13}$$

We will often refer to ϕ^{scat} and $\boldsymbol{A}^{\text{scat}}$ as the *scalar and vector Helmholtz potentials* since ϕ^{scat} and $\boldsymbol{A}^{\text{scat}}$ satisfy the Helmholtz equations with wavenumber k:

$$\Delta\phi^{\text{scat}} + k^2\phi^{\text{scat}} = 0, \quad \Delta\boldsymbol{A}^{\text{scat}} + k^2\boldsymbol{A}^{\text{scat}} = \boldsymbol{0}. \tag{14}$$

Using the representation (10) for the electric field and imposing the boundary condition (5) results in the Electric Field Integral Equation (EFIE), [4–7]:

$$\begin{aligned} i\omega\boldsymbol{n} \times \boldsymbol{A}^{\text{scat}}[\boldsymbol{J}](\boldsymbol{x}) - \boldsymbol{n} \times \nabla\phi^{\text{scat}} \left[\frac{\nabla_s \cdot \boldsymbol{J}}{i\omega}\right](\boldsymbol{x}) \\ = -\boldsymbol{n} \times \boldsymbol{E}^{\text{inc}}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial D. \end{aligned} \tag{15}$$

The representation (11) for the magnetic field and the boundary condition (8) results in the Magnetic Field Integral Equation (MFIE):

$$\frac{1}{2}\boldsymbol{J}(\boldsymbol{x}) - K[\boldsymbol{J}](\boldsymbol{x}) = \boldsymbol{n}(\boldsymbol{x}) \times \boldsymbol{H}^{\text{inc}}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial D, \tag{16}$$

where

$$K[\boldsymbol{J}](\boldsymbol{x}) = \int_{\partial D} \boldsymbol{n}(\boldsymbol{x}) \times \nabla \times g_k(\boldsymbol{x}-\boldsymbol{y}) \boldsymbol{J}(\boldsymbol{y}) dA_{\boldsymbol{y}}\,. \tag{17}$$

There is an enormous literature on the properties of these integral equations, which we will not review here, except to note that the EFIE is poorly scaled; one term in the

representation of $\boldsymbol{E}$ is of order $O(\omega)$ and one term is of the order $O(\omega^{-1})$. This makes it difficult to compute both the solenoidal and irrotational components of the current $\boldsymbol{J}$ and causes ill-conditioning in the integral equation at low frequencies — a phenomenon generally referred to as "low-frequency breakdown" [8, 9]. Both the EFIE and the MFIE are also subject to spurious resonances at a countable set of frequencies ω_j going to infinity. Below the first such resonance, the MFIE is a well-conditioned second kind Fedholm integral equation. While low-frequency breakdown is obvious in the EFIE, it is not entirely avoided by switching to the MFIE [8]. The problem is that the current $\boldsymbol{J}$ is not sufficient for computing accurately the electric field. Note for example that

$$\boldsymbol{n} \cdot \boldsymbol{E} = \rho = \frac{\nabla_s \cdot \boldsymbol{J}}{i\omega\epsilon} \,. \tag{18}$$

As $\omega \to 0$, what in numerical analysis is called *catastrophic cancellation* causes a progressive loss of digits [10, 11]. Catastrophic cancellation comes not just from the ill-conditioning associated with the evaluation of a derivative. The current $\boldsymbol{J}$ is an $O(1)$ quantity, while $\nabla_s \cdot \boldsymbol{J}$ is $O(\omega)$, amplifying the loss of digits. A variety of remedies to solve this problem have been suggested. In the widest use are methods based on specialized basis functions for the discretization of the current $\boldsymbol{J}$ itself. Loop-tree and loop-star basis functions, for example, can be used to rescale the solenoidal and the irrotational parts of the current [9, 12–15]. A second class of methods is based on using both current and charge as separate unknowns. This avoids terms of the order $O(\omega^{-1})$ (see [16–20]). Unfortunately, all of these approaches encounter a second difficulty in multiply-connected domains — a phenomenon which we refer to as "topological low-frequency breakdown" [21, 22]. At zero frequency, the MFIE, Calderon-preconditioned EFIE and charge-current based integral equations are all rank-deficient, with a nullspace of dimension related to the topology of the surface ∂D: g for the MFIE, $2g$ for the Calderon-preconditioned EFIE and $g + N$ for the charge-current based integral equations [21–23], where g is the genus of the surface ∂D and N is the number of connected components. This inevitably leads to ill-conditioning in the low-frequency regime. This problem was carefully analyzed in the paper [21], and the nullspace characterized in terms of harmonic vector fields [21–23].

Definition 1. *Assuming D is topologically equivalent to a sphere with g handles, one can choose g surfaces S_j in $\mathbb{R}^3 \backslash D$ so that $\mathbb{R}^3 \backslash (D \cup_{j=1}^{g} S_j)$ is simply connected. The boundaries of these surfaces are loops on ∂D called* B-cycles. *They go around the "holes" and form a basis for the first homology group of the domain D.*

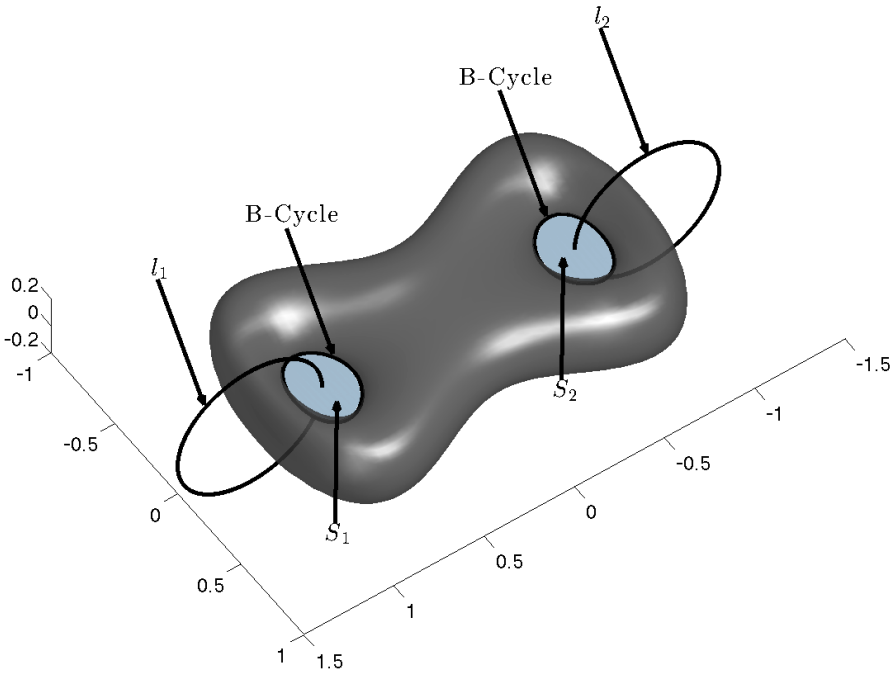

Figure 1: Double torus, number of connected components N=1, genus g=2, two B-cycles. l_1 and l_2 are the integration domain of integrals in (32). Surfaces S_1, S_2 are covering each hole of the surface.

In [24], it was shown that the addition of a consistency condition of the form

$$\int_{S_j} \boldsymbol{H}^{\text{scat}} \cdot \boldsymbol{n}\, da = -\int_{S_j} \boldsymbol{H}^{\text{inc}} \cdot \boldsymbol{n}\, da \tag{19}$$

or

$$\int_{B_j} \boldsymbol{A}^{\text{scat}} \cdot \boldsymbol{ds} = -\int_{B_j} \boldsymbol{A}^{\text{inc}} \cdot \boldsymbol{ds}, \tag{20}$$

where the line integrals represent the circulation of the vector potential, enforces uniqueness on the solution of the MFIE, assuming ω is not at a spurious resonance. This requires knowledge of, or computation of, the genus, geometry tools that identify B-cycles, and linear algebraic methods that are capable of efficiently solving integral equations subject to constraints.

Remark 1. *These topological issues can also be addressed through an analysis of the Hodge decomposition of the source current on the surface of the scatterer itself, using the generalized Debye source representation of [25]. Additional conditions are used (similar to (19)) to ensure that the problem is well-posed.*

Remark 2. *Very recently, a method was introduced that overcomes the topological low-frequency breakdown inherent in the EFIE by a clever projection of the* discretized *problem using Rao-Wilton-Glisson (RWG) basis functions into a suitable subspace [26].*

In short, the various integral equations presently available pose significant difficulties in the low-frequency regime.

1.2 A decoupled formulation

In this paper we introduce a new formulation for electromagnetic scattering from perfect conductors. Rather than imposing boundary conditions on the field quantities ($\boldsymbol{E}$, $\boldsymbol{H}$), we derive conditions on the potentials themselves. Moreover, we show that the integral equations for $\boldsymbol{A}^{\text{scat}}$ and ϕ^{scat} can be decoupled, lead to well-conditioned linear systems, and are insensitive to the genus of the scatterer. More precisely, we seek to impose the boundary conditions

$$\begin{aligned} \boldsymbol{n} \times \boldsymbol{A}^{\text{scat}}(\boldsymbol{x}) &= -\boldsymbol{n} \times \boldsymbol{A}^{\text{inc}}(\boldsymbol{x})|_{\partial D}, \\ \boldsymbol{n} \times \nabla \phi^{\text{scat}}(\boldsymbol{x}) &= -\boldsymbol{n} \times \nabla \phi^{\text{inc}}(\boldsymbol{x})|_{\partial D}. \end{aligned} \tag{21}$$

At first glance, there is an obvious difficulty with such an approach: the vector and scalar potentials are not unique, a fact generally referred to as *gauge freedom*. Even in the Lorenz gauge above, the representation is known not to be unique. That is, the condition (13) does not completely determine the potentials $\boldsymbol{A}^{\text{scat}}, \phi^{\text{scat}}$. To see this, consider the vector potentials $\boldsymbol{A}'^{\text{scat}}, \phi'^{\text{scat}}$ defined by

$$\begin{aligned} \boldsymbol{A}'^{\text{scat}}[\boldsymbol{J}](\boldsymbol{x}) &= \boldsymbol{A}^{\text{scat}}[\boldsymbol{J}](\boldsymbol{x}) + \nabla S_k[\sigma](\boldsymbol{x}), \\ \phi'^{\text{scat}}[\rho](\boldsymbol{x}) &= \phi^{\text{scat}}[\rho](\boldsymbol{x}) + i\omega S_k[\sigma](\boldsymbol{x}). \end{aligned} \tag{22}$$

Here, σ is an arbitrary source on the surface ∂D. It is straightforward to check that the fields $\boldsymbol{E}^{\text{scat}}, \boldsymbol{H}^{\text{scat}}$ induced by $\boldsymbol{A}'^{\text{scat}}, \phi'^{\text{scat}}$ are the same as those induced by $\boldsymbol{A}^{\text{scat}}, \phi^{\text{scat}}$, all the while satisfying the Lorenz gauge condition.

We will make use of this additional gauge freedom to establish a well-posed boundary value problem and a stable, well-conditioned integral equation. In Section 2, we consider the low-frequency limit of the exterior scattering from perfect conductors, both for the sake of review and to motivate our formulation. In Section 3, we discuss the relevant existence, uniqueness and stability results for what we refer to as the *decoupled potential integral equation* (DPIE). Finally, we discuss the stable representation of the incoming field in terms of scalar and vector potentials and the high-frequency behavior of the new formulation.

2 Preliminaries

In this section, we consider the low-frequency limit of the Maxwell equations, where the electric and magnetic fields are decoupled. We will refer to the electrostatic and magneto-

static fields by $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$, respectively.

2.1 Electrostatics

The electrostatic field satisfies the equations

$$\nabla \times \boldsymbol{E}_0(\boldsymbol{x}) = 0, \quad \nabla \cdot \boldsymbol{E}_0(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \mathbb{R}^3/D, \tag{23}$$

which we decompose (as above) into incoming and scattered fields. The scattered field must satisfy the radiation condition

$$\boldsymbol{E}_0^{\text{scat}}(\boldsymbol{x}) = o(1), \quad |\boldsymbol{x}| \to \infty. \tag{24}$$

The boundary condition for the electrostatic field is the same as that for any non-zero frequency,

$$\boldsymbol{n} \times \boldsymbol{E}_0(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial D, \tag{25}$$

but the solution is no longer unique.

The nullspace, that is functions satisfying (23) and (25) and the radiation condition (24), is of dimension N, where N is the number of connected components of the scatterers ∂D. They are known as *harmonic Dirichlet fields* [22] with a basis denoted by $\{\mathbf{Y}_j\}_{j=1}^N$. It is straightforward to see that there are at least N such solutions, since they correspond to the well-studied problem of capacitance. To see this, let us denote by ∂D_j the jth connected component of ∂D. Taking the static limit of (10), the scattered electrostatic field is described as the gradient of a scalar harmonic function:

$$\boldsymbol{E}_0^{\text{scat}} = -\nabla \phi_0^{\text{scat}}\,. \tag{26}$$

Imposing the boundary condition (25) and assuming the incoming field is represented in terms of an incoming potential ϕ^{inc}, we have

$$\begin{aligned} \Delta \phi_0^{\text{scat}} &= 0, \\ \boldsymbol{n} \times \nabla \phi_0^{\text{scat}} &= -\boldsymbol{n} \times \nabla \phi_0^{\text{inc}}|_{\partial D}. \end{aligned}$$

It is clear that the preceding boundary condition is satisfied by any scattered potential that satisfies the Dirichlet condition

$$\phi_0^{\text{scat}} = -\phi_0^{\text{inc}}|_{\partial D_j} + V_j\,, \tag{27}$$

where V_j is an arbitrary constant on ∂D_j that represents the voltage of each conductor (with respect to infinity). The Dirichlet field $\mathbf{Y}_j$ corresponds to the gradient of ϕ_0^{scat} obtained by setting ϕ_0^{scat} to zero on each boundary component ∂D_i for $i \neq j$ and $\phi_0^{\text{scat}} = 1$ on ∂D_j. Let us now define the scalars Q_j by

$$Q_j = \int_{\partial D_j} \frac{\partial \phi_0^{\text{scat}}}{\partial n} ds = -\int_{\partial D_j} \mathbf{n} \cdot \mathbf{E}_0^{\text{scat}} ds, \tag{28}$$

so that $-Q_j$ is the total charge on each conductor ∂D_j. The matrix that links the voltages V_j and the charges $-Q_j$ is known as the *capacitance matrix* [1]. Since we are interested here in the time-harmonic Maxwell equations and their zero-frequency limit, we must have charge neutrality on each boundary component. Thus, we are interested in studying (27) where the voltages V_j are additional unknowns, but for which N additional constraints are given of the form $Q_j = 0$, for $j = 1, \ldots, N$.

It is important to note that, in the static regime, the problem suffers from more than non-uniqueness. The boundary condition

$$\boldsymbol{n} \times \nabla \phi_0^{\text{scat}} = -\boldsymbol{n} \times \boldsymbol{E}_0^{\text{inc}}$$

cannot be satisfied unless the incoming field is also an electrostatic field. In particular, if the circulation of the incoming field $\oint_{L \subset \partial D} \boldsymbol{E}_0^{\text{inc}} \cdot d\mathbf{l}$ is not zero on every closed loop L on the surface ∂D, the solution $\boldsymbol{E}_0^{\text{scat}}$ does not exist. This follows easily from the fact that

$$\oint_{L \subset \partial D} \boldsymbol{E}_0^{\text{inc}} \cdot d\mathbf{l} = -\oint_{L \subset \partial D} \nabla \phi_0^{\text{scat}} \cdot d\mathbf{l} = 0.$$

We refer the reader to [22] for further discussion.

2.2 Magnetostatics

The magnetostatic field satisfies the equations

$$\nabla \times \boldsymbol{H}_0(\boldsymbol{x}) = 0, \quad \nabla \cdot \boldsymbol{H}_0(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \mathbb{R}^3 / D, \tag{29}$$

and the boundary condition

$$\boldsymbol{n} \cdot \boldsymbol{H}_0(\boldsymbol{x}) = 0|_{\partial D}. \tag{30}$$

The total field is again decomposed into incoming and scattered fields, with the scattered field satisfying the radiation condition (31),

$$\boldsymbol{H}_0^{\text{scat}}(\boldsymbol{x}) = o(1), \quad |\boldsymbol{x}| \to \infty. \tag{31}$$

$\boldsymbol{H}_0^{\text{scat}}$ can be described either as the curl of a harmonic vector potential $\mathbf{A_0}$ or as the gradient of a harmonic scalar potential ϕ_0^{scat}.

The magnetostatic problem also suffers from non-uniqueness. The nullspace, that is functions satisfying (29) and (30) and the radiation condition (31), is of dimension g, where g is the genus of the surface ∂D. Elements of this space are called *harmonic Neumann fields* $\{\mathbf{Z}_m\}_{m=1}^g$ [22]. In order to completely specify the solution, additional information, such as the total induced current I_m on g loops lying on the surface, must be specified:

$$\oint_{l_m} \boldsymbol{H}_0^{\text{scat}}(\boldsymbol{x}) \cdot d\mathbf{l} = I_m, \quad m = 1, \ldots, g\,. \tag{32}$$

The loops l_m here go around the "holes" (see Fig. 1). That is, they are a basis for the first homology group of $\mathbb{R}^3\backslash D$, with spanning surfaces that lie in the *interior* of the scatterer, see [22] for more details. The persistent currents I_m at zero frequency are due to the potential presence of superconducting loops (as we are considering scattering from perfect electric conductors).

2.3 Summary

To summarize, the problem of electromagnetic scattering from perfect conductors is uniquely solvable for any ω strictly greater than zero. At $\omega = 0$, however, various subtleties arise. The issue of Dirichlet fields needs to be resolved in electrostatics and the issue of Neumann fields needs to be resolved in magnetostatics. For any ω strictly greater than zero, however, it is necessary that the total charge Q_j induced on any connected component of the scatterer be zero. Enforcing this condition at $\omega = 0$ (and introducing the additional unknown constants V_j as above) uniquely determines the electrostatic field. In the magnetostatic case, however, we are obligated to introduce additional constants, such as the $\{I_m\}$ in (32), in order to account for the Neumann fields when the scatterer has non-zero genus.

3 Scattering Theory for Decoupled Potentials

We turn now to the analytic foundations of the DPIE. We first derive boundary value problems for the scattered scalar and vector potentials that are completely insensitive to the genus, although they do depend explicitly on the number of boundary components. After this reformulation of the Maxwell equations, we design integral representations that lead to well-conditioned and invertible linear systems of equations.

Definition 2. *By the* scalar Dirichlet problem*, we mean the calculation of a scalar Helmholtz or Laplace potential in $\mathbb{R}^3\backslash D$ whose boundary value equals a given function f on ∂D and which satisfies standard radiation conditions at infinity:*

$$\Delta\phi^{\text{scat}} + k^2\phi^{\text{scat}} = 0, \quad \phi^{\text{scat}}|_{\partial D} = f, \tag{33}$$

$$\tfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\cdot\nabla\phi^{\text{scat}}(\boldsymbol{x}) - ik\phi^{\text{scat}}(\boldsymbol{x}) = o\Big(\tfrac{1}{|\boldsymbol{x}|}\Big), \qquad |\boldsymbol{x}| \to \infty\,, \tag{34}$$

for the scalar Helmholtz potential, and

$$\Delta\phi_0^{\text{scat}} = 0, \quad \phi_0^{\text{scat}}|_{\partial D} = f, \tag{35}$$

$$\phi_0^{\text{scat}}(\boldsymbol{x}) = O\Big(\tfrac{1}{|\boldsymbol{x}|}\Big), \quad \nabla\phi_0^{\text{scat}}(\boldsymbol{x}) = O\Big(\tfrac{1}{|\boldsymbol{x}|^2}\Big), \qquad |\boldsymbol{x}| \to \infty, \tag{36}$$

for the scalar Laplace potential, respectively.

Definition 3. *By the* vector Dirichlet problem*, we mean the calculation of a vector Helmholtz or Laplace potential in* $\mathbb{R}^3\backslash D$ *whose tangential boundary values equal a given tangential function* $\mathbf{f}$ *on* ∂D*, and whose divergence equals a given scalar function* h *on* ∂D *and which satisfies standard radiation conditions at infinity:*

$$\Delta \boldsymbol{A}^{\text{scat}} + k^2 \boldsymbol{A}^{\text{scat}} = 0, \quad \boldsymbol{n} \times \boldsymbol{A}^{\text{scat}}|_{\partial D} = \mathbf{f}, \quad \nabla \cdot \boldsymbol{A}^{\text{scat}}|_{\partial D} = h, \tag{37}$$

$$\nabla \times \boldsymbol{A}^{\text{scat}}(\boldsymbol{x}) \times \tfrac{\boldsymbol{x}}{|\boldsymbol{x}|} + \tfrac{\boldsymbol{x}}{|\boldsymbol{x}|} \nabla \cdot \boldsymbol{A}^{\text{scat}}(\boldsymbol{x}) - ik \boldsymbol{A}^{\text{scat}}(\boldsymbol{x}) = o\Big(\tfrac{1}{|\boldsymbol{x}|}\Big), \qquad |\boldsymbol{x}| \to \infty\,, \tag{38}$$

for the vector Helmholtz potential, and

$$\Delta \boldsymbol{A}_0^{\text{scat}} = 0, \quad \boldsymbol{n} \times \boldsymbol{A}_0^{\text{scat}}|_{\partial D} = \mathbf{f}, \quad \nabla \cdot \boldsymbol{A}_0^{\text{scat}}|_{\partial D} = h, \tag{39}$$

$$\boldsymbol{A}_0^{\text{scat}}(\boldsymbol{x}) = O\Big(\tfrac{1}{|\boldsymbol{x}|}\Big), \quad \nabla \times \boldsymbol{A}_0^{\text{scat}}(\boldsymbol{x}) = O\Big(\tfrac{1}{|\boldsymbol{x}|^2}\Big), \quad \nabla \cdot \boldsymbol{A}_0^{\text{scat}}(\boldsymbol{x}) = O\Big(\tfrac{1}{|\boldsymbol{x}|^2}\Big), \qquad |\boldsymbol{x}| \to \infty\,, \tag{40}$$

for the vector Laplace potential, respectively.

For $k \neq 0$, both Dirichlet problems have unique solutions, but for $k = 0$, the vector Dirichlet problem has a nullspace — the harmonic Dirichlet fields discussed in Section 2.1. This lack of uniqueness also makes the vector Dirichlet problem ill-conditioned at low frequencies.

Unknowns: ϕ^{scat} $\boldsymbol{A}^{\text{scat}}$	Laplace	Helmholtz
Scalar:	$\left\{ \begin{array}{r} \Delta\phi^{\text{scat}} = 0, \\ \phi^{\text{scat}}\|_{\partial D} = f. \end{array} \right.$ (Yes)	$\left\{ \begin{array}{r} \Delta\phi^{\text{scat}} + k^2\phi^{\text{scat}} = 0, \\ \phi^{\text{scat}}\|_{\partial D} = f. \end{array} \right.$ (Yes)
Vector:	$\left\{ \begin{array}{r} \Delta\boldsymbol{A}^{\text{scat}} = 0, \\ \boldsymbol{n} \times \boldsymbol{A}^{\text{scat}}\|_{\partial D} = \mathbf{f}, \\ \nabla \cdot \boldsymbol{A}^{\text{scat}}\|_{\partial D} = h. \end{array} \right.$ (No)	$\left\{ \begin{array}{r} \Delta\boldsymbol{A}^{\text{scat}} + k^2\boldsymbol{A}^{\text{scat}} = 0, \\ \boldsymbol{n} \times \boldsymbol{A}^{\text{scat}}\|_{\partial D} = \mathbf{f}, \\ \nabla \cdot \boldsymbol{A}^{\text{scat}}\|_{\partial D} = h. \end{array} \right.$ (Yes)

Table 1: Uniqueness for Dirichlet problems

3.1 Modified Dirichlet problems

In order to address the non-uniqueness of the vector Dirichlet problem at zero frequency and in order to enforce that the uncoupled scalar and vector potentials define a suitable Maxwell field (enforcing the Lorenz gauge), we introduce a related set of boundary value problems, which we refer to as *the modified Dirichlet problems.*

Definition 4. *By the* scalar modified Dirichlet problem*, we mean the calculation of a scalar Helmholtz or Laplace potential in $\mathbb{R}^3\backslash D$ which satisfies standard radiation conditions at infinity. Letting N denote the number of connected components of the boundary ∂D, we introduce extra unknown degrees of freedom $\{V_j\}_{j=1}^N$ and the boundary data f is supplemented with additional (known) constants $\{Q_j\}_{j=1}^N$. For the scalar Helmholtz potential,*

$$\Delta\phi^{\text{scat}} + k^2\phi^{\text{scat}} = 0, \quad \phi^{\text{scat}}|_{\partial D_j} = f + V_j, \tag{41}$$

$$\tfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\cdot\nabla\phi^{\text{scat}}(\boldsymbol{x}) - ik\phi^{\text{scat}}(\boldsymbol{x}) = o\Big(\tfrac{1}{|\boldsymbol{x}|}\Big), \qquad |\boldsymbol{x}|\to\infty\,, \tag{42}$$

with

$$\int_{\partial D_j}\frac{\partial\phi^{\text{scat}}}{\partial n}ds = Q_j.$$

For the scalar Laplace potential,

$$\Delta\phi_0^{\text{scat}} = 0, \quad \phi_0^{\text{scat}}|_{\partial D_j} = f + V_j, \tag{43}$$

$$\phi_0^{\text{scat}}(\boldsymbol{x}) = O\Big(\tfrac{1}{|\boldsymbol{x}|}\Big), \quad \nabla\phi_0^{\text{scat}}(\boldsymbol{x}) = O\Big(\tfrac{1}{|\boldsymbol{x}|^2}\Big), \qquad |\boldsymbol{x}|\to\infty, \tag{44}$$

with

$$\int_{\partial D_j}\frac{\partial\phi^{\text{scat}}}{\partial n}ds = Q_j.$$

Definition 5. *By the* vector modified Dirichlet problem*, we mean the calculation of a vector Helmholtz or Laplace potential in $\mathbb{R}^3\backslash D$ which satisfies standard radiation conditions at infinity. Letting N denote the number of connected components of the boundary ∂D, we introduce extra unknown degrees of freedom $\{v_j\}_{j=1}^N$ and the boundary data f is supplemented with additional (known) constants $\{q_j\}_{j=1}^N$. For the vector Helmholtz potential,*

$$\Delta\boldsymbol{A}^{\text{scat}} + k^2\boldsymbol{A}^{\text{scat}} = 0, \quad \boldsymbol{n}\times\boldsymbol{A}^{\text{scat}}|_{\partial D} = \mathbf{f}, \quad \nabla\cdot\boldsymbol{A}^{\text{scat}}|_{\partial D_j} = h + v_j, \tag{45}$$

$$\nabla\times\boldsymbol{A}^{\text{scat}}(\boldsymbol{x})\times\tfrac{\boldsymbol{x}}{|\boldsymbol{x}|} + \tfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\nabla\cdot\boldsymbol{A}^{\text{scat}}(\boldsymbol{x}) - ik\boldsymbol{A}^{\text{scat}}(\boldsymbol{x}) = o\Big(\tfrac{1}{|\boldsymbol{x}|}\Big), \qquad |\boldsymbol{x}|\to\infty\,, \tag{46}$$

with

$$\int_{\partial D_j}\boldsymbol{n}\cdot\boldsymbol{A}^{\text{scat}}ds = q_j.$$

For the vector Laplace potential,

$$\Delta\boldsymbol{A}_0^{\text{scat}} = 0, \quad \boldsymbol{n}\times\boldsymbol{A}_0^{\text{scat}}|_{\partial D} = \mathbf{f}, \quad \nabla\cdot\boldsymbol{A}_0^{\text{scat}}|_{\partial D_j} = h + v_j, \tag{47}$$

$$\boldsymbol{A}_0^{\text{scat}}(\boldsymbol{x}) = O\Big(\tfrac{1}{|\boldsymbol{x}|}\Big), \quad \nabla\times\boldsymbol{A}_0^{\text{scat}}(\boldsymbol{x}) = O\Big(\tfrac{1}{|\boldsymbol{x}|^2}\Big), \quad \nabla\cdot\boldsymbol{A}_0^{\text{scat}}(\boldsymbol{x}) = O\Big(\tfrac{1}{|\boldsymbol{x}|^2}\Big), \qquad |\boldsymbol{x}|\to\infty\,, \tag{48}$$

with

$$\int_{\partial D_j}\boldsymbol{n}\cdot\boldsymbol{A}^{\text{scat}}ds = q_j.$$

We summarize the modified Dirichlet boundary value problems in Table 2.

Unknowns: ϕ^{scat}, $\{V_j\}_{j=1}^N$ $\boldsymbol{A}^{\text{scat}}$, $\{v_j\}_{j=1}^N$	Laplace	Helmholtz
Scalar:	$\begin{cases} \Delta\phi^{\text{scat}} = 0, \\ \phi^{\text{scat}}\|_{\partial D_j} = f + V_j, \\ \int_{\partial D_j} \frac{\partial \phi^{\text{scat}}}{\partial n} ds = Q_j. \end{cases}$ (Yes)	$\begin{cases} \Delta\phi^{\text{scat}} + k^2\phi^{\text{scat}} = 0, \\ \phi^{\text{scat}}\|_{\partial D_j} = f + V_j, \\ \int_{\partial D_j} \frac{\partial \phi^{\text{scat}}}{\partial n} ds = Q_j. \end{cases}$ (Yes)
Vector:	$\begin{cases} \Delta\boldsymbol{A}^{\text{scat}} = 0, \\ \boldsymbol{n} \times \boldsymbol{A}^{\text{scat}}\|_{\partial D} = \mathbf{f}, \\ \nabla \cdot \boldsymbol{A}^{\text{scat}}\|_{\partial D_j} = h + v_j, \\ \int_{\partial D_j} \boldsymbol{n} \cdot \boldsymbol{A}^{\text{scat}} ds = q_j. \end{cases}$ (Yes)	$\begin{cases} \Delta\boldsymbol{A}^{\text{scat}} + k^2\boldsymbol{A}^{\text{scat}} = 0, \\ \boldsymbol{n} \times \boldsymbol{A}^{\text{scat}}\|_{\partial D} = \mathbf{f}, \\ \nabla \cdot \boldsymbol{A}^{\text{scat}}\|_{\partial D_j} = h + v_j, \\ \int_{\partial D_j} \boldsymbol{n} \cdot \boldsymbol{A}^{\text{scat}} ds = q_j. \end{cases}$ (Yes)

Table 2: Uniqueness for modified Dirichlet problems

We now define the scattered scalar and vector potentials in terms of modified Dirichlet problems.

Definition 6. *Let $\phi^{\text{inc}}, \boldsymbol{A}^{\text{inc}}$ denote incoming scalar and vector potentials and assume that D is a perfect conductor. The* scattered scalar potential ϕ^{scat} *is the solution to the scalar modified Dirichlet problem with boundary data:*

$$f := -\phi^{\text{inc}}|_{\partial D_j}, \quad Q_j := -\int_{\partial D_j} \frac{\phi^{\text{inc}}}{\partial n} ds\,. \tag{49}$$

Likewise, the scattered vector potential $\boldsymbol{A}^{\text{scat}}$ *is the solution to the vector modified Dirichlet problem with boundary data:*

$$\mathbf{f} := -\boldsymbol{n} \times \boldsymbol{A}^{\text{inc}}|_{\partial D}, \quad h := -\nabla \cdot \boldsymbol{A}^{\text{inc}}|_{\partial D}, \quad q_j := -\int_{\partial D_j} \boldsymbol{n} \cdot \boldsymbol{A}^{\text{inc}} ds\,. \tag{50}$$

3.2 Uniqueness

We begin with two well-known theorems from scattering theory.

Theorem 1. *[22] Let ϕ^{scat} be a scalar Helmholtz potential with wavenumber k, $(k \neq 0)$ in the exterior domain $\mathbb{R}^3 \backslash \overline{D}$, satisfying the radiation condition (34) and the condition*

$$I_s = \Im\left\{ k \int_{\partial D} \phi^{\text{scat}} \frac{\partial \overline{\phi}^{\text{scat}}}{\partial n} ds \right\} \geq 0. \tag{51}$$

Here, $\Im\{f\}$ denotes the imaginary part of f. Then, $\phi^{\text{scat}} = 0$ in $\mathbb{R}^3 \backslash \overline{D}$.

Theorem 2. *[22] Let $\boldsymbol{A}^{\text{scat}}$ be a vector Helmholtz potential with wavenumber k, $(k \neq 0)$ in the exterior domain $\mathbb{R}^3 \backslash \overline{D}$, satisfying the radiation condition (38) and the condition*

$$I_v = \Im\left\{ k \int_{\partial D} \boldsymbol{n} \times \boldsymbol{A}^{\text{scat}} \cdot \nabla \times \overline{\boldsymbol{A}}^{\text{scat}} + \boldsymbol{n} \cdot \boldsymbol{A}^{\text{scat}} \nabla \cdot \overline{\boldsymbol{A}}^{\text{scat}} ds \right\} \geq 0\,. \tag{52}$$

Then, $\boldsymbol{A}^{\text{scat}} = 0$ in $\mathbb{R}^3 \backslash \overline{D}$.

We now show that the modified Dirichlet problems have unique solutions in all regimes. For simplicity, we assume that k is real. The proofs are analogous when the wavenumber k has a positive imaginary part, which adds dissipation.

Theorem 3. *The scalar modified Dirichlet problem has at most one solution for any $k > 0$.*

Proof. Consider a solution of the homogeneous problem ($f = 0, Q_j = 0$):

$$\begin{cases} \Delta \phi^{\text{scat}} + k^2 \phi^{\text{scat}} = 0, \\ \phi^{\text{scat}}|_{\partial D} = 0 + V_j, \\ \int_{\partial D_j} \frac{\partial \phi^{\text{scat}}}{\partial n} ds = 0. \end{cases} \tag{53}$$

The quantity I_s in (51) is then given by

$$I_s = \Im\left\{ k \int_{\partial D} \phi^{\text{scat}} \frac{\partial \overline{\phi}^{\text{scat}}}{\partial n} ds \right\} = \Im\left\{ k \sum_{j=1}^{N} V_j \int_{\partial D_j} \frac{\partial \overline{\phi}^{\text{scat}}}{\partial n} \right\} = 0. \tag{54}$$

Thus, by Theorem 1, $\phi^{\text{scat}} = 0$ and using the boundary condition $\phi^{\text{scat}}|_{\partial D_j} = 0 + V_j$, we get $V_j = 0$. □

Theorem 4. *The vector modified Dirichlet problem has at most one solution for any $k > 0$.*

Proof. Consider a solution of the homogeneous problem ($\mathbf{f} = 0, h = 0, q_j = 0$):

$$\begin{cases} \Delta \boldsymbol{A}^{\text{scat}} + k^2 \boldsymbol{A}^{\text{scat}} = 0, \\ \boldsymbol{n} \times \boldsymbol{A}^{\text{scat}}|_{\partial D} = \mathbf{0}, \\ \nabla \cdot \boldsymbol{A}^{\text{scat}}|_{\partial D_j} = 0 + v_j, \\ \int_{\partial D_j} \boldsymbol{n} \cdot \boldsymbol{A}^{\text{scat}} ds = 0. \end{cases} \tag{55}$$

The quantity I_v in 52 is then given by

$$\begin{aligned} I_v &= \Im\left\{ k \int_{\partial D} \boldsymbol{n} \times \boldsymbol{A}^{\text{scat}} \cdot \nabla \times \overline{\boldsymbol{A}}^{\text{scat}} + \boldsymbol{n} \cdot \boldsymbol{A}^{\text{scat}} \nabla \cdot \overline{\boldsymbol{A}}^{\text{scat}} ds \right\} = \\ &= \Im\left\{ k \sum_{j=1}^{N} \overline{v_j} \int_{\partial D_j} \boldsymbol{n} \cdot \boldsymbol{A}^{\text{scat}} ds \right\} = 0. \end{aligned} \tag{56}$$

Thus, by Theorem 2, $\mathbf{A}^{\text{scat}} = 0$ and using the boundary condition $\nabla \cdot \boldsymbol{A}^{\text{scat}}|_{\partial D_j} = 0 + v_j$, we obtain $v_j = 0$. □

Theorem 5. *The scalar modified Dirichlet problem for the Laplace equation has at most one solution.*

Proof. This is a well-known result. When $f = 0$, the relation between Q_j and V_j is the capacitance matrix [1] and this matrix is always invertible (Theorem 5.6 in [22]). Thus, if $Q_j = 0$, then $V_j = 0$. The fact that $\phi^{\text{scat}} = 0$ follows from the maximum principle [27]. □

Before proving the uniqueness of the vector modified Dirichlet problem for the Laplace equation we need the following technical result that shows a relation between the vector and scalar modified Dirichlet problems. This Lemma will also be used in section 4 to prove the connection between electromagnetic scattering and modified Dirichlet problems.

Lemma 1. *Let $\boldsymbol{A}^{\text{scat}}, \{v_j\}_{j=1}^N$ be a solution of the vector modified Dirichlet problem with boundary data $\mathbf{f}, h, \{q_j\}_{j=1}^N$ for $(k \geq 0)$. Then,*

$$\psi^{\text{scat}} := \nabla \cdot \boldsymbol{A}^{\text{scat}}, \quad \{V_j = v_j\}_{j=1}^N \tag{57}$$

satisfies the scalar modified Dirichlet problem with boundary data:

$$f := h, \quad \{Q_j = -k^2 q_j\}_{j=1}^N . \tag{58}$$

Proof. By hypothesis, $\boldsymbol{A}^{\text{scat}}$ satisfies

$$\Delta \boldsymbol{A}^{\text{scat}} + k^2 \boldsymbol{A}^{\text{scat}} = 0 \tag{59}$$

$$\boldsymbol{n} \times \boldsymbol{A}^{\text{scat}}|_{\partial D} = \mathbf{f} \tag{60}$$

$$\nabla \cdot \boldsymbol{A}^{\text{scat}}|_{\partial D_j} = h + v_j \tag{61}$$

$$\int_{\partial D_j} \boldsymbol{n} \cdot \boldsymbol{A}^{\text{scat}} ds = q_j . \tag{62}$$

Taking the divergence of (59), we get

$$\Delta \nabla \cdot \boldsymbol{A}^{\text{scat}} + k^2 \nabla \cdot \boldsymbol{A}^{\text{scat}} = 0 . \tag{63}$$

Therefore, $\psi^{\text{scat}} = \nabla \cdot \boldsymbol{A}^{\text{scat}}$ satisfies the Helmholtz equation. From 61, we get

$$\psi^{\text{scat}} = \nabla \cdot \boldsymbol{A}^{\text{scat}} = h + v_j|_{\partial D_j} . \tag{64}$$

Finally, we may write

$$\begin{aligned}
\nabla \times \nabla \times \boldsymbol{A}^{\text{scat}} &= k^2 \boldsymbol{A}^{\text{scat}} + \nabla \nabla \cdot \boldsymbol{A}^{\text{scat}} \Rightarrow \\
\boldsymbol{n} \cdot \nabla \times \nabla \times \boldsymbol{A}^{\text{scat}} &= k^2 \boldsymbol{n} \cdot \boldsymbol{A}^{\text{scat}} + \boldsymbol{n} \cdot \nabla \nabla \cdot \boldsymbol{A}^{\text{scat}} \Rightarrow \\
-\nabla_s \cdot (\boldsymbol{n} \times \nabla \times \boldsymbol{A}^{\text{scat}}) &= k^2 \boldsymbol{n} \cdot \boldsymbol{A}^{\text{scat}} + \boldsymbol{n} \cdot \nabla \frac{\partial \psi^{\text{scat}}}{\partial n} \Rightarrow \\
-\int_{\partial D_j} \nabla_s \cdot (\boldsymbol{n} \times \nabla \times \boldsymbol{A}^{\text{scat}}) ds = 0 &= k^2 \int_{\partial D_j} \boldsymbol{n} \cdot \boldsymbol{A}^{\text{scat}} ds + \int_{\partial D_j} \frac{\partial \psi^{\text{scat}}}{\partial n} ds .
\end{aligned} \tag{65}$$

Using the boundary condition (62), we obtain

$$\int_{\partial D_j} \frac{\partial \psi^{\text{scat}}}{\partial n} ds = -k^2 q_j \,, \tag{66}$$

and the result follows. □

Theorem 6. *The vector modified Dirichlet problem for the Laplace equation has at most one solution.*

Proof. Let $(\boldsymbol{A}^{\text{scat}}, v_j)$ be a solution of the homogeneous vector modified Dirichlet problem. Then, applying Lemma 1, $(\nabla \cdot \boldsymbol{A}^{\text{scat}}, \{v_j\}_{j=1}^N)$ satisfies the homogeneous scalar modified Dirichlet problem By Theorem 5, $\psi^{\text{scat}} = \nabla \cdot \boldsymbol{A}^{\text{scat}} = 0$ and $v_j = 0$. By theorem 5.9 in [22], $\boldsymbol{A}^{\text{scat}}$ is a harmonic Dirichlet field, and thus a linear combination of the basis functions $\{\mathbf{Y}_j\}_{j=1}^N$. It follows from the flux conditions $\int_{\partial D_j} \boldsymbol{n} \cdot \boldsymbol{A}^{\text{scat}} ds = 0$ that $\boldsymbol{A}^{\text{scat}} = 0$. □

3.3 Existence and stability

In this section, we use the Fredholm alternative to obtain existence results for the modified Dirichlet problems, making use of the single and double layer potentials, S_k and D_k, of classical potential theory. We also show that the solution depends continuously on the boundary data, uniformly in k in a neighborhood of $k = 0$. Next we define classical operators in potential theory:

$$\begin{aligned}
S_k\sigma &= \int_{\partial D} g_k(\boldsymbol{x}-\boldsymbol{y})\sigma(\boldsymbol{y}) dA_{\boldsymbol{y}}, \\
D_k\sigma &= \int_{\partial D} \frac{\partial g_k}{\partial n_y}(\boldsymbol{x}-\boldsymbol{y})\sigma(\boldsymbol{y}) dA_{\boldsymbol{y}}, \\
S_k'\sigma &= \int_{\partial D} \frac{\partial g_k}{\partial n_x}(\boldsymbol{x}-\boldsymbol{y})\sigma(\boldsymbol{y}) dA_{\boldsymbol{y}}, \\
D_k'\sigma &= \frac{\partial}{\partial n_x}\int_{\partial D} \frac{\partial g_k}{\partial n_y}(\boldsymbol{x}-\boldsymbol{y})\sigma(\boldsymbol{y}) dA_{\boldsymbol{y}},
\end{aligned} \tag{67}$$

where $\boldsymbol{x} \in \partial D$ and the Green's function on the free space is:

$$g_k(\boldsymbol{x}) = \frac{e^{ik|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|}. \tag{68}$$

For off-surface evaluations $\boldsymbol{x} \in \mathbb{R}^3 \backslash \partial D$ we have:

$$\begin{aligned}
S_k[\sigma](\boldsymbol{x}) &= \int_{\partial D} g_k(\boldsymbol{x}-\boldsymbol{y})\sigma(\boldsymbol{y}) dA_{\boldsymbol{y}}, \\
D_k[\sigma](\boldsymbol{x}) &= \int_{\partial D} \frac{\partial g_k}{\partial n_y}(\boldsymbol{x}-\boldsymbol{y})\sigma(\boldsymbol{y}) dA_{\boldsymbol{y}}.
\end{aligned} \tag{69}$$

Theorem 7. *Suppose that we represent the solution to the scalar modified Dirichlet problem with $k > 0$ in the form*

$$\phi^{\text{scat}}(\boldsymbol{x}) = D_k[\sigma](\boldsymbol{x}) - i\eta S_k[\sigma](\boldsymbol{x})\,, \tag{70}$$

with $\eta \in \mathbb{R}\backslash\{0\}$. Then, imposing the desired boundary conditions and constraints leads to a Fredholm equation of the second kind:

$$\begin{aligned} &\frac{\sigma}{2} + D_k\sigma - i\eta S_k\sigma - \sum_{j=1}^{N} V_j\chi_j = f, \\ &\int_{\partial D_j} (D_k'\sigma + i\eta\frac{\sigma}{2} - i\eta S_k'\sigma)ds = Q_j\,, \end{aligned} \tag{71}$$

where χ_j denotes the characteristic function for boundary ∂D_j. Here, σ and the constants $\{V_j\}_{j=1}^N$ are unknowns. Moreover, (71) is invertible and the result holds for the modified Dirichlet problem governed by the Laplace equation ($k = 0$) as well.

Proof. See Appendix A. □

In order to study the vector modified Dirichlet problem, we define the following dyadic operators:

$$\overline{\overline{L}}\begin{pmatrix} \mathbf{a} \\ \rho \end{pmatrix} = \begin{pmatrix} L_{11}\mathbf{a} + L_{12}\rho \\ L_{21}\mathbf{a} + L_{22}\rho \end{pmatrix}, \tag{72}$$

where

$$\begin{aligned} L_{11}\mathbf{a} &= n \times S_k\mathbf{a}, \\ L_{12}\rho &= -\boldsymbol{n} \times S_k(\boldsymbol{n}\rho), \\ L_{21}\mathbf{a} &= 0, \\ L_{22}\rho &= D_k\rho, \end{aligned} \tag{73}$$

and

$$\overline{\overline{R}}\begin{pmatrix} \mathbf{a} \\ \rho \end{pmatrix} = \begin{pmatrix} R_{11}\mathbf{a} + R_{12}\rho \\ R_{21}\mathbf{a} + R_{22}\rho \end{pmatrix}, \tag{74}$$

where

$$\begin{aligned} R_{11}\mathbf{a} &= \boldsymbol{n} \times S_k(\boldsymbol{n} \times \mathbf{a}), \\ R_{12}\rho &= \boldsymbol{n} \times \nabla S_k(\rho), \\ R_{21}\mathbf{a} &= \nabla \cdot S_k(\boldsymbol{n} \times \mathbf{a}), \\ R_{22}\rho &= -k^2 S_k\rho. \end{aligned} \tag{75}$$

Theorem 8. *Suppose that we represent the solution to the vector modified Dirichlet problem with $k > 0$ in the form*

$$\boldsymbol{A}^{\text{scat}} = \nabla \times S_k[\mathbf{a}](\boldsymbol{x}) - S_k[\boldsymbol{n}\rho](\boldsymbol{x}) + i\eta(S_k[\boldsymbol{n} \times \mathbf{a}](\boldsymbol{x}) + \nabla S_k[\rho](\boldsymbol{x})), \tag{76}$$

with $\eta \in \mathbb{R}\backslash\{0\}$. Then, for $|\eta|$ sufficiently small, imposing the desired boundary conditions and constraints leads to a Fredholm equation of the second kind:

$$\begin{gathered}\frac{1}{2}\begin{pmatrix}\mathbf{a}\\ \rho\end{pmatrix}+\overline{\overline{L}}\begin{pmatrix}\mathbf{a}\\ \rho\end{pmatrix}+i\eta\overline{\overline{R}}\begin{pmatrix}\mathbf{a}\\ \rho\end{pmatrix}+\begin{pmatrix}0\\ \sum_{j=1}^{N}v_j\chi_j\end{pmatrix}=\begin{pmatrix}\mathbf{f}\\ h\end{pmatrix},\\ \int_{\partial D_j}\Big(\boldsymbol{n}\cdot\nabla\times S_k\mathbf{a}-\boldsymbol{n}\cdot S_k(\boldsymbol{n}\rho)+i\eta\big(\boldsymbol{n}\cdot S_k(\mathbf{n}\times\mathbf{a})-\frac{\rho}{2}+S_k'\rho\big)\Big)ds=q_j,\end{gathered}\tag{77}$$

where χ_j denotes the characteristic function for boundary ∂D_j. Here, $\mathbf{a}, \rho$ and the constants $\{v_j\}_{j=1}^N$ are unknowns. Moreover, (77) is invertible and the result holds for the vector modified Dirichlet problem governed by the vector Laplace equation ($k = 0$) as well.

Proof. See Appendix A. □

Definition 7. *We will refer to (71) and (77) as the scalar and vector decoupled potential integral equations. The former will be abbreviated by* DPIEs *and the latter by* DPIEv*. Together, they form the* DPIE*.*

The following two theorems show that the solutions to the modified Dirichlet problems are continuous functions of the boundary data all the way to $k = 0$. In particular, they are independent of the genus of ∂D.

Theorem 9. *The scalar modified Dirichlet problem has a unique solution for $k \geq 0$. Moreover, the solution depends continuously on the boundary data $f, \{Q_j\}_{j=1}^N$ in the sense that the operator mapping the given boundary data onto the solution is uniformly continuous from*

$$f, \{Q_j\}_{j=1}^N \in C^{0,\alpha}(\partial D)\times\mathbb{C}^N \quad\rightarrow\quad \phi^{\text{scat}}, \{V_j\}_{j=1}^N \in C^{0,\alpha}(\mathbb{R}^3/D)\times\mathbb{C}^N$$

for any $k \in [0, k_{max}]$, with fixed k_{max}. $C^{0,\alpha}(X)$ here is equipped with the usual Hölder norm [22].

Proof. See Appendix A. □

Theorem 10. *The vector modified Dirichlet problem has a unique solution for $k \geq 0$. Moreover, the solution depends continuously on the boundary data $\mathbf{f}, h, \{q_j\}_{j=1}^N$ in the sense that the operator mapping the given boundary data onto the solution is uniformly continuous from*

$$\mathbf{f}, h, \{Q_j\}_{j=1}^N \in T^{0,\alpha}(\partial D)\times C^{0,\alpha}(\partial D)\times\mathbb{C}^N \quad\rightarrow\quad \boldsymbol{A}^{\text{scat}}, \{v_j\}_{j=1}^N \in C^{0,\alpha}(\mathbb{R}^3/D)\times\mathbb{C}^N$$

for any $k \in [0, k_{max}]$, with fixed k_{max}. Here, $T^{0,\alpha}(\partial D)$ is equipped with the usual Holder norm [22].

Proof. See Appendix A. □

4 Electromagnetic scattering and modified Dirichlet problems

In this section, we explain the connection between the scalar and vector modified Dirichlet problems and the Maxwell equations. It is evident from Theorems 9 and 10 that, if such a reformulation exists, then we have overcome the topological low-frequency breakdown that makes electromagnetic scattering from surfaces with nontrivial genus so difficult at low frequency.

We will first show that the vector and scalar modified Dirichlet problems preserve the Lorenz gauge, so that the induced $\boldsymbol{E}$ and $\boldsymbol{H}$ fields are Maxwellian. We will also show that the calculation is stable, in the sense that bounded "incoming" data leads to bounded "outgoing" data, independent of the frequency. We will then show, in Theorem 12, that the modified Dirichlet problems lead directly to the solution of the desired scattering problem.

Theorem 11. *Let $\boldsymbol{A}^{\text{inc}}, \phi^{\text{inc}}$ be bounded (for $\omega \to 0$) incoming vector and scalar Helmholtz potentials in the Lorenz gauge:*

$$\nabla \cdot \mathbf{A}^{\text{inc}} = i\omega\mu\epsilon\phi^{\text{inc}}\,. \tag{78}$$

Then, the associated vector and scalar scattered Helmholtz potentials $\boldsymbol{A}^{\text{scat}}, \phi^{\text{scat}}$ (see Definition 6) are also bounded and satisfy the Lorenz gauge condition.

Proof. By Lemma 1, the scalar Helmholtz potential $\psi^{\text{scat}} = \nabla \cdot \boldsymbol{A}^{\text{scat}}$ satisfies

$$\begin{aligned} \psi^{\text{scat}} &= h + v_j = -\nabla \cdot \boldsymbol{A}^{\text{inc}} + v_j\big|_{\partial D} \\ \int_{\partial D_j} \frac{\partial \psi^{\text{scat}}}{\partial n} ds &= -k^2 q_j = k^2 \int_{\partial D_j} \mathbf{n} \cdot \mathbf{A}^{\text{inc}} ds\,. \end{aligned} \tag{79}$$

Using the Lorenz gauge condition on the boundary itself, we may write

$$\psi^{\text{scat}} = h + v_j = -i\omega\mu\epsilon\phi^{\text{inc}} + v_j\big|_{\partial D}\,. \tag{80}$$

Since

$$\nabla \cdot \left(i\omega \boldsymbol{A}^{\text{inc}} - \nabla\phi^{\text{inc}}\right) = 0\,, \tag{81}$$

we have

$$\int_{\partial D_j} \frac{\partial \phi^{\text{inc}}}{\partial n} ds = i\omega \int_{\partial D_j} \mathbf{n} \cdot \mathbf{A}^{\text{inc}} ds\,. \tag{82}$$

Thus,

$$\int_{\partial D_j} \frac{\partial \psi^{\text{scat}}}{\partial n} ds = k^2 \int_{\partial D_j} \mathbf{n} \cdot \mathbf{A}^{\text{inc}} ds = \frac{k^2}{i\omega} \int_{\partial D_j} \frac{\partial \phi^{\text{inc}}}{\partial n} ds = -i\omega\mu\epsilon \int_{\partial D_j} \frac{\partial \phi^{\text{inc}}}{\partial n} ds\,. \tag{83}$$

From (80) and (83), we see that ψ^{scat} and $i\omega\mu\epsilon\phi^{\text{scat}}$ satisfy the same scalar modified Dirichlet problem. By uniqueness (Theorem 3), we find that

$$i\omega\mu\epsilon\phi^{\text{scat}} = \psi^{\text{scat}} = \nabla \cdot \boldsymbol{A}^{\text{scat}}\,,$$

so that $\boldsymbol{A}^{\text{scat}}$ and ϕ^{scat} are in the Lorenz gauge. By Theorems 9 and 10, $\boldsymbol{A}^{\text{scat}}, \phi^{\text{scat}}$ are uniformly continuous functions of $\boldsymbol{A}^{\text{inc}}, \phi^{\text{inc}}$ for $k \in [0, k_{max}]$. Since $\boldsymbol{A}^{\text{inc}}, \phi^{\text{inc}}$ are bounded, $\boldsymbol{A}^{\text{scat}}, \phi^{\text{scat}}$ are also bounded. □

The next theorem is the main result of the present paper.

Theorem 12. *For any $k \geq 0$, let $\mathbf{E}^{\text{inc}}, \mathbf{H}^{\text{inc}}$ be an incoming electromagnetic field described by the potentials $\boldsymbol{A}^{\text{inc}}, \phi^{\text{inc}}$ in the Lorenz gauge:*

$$\begin{aligned}
\boldsymbol{H}^{\text{inc}} &= \frac{1}{\mu}\nabla \times \boldsymbol{A}^{\text{inc}},\\
\boldsymbol{E}^{\text{inc}} &= i\omega\boldsymbol{A}^{\text{inc}} - \nabla\phi^{\text{inc}},\\
\nabla \cdot \boldsymbol{A}^{\text{inc}} &= i\omega\mu\epsilon\phi^{\text{inc}}\,,
\end{aligned}$$

and let $\boldsymbol{A}^{\text{scat}}, \phi^{\text{scat}}$ denote the corresponding scattered vector and scalar potentials (Definition 6). Then the electromagnetic fields $\mathbf{E}^{\text{scat}}, \mathbf{H}^{\text{scat}}$ scattered from a perfect conductor are given by

$$\begin{aligned}
\boldsymbol{H}^{\text{scat}} &= \frac{1}{\mu}\nabla \times \boldsymbol{A}^{\text{scat}},\\
\boldsymbol{E}^{\text{scat}} &= i\omega\boldsymbol{A}^{\text{scat}} - \nabla\phi^{\text{scat}}.
\end{aligned}$$

with

$$\begin{aligned}
\nabla \cdot \boldsymbol{A}^{\text{scat}} &= i\omega\mu\epsilon\phi^{\text{scat}},\\
\boldsymbol{n} \times \boldsymbol{E}^{\text{scat}} = -\boldsymbol{n} \times \boldsymbol{E}^{\text{inc}}|_{\partial D}, \quad \boldsymbol{n} \cdot \boldsymbol{H}^{\text{scat}} &= -\boldsymbol{n} \cdot \boldsymbol{H}^{\text{inc}}|_{\partial D}.
\end{aligned}$$

Proof. Since $\boldsymbol{A}^{\text{scat}}, \phi^{\text{scat}}$ are Helmholtz potentials in the Lorenz gauge, the associated $\boldsymbol{E}^{\text{scat}}, \boldsymbol{H}^{\text{scat}}$ are valid Maxwell fields that satisfy the necessary radiation condition. We need only check that the desired boundary conditions are satisfied. From the boundary conditions on $\boldsymbol{A}^{\text{scat}}, \phi^{\text{scat}}$ we have

$$\begin{aligned}
\boldsymbol{n} \times \boldsymbol{A}^{\text{scat}} &= -\boldsymbol{n} \times \boldsymbol{A}^{\text{inc}}|_{\partial D}\\
\Rightarrow i\omega\boldsymbol{n} \times \boldsymbol{A}^{\text{scat}} &= -i\omega\boldsymbol{n} \times \boldsymbol{A}^{\text{inc}}|_{\partial D}\,,
\end{aligned} \tag{84}$$

$$\begin{aligned}
\phi^{\text{scat}} &= -\phi^{\text{inc}} + V_j|_{\partial D_j}\\
\Rightarrow \boldsymbol{n} \times \nabla\phi^{\text{scat}} &= -\boldsymbol{n} \times \nabla\phi^{\text{inc}}|_{\partial D}\,.
\end{aligned} \tag{85}$$

Adding (84) and (85), we have

$$\begin{aligned}
i\omega\boldsymbol{n} \times \boldsymbol{A}^{\text{scat}} - \boldsymbol{n} \times \nabla\phi^{\text{scat}} &= -i\omega\boldsymbol{n} \times \boldsymbol{A}^{\text{inc}} + \boldsymbol{n} \times \nabla\phi^{\text{inc}}|_{\partial D}\\
\Rightarrow \boldsymbol{n} \times \boldsymbol{E}^{\text{scat}} &= -\boldsymbol{n} \times \boldsymbol{E}^{\text{inc}}|_{\partial D}\,.
\end{aligned} \tag{86}$$

Taking the surface divergence of (84), we also have that

$$\begin{aligned} \boldsymbol{n} \times \boldsymbol{A}^{\mathrm{scat}} &= -\boldsymbol{n} \times \boldsymbol{A}^{\mathrm{inc}}|_{\partial D} \\ \Rightarrow \nabla_s \cdot \boldsymbol{n} \times \boldsymbol{A}^{\mathrm{scat}} &= -\nabla_s \cdot \boldsymbol{n} \times \boldsymbol{A}^{\mathrm{inc}}|_{\partial D} \\ \Rightarrow \boldsymbol{n} \cdot \boldsymbol{H}^{\mathrm{scat}} &= -\boldsymbol{n} \cdot \boldsymbol{H}^{\mathrm{inc}}|_{\partial D}\,. \end{aligned} \tag{87}$$

Thus, for $k > 0$ we have the correct solution. While continuity arguments are sufficient to verify that the zero frequency solution is the desired one, it is worth checking that the net charge still vanishes at $k = 0$ and that the consistency conditions (19) are satisfied. For this, note first that $\mathbf{A}, \phi$ are bounded, so that at $k = 0$,

$$\begin{aligned} \boldsymbol{H}_0^{\mathrm{inc}} &= \lim_{k\to 0} \boldsymbol{H}^{\mathrm{inc}} = \nabla \times \boldsymbol{A}_0^{\mathrm{inc}}, \\ \boldsymbol{E}_0^{\mathrm{inc}} &= \lim_{k\to 0} \boldsymbol{E}^{\mathrm{inc}} = -\nabla \phi_0^{\mathrm{inc}}, \\ \boldsymbol{H}_0^{\mathrm{scat}} &= \lim_{k\to 0} \boldsymbol{H}^{\mathrm{scat}} = \nabla \times \boldsymbol{A}_0^{\mathrm{scat}}, \\ \boldsymbol{E}_0^{\mathrm{scat}} &= \lim_{k\to 0} \boldsymbol{E}^{\mathrm{scat}} = -\nabla \phi_0^{\mathrm{scat}}\,. \end{aligned} \tag{88}$$

The net charge is computed as the surface integral of $\boldsymbol{n} \cdot \boldsymbol{E}$, and we have

$$\begin{aligned} \int_{\partial D_j} \boldsymbol{n} \cdot \boldsymbol{E}_0^{\mathrm{scat}} ds &= -\int_{\partial D_j} \frac{\partial \phi_0^{\mathrm{scat}}}{\partial n} ds = \int_{\partial D_j} \frac{\partial \phi_0^{\mathrm{inc}}}{\partial n} ds = \int_{D_j} \Delta \phi_0^{\mathrm{inc}} dv = 0, \\ \int_{\partial D_j} \boldsymbol{n} \cdot \boldsymbol{E}^{\mathrm{scat}} ds &= \int_{\partial D_j} i\omega \boldsymbol{n} \cdot \boldsymbol{A}^{\mathrm{scat}} - \frac{\partial \phi^{\mathrm{scat}}}{\partial n} ds = \\ &= -\int_{\partial D_j} i\omega \boldsymbol{n} \cdot \boldsymbol{A}^{\mathrm{inc}} - \frac{\partial \phi^{\mathrm{inc}}}{\partial n} ds = \int_{D_j} \nabla \cdot \left(i\omega \boldsymbol{A}^{\mathrm{inc}} - \nabla \phi^{\mathrm{inc}} \right) dv = 0\,. \end{aligned} \tag{89}$$

The last equality follows from the fact that the incoming potentials are assumed to be specified in the Lorenz gauge. In short,

$$\int_{\partial D_j} \boldsymbol{n} \cdot \boldsymbol{E}_0^{\mathrm{scat}} ds = \lim_{k\to 0} \int_{\partial D_j} \boldsymbol{n} \cdot \boldsymbol{E}^{\mathrm{scat}} ds = 0\,, \tag{90}$$

as expected. The consistency conditions (19) on the flux of the magnetic field through each hole S_j [24] are also easily verified for all $\omega \geq 0$:

$$\begin{aligned} \boldsymbol{n} \times \boldsymbol{A}^{\mathrm{scat}} &= -\boldsymbol{n} \times \boldsymbol{A}^{\mathrm{inc}}|_{\partial D} \\ \Rightarrow \oint_{B_j} \boldsymbol{A}^{\mathrm{scat}} \cdot d\mathbf{l} &= -\oint_{B_j} \boldsymbol{A}^{\mathrm{inc}} \cdot d\mathbf{l}\,. \end{aligned} \tag{91}$$

Here, B_j is a B-cycle, namely a loop on ∂D which goes around some "hole" and whose spanning surface S_j lies in the *exterior* of the domain (see Figure 1 and related discussion). □

5 Incoming Potentials

In a stable DPIE approach, the vector and scalar potentials must be defined in the Lorenz gauge and be bounded as $\omega \to 0$. We will need to find a representation for the incoming fields that will permit the stable uncoupling of the vector and scalar potentials. Assuming we are given the "impressed" free current and charge $\boldsymbol{J}^{imp}, \rho^{imp}$, the incoming potentials

$$\begin{aligned} \boldsymbol{A}^{\text{inc}}(\boldsymbol{x}) &= \mu S_k[\boldsymbol{J}^{imp}](\boldsymbol{x}), \\ \phi^{\text{inc}}(\boldsymbol{x}) &= \frac{1}{\epsilon} S_k[\rho^{imp}](\boldsymbol{x}), \end{aligned} \tag{92}$$

satisfy these requirements. For an incoming plane wave with a polarization vector $\boldsymbol{E}_p$ and a direction of propagation $\mathbf{u}$, given by

$$\boldsymbol{E}^{\text{inc}} = \boldsymbol{E}_p e^{ik\mathbf{u}\cdot\mathbf{x}}, \quad \boldsymbol{H}^{\text{inc}} = \boldsymbol{H}_p e^{ik\mathbf{u}\cdot\mathbf{x}} = \frac{\mathbf{u}\times\boldsymbol{E}_p}{Z} e^{ik\mathbf{u}\cdot\mathbf{x}}, \tag{93}$$

where $Z = \sqrt{\frac{\mu}{\epsilon}}$, the standard representation of incoming vector and scalar potentials

$$\boldsymbol{A}^{\text{inc}} = \frac{1}{i\omega}\boldsymbol{E}^{\text{inc}}, \quad \phi^{\text{inc}} = 0, \tag{94}$$

does not lead to stable uncoupling, since the vector potential is unbounded, as $\omega \to 0$. But, as mentioned above, the Lorenz gauge does not, by itself, impose uniqueness on the governing potentials. It is easy to check that the vector and scalar potentials defined by

$$\begin{aligned} \boldsymbol{A'}^{\text{inc}} &= -\mathbf{u}(\boldsymbol{x}\cdot\boldsymbol{E}_p)\sqrt{\mu\epsilon}e^{ik\mathbf{u}\cdot\boldsymbol{x}}, \\ \phi'^{\text{inc}} &= -\boldsymbol{x}\cdot\boldsymbol{E}_p e^{ik\mathbf{u}\cdot\boldsymbol{x}}, \end{aligned} \tag{95}$$

satisfy the Lorenz gauge condition

$$\nabla\cdot\boldsymbol{A'}^{\text{inc}}(\boldsymbol{x}) = i\omega\mu\epsilon\phi'^{\text{inc}}(\boldsymbol{x}), \tag{96}$$

both $\boldsymbol{A'}^{\text{inc}}$ and ϕ'^{inc} are bounded Helmholtz potentials, as $\omega \to 0$, and represent the same incoming plane wave (93). See Appendix B for more details how to stably decompose incoming/outgoing electric and magnetic multipole fields (Debye sources).

6 The DPIE and the Aharonov-Bohm effect

In classical physics, the Maxwell equations are described in terms of the components of the electric and magnetic fields, with the vector and scalar potentials viewed as matters of computational convenience. In quantum mechanics, however, it was shown by Aharonov and Bohm [28] that an electron is sensitive to the vector potential $\boldsymbol{A}$ itself, in regions where $\boldsymbol{E}$ and $\boldsymbol{H}$ are identically zero (the Aharonov-Bohm effect).

Let us first recall that the two pairs of potentials $\{\boldsymbol{A}, \phi\}$ and $\{\boldsymbol{A}', \phi'\}$ produce the same electromagnetic field, so long as they satisfy the condition

$$\begin{aligned} \mathbf{A}' &= \mathbf{A} + \nabla\psi, \\ \phi' &= \phi + i\omega\psi\,. \end{aligned} \tag{97}$$

In multiply connected regions at zero frequency, however, the situation is more complex. There exist potentials which give rise to identical fields that are not related according to (97). In particular, the potentials

$$\begin{aligned} \boldsymbol{A}_0 &= \mathbf{Z_1}, \quad \phi_0 = 0, \\ \boldsymbol{A}_0' &= \mathbf{0}, \quad\;\; \phi_0' = 0, \end{aligned} \tag{98}$$

where $\mathbf{Z_1}$ is an exterior harmonic Neumann field, give rise to zero electromagnetic fields in the exterior. $\mathbf{Z_1}$, however, is not the gradient of a single-valued harmonic function.

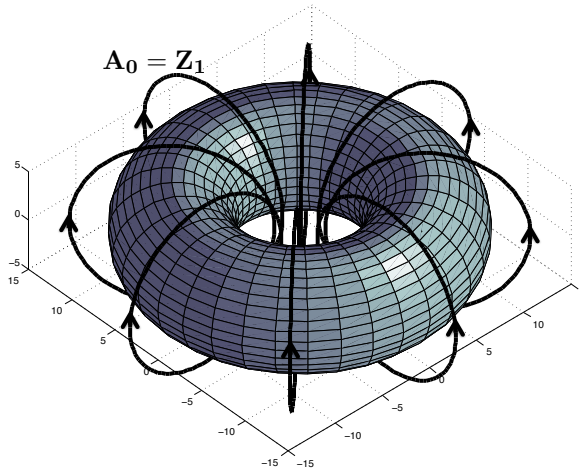

Figure 2: In the exterior of a torus, a harmonic Neumann field Z_1 serves as a vector potential $\boldsymbol{A}_0$ with the corresponding scalar potential set equal to zero. For $\boldsymbol{x} \in \mathbb{R}^3 \backslash D$, the associated electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are identically zero.

The Aharonov-Bohm effect is based on an experiment that is able to distinguish between the physical states $\boldsymbol{A}_0, \phi_0$ and $\boldsymbol{A}_0', \phi_0'$. We have taken some liberties with the actual experiment in [28] but the physical idea is the same. In essence, quantum mechanical tunneling permits an electron to be aware of the electromagnetic field in the *interior* of D, even though it is a perfect conductor. For $\boldsymbol{A}_0', \phi_0'$, the field is identically zero in the interior, but for $\boldsymbol{A}_0, \phi_0$, it is not. As discussed in [21–23, 25], $\boldsymbol{A}_0$ can be viewed as the field induced by an axisymmetric current density flowing on the surface in the direction of the

arrows in Fig. 2. This induces a non-trivial magnetic field within the torus. Electrons, as a result, sense whether they traveled through the hole of the torus or passed by the torus on the outside. The DPIE formalism easily distinguishes between these two cases, since we deal with the vector and scalar potentials directly. Thus, $\boldsymbol{A}_0, \phi_0, \boldsymbol{A}_0'$ and ϕ_0' in (98) satisfy

$$\begin{array}{lll} \boldsymbol{n} \times \boldsymbol{A}_0|_{\partial D} = \boldsymbol{n} \times \mathbf{Z_1}, & \nabla \cdot \boldsymbol{A}_0|_{\partial D} = 0, & \phi_0|_{\partial D} = 0, \\ \boldsymbol{n} \times \boldsymbol{A}_0'|_{\partial D} = \mathbf{0}, & \nabla \cdot \boldsymbol{A}_0'|_{\partial D} = 0, & \phi_0'|_{\partial D} = 0\,. \end{array} \tag{99}$$

7 The DPIE in the high frequency regime

Theorems 7 and 8 suggest that the numerical solution of scattering problems in the presence of perfect conductors can be effectively solved through the use of the DPIE, defined by equations (71) and (77). The scalar part of these equations (71) is a Fredholm equation of the second kind, invertible for all frequencies. As stated in Theorem 8, however, the vector part (77) is a Fredholm equation only for sufficiently small coupling constant $|\eta| < \|\overline{\overline{R}}\|^{-1}$. The difficulty is that the operator $\overline{\overline{R}}$ is continuous and bounded, but not compact. In fact, its spectrum has three cluster points: $\lambda = 0.5, \lambda = 0.5 + i0.5$ and $\lambda = 0.5 - i0.5$ (see [29, 30]). While the uniqueness proof holds for arbitrary values of η, existence requires further analysis. We suspect that the vector part of DPIE is invertible for all frequencies and leave the formal result as a conjecture. In fact, numerical experiments suggest that η should not be chosen too small when seeking to optimize the condition number of the DPIE.

Our interest in the DPIE formulation grew out of issues in low-frequency scattering. Nevertheless, we would like to find a representation that is effective at all frequencies, and this will involve a slight rescaling of the equations. In order to carry out a suitable analysis, we follow [31] and study scattering from the unit sphere $\partial D = \{\boldsymbol{x} : \|\boldsymbol{x}\| = 1\}$. For $k \leq 1$, setting $\eta = 1$ works well, while for $k > 1$ the optimal scaling factor $\eta \approx k$ (see [31]). Setting $\eta = k$, instead of (71), we have the *scaled* DPIEs integral equation:

$$\begin{aligned} \frac{\sigma}{2} + D_k\sigma - ikS_k\sigma - \sum_{j=1}^{N} V_j\chi_j &= f, \\ \int_{\partial D_j} (\frac{1}{k}D_k'\sigma + i\frac{\sigma}{2} - iS_k'\sigma)ds &= \frac{1}{k}Q_j\,, \end{aligned} \tag{100}$$

where the second set of equations has been multiplied by a factor of $\frac{1}{k}$. For the vector modified Dirichlet problem, when $k > 1$, we replace (76) with

$$\boldsymbol{A}^{\text{scat}} = \nabla \times S_k[\mathbf{a}](\boldsymbol{x}) - kS_k[\boldsymbol{n}\varrho](\boldsymbol{x}) + i\big(kS_k[\boldsymbol{n} \times \mathbf{a}](\boldsymbol{x}) + \nabla S_k[\varrho](\boldsymbol{x})\big), \tag{101}$$

where we have multiplied the single-layer potential terms by k, and set $\eta = 1$. We also rescale the boundary condition $\nabla \cdot \boldsymbol{A}^{\text{scat}} = -\nabla \cdot \boldsymbol{A}^{\text{scat}} + v_n$ in the modified Dirichlet problem,

dividing each side by k. These changes lead to the *scaled DPIEv* integral equation:

$$\begin{aligned} \frac{1}{2}\begin{pmatrix} \mathbf{a} \\ \rho \end{pmatrix} + \overline{\overline{L_s}}\begin{pmatrix} \mathbf{a} \\ \rho \end{pmatrix} + i\overline{\overline{R_s}}\begin{pmatrix} \mathbf{a} \\ \rho \end{pmatrix} + \begin{pmatrix} 0 \\ \sum_{j=1}^{N} v_j \chi_j \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \frac{1}{k}h \end{pmatrix}, \\ \int_{\partial D_j} \Big(\boldsymbol{n} \cdot \nabla \times S_k \mathbf{a} - k\boldsymbol{n} \cdot S_k(\boldsymbol{n}\varrho) + i\big(k\boldsymbol{n} \cdot S_k(\mathbf{n} \times \mathbf{a}) - \frac{\varrho}{2} + S_k'\varrho\big)\Big) ds = q_j, \end{aligned} \tag{102}$$

where

$$\overline{\overline{L_s}} = \begin{pmatrix} L_{11} & kL_{12} \\ \frac{1}{k}L_{21} & L_{22} \end{pmatrix}, \quad \overline{\overline{R_s}} = \begin{pmatrix} kR_{11} & R_{12} \\ R_{21} & \frac{1}{k}R_{22} \end{pmatrix}, \tag{103}$$

with L_{ij}, R_{ij} defined in (73).

We turn now to the analysis of the DPIE on the unit sphere, where exact expressions for the various integral operators have been worked out in detail [32]. More precisely, using scalar and vector spherical harmonics, each integral operator has a simple signature, which has been tabulated in [32]. This permits us to compute the condition number and spectrum of the DPIEs and DPIEv integral equations. In Figs. 3 and 4, we plot the spectrum and the singular values of the *scaled* DPIEv (102).

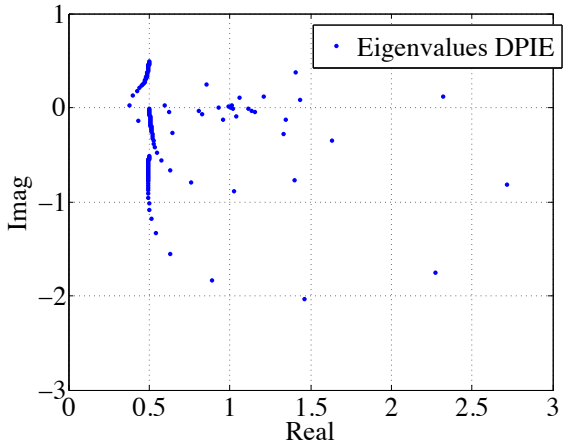

Figure 3: Spectrum of the scaled DPIEv integral equation (102) for a spherical scatterer of radius 1 at $k = 10$. As discussed in the text, there are three different cluster points: at $\lambda = 0.5, \lambda = 0.5 + i0.5$ and $\lambda = 0.5 - i0.5$.

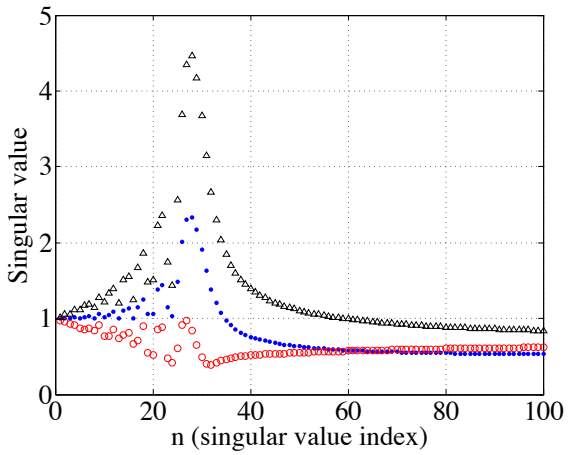

Figure 4: Singular values of the scaled DPIEv integral equation (102) for a spherical scatterer of radius 1 at $k = 30$. Three curves are shown, since (102) is a vector integral equation with three unknowns (the scalar ϱ and the tangential surface vector $\mathbf{a}$).

In Fig. 5, we compare the condition numbers of the DPIEv and the scaled DPIEv. Equally revealing is to plot the spectrum of the DPIEv and the scaled DPIEv (Fig. 6).

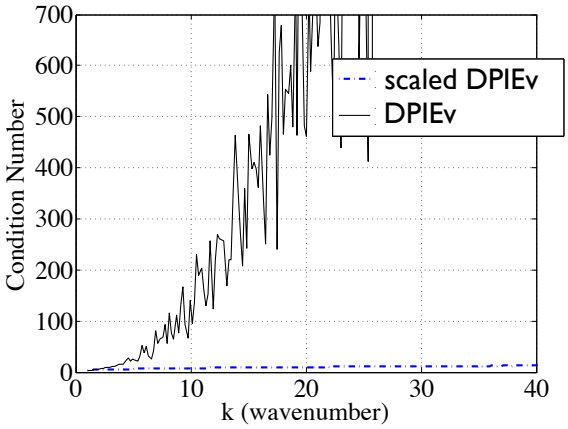

Figure 5: Condition number of the scaled DPIEv (102) and the original DPIEv (77) for a spherical scatterer of radius 1 as a function of wavenumber k.

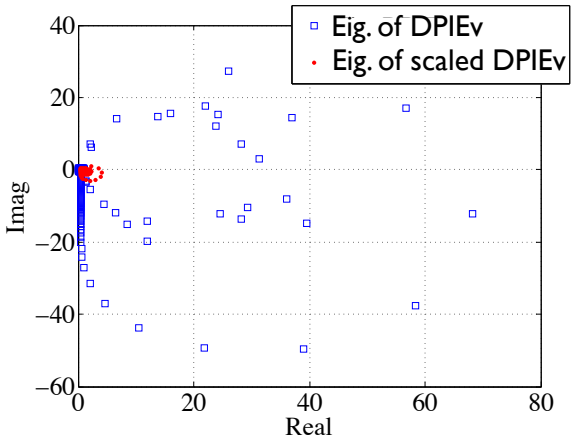

Figure 6: The spectrum of the original DPIEv and the scaled DPIEv (102) for a spherical scatterer of radius 1 at $k = 10$.

While the analysis above has been carried out only for a spherical scatterer, it is reasonable to expect that the scaled DPIEv is likely to have a big impact even for surfaces of arbitrary shape. As a final point of comparison, Fig. 7 plots the condition number of various integral equations that have been suggested for the solution of the Maxwell equations over a wide range of frequencies. Note that the scaled DPIE produces slightly worse condition numbers for $1 \ll k$. We suspect that improvements in the DPIE representation may lead to better performance, but leave that for future research. Note, however, that the scaled DPIE is surprisingly effective in the high-frequency regime, despite the fact that it was conceived in order to overcome topological low-frequency breakdown for scatterers of non-zero genus.

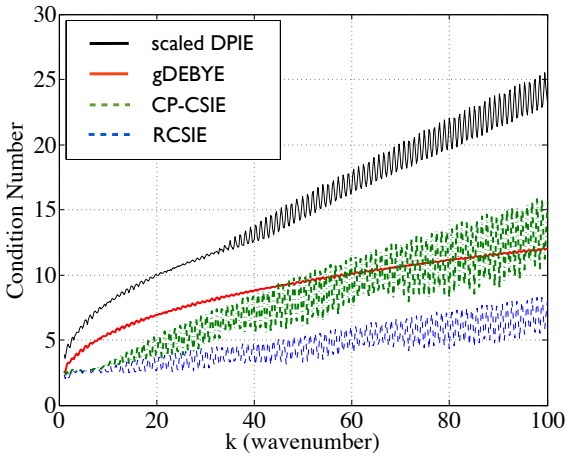

Figure 7: Comparison of the condition numbers of several resonance-free integral equations: the scaled DPIE, the generalized Debye source equation (gDEBYE) [25], the Calderon preconditioning combined source integral equation (CP-CSIE) [33], and the regularized combined source integral equation (RCSIE) [29].

8 Conclusions

We have presented a new formulation of the problem of electromagnetic scattering from perfect electric conductors. Rather than imposing boundary conditions on the field quantities themselves, we have derived well-posed boundary value problems for the vector and scalar potentials themselves, in the Lorenz gauge. This requires that we describe incoming fields in the same gauge, but that poses no fundamental obstacle. We have explained, in section 5, how to do this for partial wave expansions, plane waves, and (of course) the potentials induced by known impressed currents and charges. We have also developed integral representations for the vector and scalar potentials that lead to well-conditioned integral equations (the decoupled potential integral equations or DPIE). Most importantly, we have shown that the DPIE is insensitive to the genus of the scatterer. This is one of the few schemes of which we are aware that does not suffer from *topological* low-frequency breakdown without substantial complications (including the computation of special basis functions that span the space of surface harmonic vector fields [25]).

Careful analysis of scattering from a unit sphere has demonstrated that the method works well across a range of frequencies, but the DPIE is likely to be of particular utility in the low frequency regime — especially for structures with complicated multiply-connected geometry. We will report detailed numerical experiments at a later date.

Acknowledgements

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A Proofs of existence and stability theorems

Proof of Theorem 7:
Consider a solution $\tilde{\sigma}, \{\tilde{V}_j\}_{j=1}^N$ of the homogeneous equation (71):

$$\begin{aligned} &\frac{\tilde{\sigma}}{2} + D_k\tilde{\sigma} - i\eta S_k\tilde{\sigma} - \sum_{j=1}^{N} \tilde{V}_j\chi_j = 0, \\ &\int_{\partial D_j} (D_k'\tilde{\sigma} + i\eta\frac{\tilde{\sigma}}{2} - i\eta S_k'\tilde{\sigma})ds = 0\,. \end{aligned} \tag{104}$$

For this solution, the scalar function and constants

$$\phi^{\text{scat}}(\boldsymbol{x}) = D_k[\tilde{\sigma}](\boldsymbol{x}) - i\eta S_k[\tilde{\sigma}](\boldsymbol{x}), \quad \{\tilde{V}_j\}_{j=1}^N \tag{105}$$

satisfy the scalar modified Dirichlet problem with right-hand side $f = 0, \{Q_j = 0\}_{j=1}^N$. By Theorem 3, we have $\phi^{\text{scat}} = 0, \{\tilde{V}_j = 0\}_{j=1}^N$. As ϕ^{scat} is represented by a combination of single and double layers, it is known that $\tilde{\sigma} = 0$. This proves uniqueness (see [3, 22]). Note that the operators D_k and S_k defined on $C^{0,\alpha}(\partial D)$ are compact (see [22]). The rest of the operators in (71) are finite rank, so that (100) is a second kind equation when acting on the space $C^{0,\alpha}(\partial D) \times \mathbb{C}^N$, where $\mathbb{C}^N$ is equipped with the usual finite-dimensional topology. Note also that, as a function of k, the operators involved are continuous in the range $k \in [0, k_{\max}]$ for any fixed $k_{\max}$. This implies that the operators involved in equation (71) are not only compact, but collectively compact as well (see [34, 35]). By the Fredholm theorem, for any right hand side $f, \{\tilde{Q}_j\}_{j=1}^N \in C^{0,\alpha}(\partial D) \times \mathbb{C}^N$, there exists a solution $\sigma, \{\tilde{V}_j\}_{j=1}^N \in C^{0,\alpha}(\partial D) \times \mathbb{C}^N$.

Proof of Theorem 8:

Consider a solution $\tilde{\mathbf{a}}, \tilde{\varrho}, \{\tilde{v}_j\}_{j=1}^N$ of the homogeneous equation (106):

$$\begin{aligned} \frac{1}{2}\begin{pmatrix} \tilde{\mathbf{a}} \\ \tilde{\varrho} \end{pmatrix} + \overline{\overline{L}}\begin{pmatrix} \tilde{\mathbf{a}} \\ \tilde{\varrho} \end{pmatrix} + i\eta\overline{\overline{R}}\begin{pmatrix} \tilde{\mathbf{a}} \\ \tilde{\varrho} \end{pmatrix} - \begin{pmatrix} 0 \\ \sum_{j=1}^N v_j\chi_j \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}, \\ \int_{\partial D_j}\Big(\boldsymbol{n}\cdot\nabla\times S_k\tilde{\mathbf{a}} - \boldsymbol{n}\cdot S_k(\boldsymbol{n}\tilde{\varrho}) + i\eta\big(\boldsymbol{n}\cdot S_k(\mathbf{n}\times\tilde{\mathbf{a}}) - \frac{\tilde{\varrho}}{2} + S_k'V_0\tilde{\varrho}\big)\Big)ds = 0\,. \end{aligned} \tag{106}$$

For this solution, the vector field and constants

$$\boldsymbol{A}^{\mathrm{scat}} = \nabla\times S_k[\tilde{\mathbf{a}}](\boldsymbol{x}) - S_k[\boldsymbol{n}\tilde{\varrho}](\boldsymbol{x}) + i\eta\big(S_k[\mathbf{n}\times\tilde{\mathbf{a}}](\boldsymbol{x}) + \nabla S_k[\tilde{\varrho}](\boldsymbol{x})\big),\quad \{\tilde{v}_j\}_{j=1}^N \tag{107}$$

satisfy the vector modified Dirichlet problem, with right hand side $\mathbf{f} = 0, h = 0, \{q_j = 0\}_{j=1}^N$. By Theorem 4, we have $\boldsymbol{A}^{\mathrm{scat}} = 0, \{\tilde{V}_j = 0\}_{j=1}^N$. It is known that a zero field $\boldsymbol{A}^{\mathrm{scat}} = 0$ with this representation (107) must have trivial sources (see [22]). Thus, $\tilde{\mathbf{a}} = 0$ and $\tilde{\varrho} = 0$, which proves uniqueness (see [3]).

Note that the operator $\overline{\overline{L}}$ defined on $T^{0,\alpha}(\partial D)\times C^{0,\alpha}(\partial D)$ is compact (see Table [22]). The operator $\overline{\overline{R}}$ is continuous and bounded on $T^{0,\alpha}(\partial D)\times C^{0,\alpha}(\partial D)$, but not compact. However, for a choice of the constant $|\eta| < \|\overline{\overline{R}}\|^{-1}$, $I + i\eta\overline{\overline{R}}$ has a bounded inverse given by its Neumann series. The rest of the operators in equation (106) are finite rank, so that equation (106) is second kind when acting on the space $T^{0,\alpha}(\partial D)\times C^{0,\alpha}(\partial D)\times\mathbb{C}^N$, where $\mathbb{C}^N$ is equipped with the usual finite-dimensional topology. Note also that, as a function of k, the operators involved are continuous in the range $k\in[0,k_{\max}]$ for any fixed $k_{\max}$. This implies that the operators involved in equation 106 are not only compact, but collectively compact as well (see [34]). By Fredholm theory, for any right hand side $\mathbf{f}, h, \{\tilde{q}_j\}_{j=1}^N \in T^{0,\alpha}(\partial D)\times C^{0,\alpha}(\partial D)\times\mathbb{C}^N$, there exists a solution $\mathbf{a}, \varrho, \{\tilde{v}_j\}_{j=1}^N \in T^{0,\alpha}(\partial D)\times C^{0,\alpha}(\partial D)\times\mathbb{C}^N$.

Proof of Theorem 9:
The solution of the integral equation (71) depends continuously on the right hand side $f, \{\tilde{Q}_j\}_{j=1}^N$, with the corresponding Hölder topology. Due to the collective compactness of the operators involved in (71), the continuity is uniform in $k\in[0,k_{\max}]$. The map $\phi^{\mathrm{scat}}(\sigma,\{V_j\}_{j=1}^N)$ is continuous on $C^{0,\alpha}(\partial D)\times\mathbb{C}^N \to C^{0,\alpha}(\mathbf{R}^3\backslash D)\times\mathbb{C}^N$ (see [22]). By composition, the map $\phi^{\mathrm{scat}}(f,\{Q_j\}_{j=1}^N)$ is continuous as a map from $C^{0,\alpha}(\partial D)\times\mathbb{C}^N \to C^{0,\alpha}(\mathbf{R}^3\backslash D)\times\mathbb{C}^N$, uniformly on $k\in[0,k_{\max}]$. That is,

$$\|\phi^{\mathrm{scat}}\|_{0,\alpha,\mathbf{R}^3\backslash D} \le K_{(\alpha,\partial D,k_{max})}\Big(\|f\|_{0,\alpha,\partial D} + \sum_{j=1}^N |Q_j|^2\Big)\,, \tag{108}$$

where the constant $K_{(\alpha,\partial D,k_{max})}$ depends on α, the surface ∂D, and the maximum frequency k_{max}. The result is valid uniformly down to zero frequency $k=0$.

Proof of Theorem 10:
The solution of (77) depends continuously on the right hand side with the corresponding Hölder topology. Due to the collective compactness of the operators involved in (77), the continuity is uniform in $k \in [0, k_{\max}]$. The map $\boldsymbol{A}^{\text{scat}}(\mathbf{a}, \varrho, \{v_j\}_{j=1}^N) : T^{0,\alpha}(\partial D) \times C^{0,\alpha}(\partial D) \times \mathbb{C}^N \to T^{0,\alpha}(\mathbf{R}^3 \backslash D) \times C^{0,\alpha}(\mathbf{R}^3 \backslash D) \times \mathbb{C}^N$ is continuous (see [22]). By composition, the map $\boldsymbol{A}^{\text{scat}}(\mathbf{f}, h, \{q_j\}_{j=1}^N) : T^{0,\alpha}(\partial D) \times C^{0,\alpha}(\partial D) \times \mathbb{C}^N \to T^{0,\alpha}(\mathbf{R}^3 \backslash D) \times C^{0,\alpha}(\mathbf{R}^3 \backslash D) \times \mathbb{C}^N$ is continuous, uniformly in k for $k \in [0, k_{\max}]$. That is,

$$\|\boldsymbol{A}^{\text{scat}}\|_{0,\alpha,\mathbf{R}^3 \backslash D} \leq K_{(\alpha,\partial D,k_{max})} \Big(\|\mathbf{f}\|_{0,\alpha,\partial D} + \|h\|_{0,\alpha,\partial D} + \sum_{j=1}^{N} |q_j|^2 \Big), \tag{109}$$

where the constant $K_{(\alpha,\partial D,k_{max})}$ depends on α, the surface ∂D and the maximum frequency k_{max}. The result is valid uniformly down to zero frequency $k = 0$.

B Partial wave expansions

An important representation of the electromagnetic field is that based on separation of variables in spherical coordinates. As shown independently by Lorenz, Debye and Mie [2, 25], the fields induced by sources in the interior of a sphere can always be expressed in the exterior of the sphere according to the representation:

$$\begin{aligned} \boldsymbol{E}^{far} &= \sum_{m,n} \big[a_{mn} \nabla \times \nabla \times (\boldsymbol{x} h_n(k|\boldsymbol{x}|) Y_n^m) + i\omega\mu b_{mn} \nabla \times (\boldsymbol{x} f_n(k|\boldsymbol{x}|) Y_n^m) \big], \\ \boldsymbol{H}^{far} &= \sum_{m,n} \big[b_{mn} \nabla \times \nabla \times (\boldsymbol{x} h_n(k|\boldsymbol{x}|) Y_n^m) - i\omega\epsilon a_{mn} \nabla \times (\boldsymbol{x} f_n(k|\boldsymbol{x}|) Y_n^m) \big], \end{aligned} \tag{110}$$

where h_n is the spherical Hankel function of the first kind. For sources in the exterior of the sphere, we have

$$\begin{aligned} \boldsymbol{E}^{loc} &= \sum_{m,n} \big[a_{mn} \nabla \times \nabla \times (\boldsymbol{x} j_n(k|\boldsymbol{x}|) Y_n^m) + i\omega\mu b_{mn} \nabla \times (\boldsymbol{x} f_n(k|\boldsymbol{x}|) Y_n^m) \big], \\ \boldsymbol{H}^{loc} &= \sum_{m,n} \big[b_{mn} \nabla \times \nabla \times (\boldsymbol{x} j_n(k|\boldsymbol{x}|) Y_n^m) - i\omega\epsilon a_{mn} \nabla \times (\boldsymbol{x} f_n(k|\boldsymbol{x}|) Y_n^m) \big], \end{aligned} \tag{111}$$

where j_n is the spherical Bessel function [36]. In order to obtain a finite static limit, we renormalize and define the modified spherical Hankel/Bessel function by

$$\widetilde{f}_n(k,r) := \begin{cases} \widetilde{h}_n(k,r) = h_n(kr) \frac{k^{n+1}}{-i(2n-1)(2n-3)\ldots 5\cdot 3\cdot 1}, \\ \widetilde{j}_n(k,r) = j_n(kr) \frac{(2n+1)(2n-1)\ldots 5\cdot 3\cdot 1}{k^{n+1}}. \end{cases} \tag{112}$$

It is easy to check that

$$\lim_{k\to 0} \widetilde{f}_n(k,r) = \begin{cases} \lim_{k\to 0} \widetilde{h}_n(k,r) = \frac{1}{r^{n+1}}, \\ \lim_{k\to 0} \widetilde{j}_n(k,r) = r^n. \end{cases} \tag{113}$$

With a slight abuse of notation we will refer to both $\boldsymbol{E}^{far}$ and $\boldsymbol{E}^{loc}$ as $\boldsymbol{E}^{\text{inc}}$, and to both $\boldsymbol{H}^{far}$ and $\boldsymbol{H}^{loc}$ as $\boldsymbol{H}^{\text{inc}}$. When the distinction is important, we will specify the use of $\widetilde{h}_n(k,r)$ or $\widetilde{j}_n(k,r)$ as the radial function of interest.

Normalizing the coefficients a_{mn}, b_{mn} by the inverse of the scaling factor in (112), we write:

$$\begin{aligned}\boldsymbol{E}^{\text{inc}} &= \sum_{m,n}\left[\widetilde{a}_{mn}\nabla\times\nabla\times(\boldsymbol{x}\widetilde{f}_n(k,|\boldsymbol{x}|)Y_n^m) + i\omega\mu\widetilde{b}_{mn}\nabla\times(\boldsymbol{x}\widetilde{f}_n(k,|\boldsymbol{x}|)Y_n^m)\right],\\ \boldsymbol{H}^{\text{inc}} &= \sum_{m,n}\left[\widetilde{b}_{mn}\nabla\times\nabla\times(\boldsymbol{x}\widetilde{f}_n(k,|\boldsymbol{x}|)Y_n^m) - i\omega\epsilon\widetilde{a}_{mn}\nabla\times(\boldsymbol{x}\widetilde{f}_n(k,|\boldsymbol{x}|)Y_n^m)\right].\end{aligned} \tag{114}$$

The fields of a magnetic multipole of degree n and order m are defined to be

$$\begin{aligned}\dot{\boldsymbol{E}}^{\text{inc}}_{nm} &= i\omega\mu\nabla\times(\boldsymbol{x}\widetilde{f}_n(k,|\boldsymbol{x}|)Y_n^m),\\ \dot{\boldsymbol{H}}^{\text{inc}}_{nm} &= \nabla\times\nabla\times(\boldsymbol{x}\widetilde{f}_n(k,|\boldsymbol{x}|)Y_n^m).\end{aligned} \tag{115}$$

The corresponding vector and scalar potentials can be defined by

$$\begin{aligned}\dot{\boldsymbol{A}}^{\text{inc}}_{nm} &= \mu\nabla\times(\boldsymbol{x}\widetilde{f}_n(k,|\boldsymbol{x}|)Y_n^m),\\ \dot{\phi}^{\text{inc}}_{nm} &= 0.\end{aligned} \tag{116}$$

They clearly satisfy

$$\begin{aligned}\Delta\dot{\phi}^{\text{inc}}_{nm} + k^2\dot{\phi}^{\text{inc}}_{nm} &= 0,\\ \Delta\dot{\boldsymbol{A}}^{\text{inc}}_{nm} + k^2\dot{\boldsymbol{A}}^{\text{inc}}_{nm} &= 0,\\ \nabla\cdot\dot{\boldsymbol{A}}^{\text{inc}}_{nm} &= i\omega\mu\epsilon\dot{\phi}^{\text{inc}}_{nm}.\end{aligned} \tag{117}$$

Moreover, $\dot{\boldsymbol{A}}^{\text{inc}}_{mn}$ and $\dot{\phi}^{\text{inc}}_{mn}$ are bounded. The fields of an electric multipole of degree n and order m are defined to be

$$\begin{aligned}\ddot{\boldsymbol{E}}^{\text{inc}}_{nm} &= \nabla\times\nabla\times(\boldsymbol{x}\widetilde{f}_n(k,|\boldsymbol{x}|)Y_n^m),\\ \ddot{\boldsymbol{H}}^{\text{inc}}_{nm} &= -i\omega\epsilon\nabla\times(\boldsymbol{x}\widetilde{f}_n(k,|\boldsymbol{x}|)Y_n^m).\end{aligned} \tag{118}$$

In this case, however, it is easy to verify that the function which serves as the obvious vector potential, namely $\boldsymbol{x}\widetilde{f}_n(k,|\boldsymbol{x}|)Y_n^m$, is *not* in the Lorenz gauge. To find a suitable replacement, we compute:

$$\nabla\times\nabla\times(\boldsymbol{x}\widetilde{f}_n(k,|\boldsymbol{x}|)Y_n^m) = k^2\boldsymbol{x}\widetilde{f}_n(k,|\boldsymbol{x}|)Y_n^m + \nabla\frac{\partial}{\partial r}(r\widetilde{f}_n(k,r)Y_n^m)_{r=|\boldsymbol{x}|}. \tag{119}$$

Note that

$$\frac{\partial}{\partial r}(r\widetilde{f}_n(k,r)Y_n^m) = \widetilde{f}_n(k,r)Y_n^m + r\frac{\partial}{\partial r}\widetilde{f}_n(k,r)Y_n^m. \tag{120}$$

The first term $\widetilde{f}_n(k,r)Y_n^m$ is a Helmholtz potential. Making use of the following identity for spherical Hankel and Bessel functions [36]

$$\begin{aligned} \frac{n+1}{z}h_n(z) + h_n'(z) &= h_{n-1}(z), \\ \frac{n}{z}j_n(z) - j_n'(z) &= j_{n+1}(z), \end{aligned} \tag{121}$$

we have

$$r\frac{\partial}{\partial r}\widetilde{h}_n(k,r) = \frac{rk^2}{2n-1}\widetilde{h}_{n-1}(k,r) - (n+1)\widetilde{h}_n(k,r) \tag{122}$$

and

$$r\frac{\partial}{\partial r}\widetilde{j}_n(k,r) = -\frac{rk^2}{2n+3}\widetilde{j}_{n+1}(k,r) + n\widetilde{j}_n(k,r)\,. \tag{123}$$

Multiplying by Y_n^m,

$$\begin{aligned} r\frac{\partial}{\partial r}\widetilde{h}_n(k,r)Y_n^m &= \frac{rk^2}{2n-1}\widetilde{h}_{n-1}(k,r)Y_n^m - (n+1)\widetilde{h}_n(k,r)Y_n^m, \\ r\frac{\partial}{\partial r}\widetilde{j}_n(k,r)Y_n^m &= -\frac{rk^2}{2n+3}\widetilde{j}_{n+1}(k,r)Y_n^m + n\widetilde{j}_n(k,r)Y_n^m. \end{aligned} \tag{124}$$

The first term on the right-hand side is of the order $O(k)$, while the second term is a Helmholtz potential and of magnitude $O(1)$. Using (124) and (120) in (119), we obtain

$$\begin{aligned} \nabla\times\nabla\times(\boldsymbol{x}\widetilde{h}_n(k,|\boldsymbol{x}|)Y_n^m) = k^2\boldsymbol{x}\widetilde{h}_n(k,|\boldsymbol{x}|)Y_n^m + \frac{k^2}{2n-1}\nabla\big(r\widetilde{h}_{n-1}(k,r)Y_n^m\big)_{r=|\boldsymbol{x}|} \\ - \nabla\big(n\widetilde{h}_n(k,r)Y_n^m\big)_{r=|\boldsymbol{x}|}\,, \end{aligned} \tag{125}$$

for the outgoing waves, and

$$\begin{aligned} \nabla\times\nabla\times(\boldsymbol{x}\widetilde{j}_n(k,|\boldsymbol{x}|)Y_n^m) = k^2\boldsymbol{x}\widetilde{j}_n(k,|\boldsymbol{x}|)Y_n^m - \frac{k^2}{2n+3}\nabla\big(r\widetilde{j}_{n+1}(k,r)Y_n^m\big)_{r=|\boldsymbol{x}|} \\ + \nabla\big((n+1)\widetilde{j}_n(k,r)Y_n^m\big)_{r=|\boldsymbol{x}|}\,, \end{aligned} \tag{126}$$

for the incoming waves. Note that the last term on the right-hand side of (126) and the left-hand side of (126) both satisfy the vector Helmholtz equation. Thus, the first two terms on the right-hand side of (126) must together satisfy the vector Helmholtz equation as well. Dividing those two terms by k^2 and multiplying by $-i\omega\mu\epsilon$, we define the corresponding vector and scalar potentials by

$$\begin{aligned} \ddot{\boldsymbol{A}}_{nm}^{\text{inc}} &= -i\omega\mu\epsilon\boldsymbol{x}\widetilde{h}_n(k,|\boldsymbol{x}|)Y_n^m + \frac{-i\omega\mu\epsilon}{2n-1}\nabla\big(r\widetilde{h}_{n-1}(k,r)Y_n^m\big)_{r=|\boldsymbol{x}|}, \\ \ddot{\phi}_{nm}^{\text{inc}} &= n\widetilde{h}_n(k,|\boldsymbol{x}|)Y_n^m\,, \end{aligned} \tag{127}$$

for the outgoing waves, and

$$\begin{aligned}\ddot{\boldsymbol{A}}^{\rm inc}_{nm} &= -i\omega\mu\epsilon\boldsymbol{x}\widetilde{j}_n(k,|\boldsymbol{x}|)Y^m_n + \frac{i\omega\mu\epsilon}{2n+3}\nabla\big(r\widetilde{j}_{n-1}(k,r)Y^m_n\big)_{r=|\boldsymbol{x}|},\\ \ddot{\phi}^{\rm inc}_{nm} &= -(n+1)\widetilde{j}_n(k,|\boldsymbol{x}|)Y^m_n\,,\end{aligned} \tag{128}$$

for the incoming waves. It is easy to verify that

$$\begin{aligned}\Delta\ddot{\phi}^{\rm inc}_{nm} + k^2\ddot{\phi}^{\rm inc}_{nm} &= 0,\\ \Delta\ddot{\boldsymbol{A}}^{\rm inc}_{nm} + k^2\ddot{\boldsymbol{A}}^{\rm inc}_{nm} &= 0,\\ \nabla\cdot\ddot{\boldsymbol{A}}^{\rm inc}_{nm} &= i\omega\mu\epsilon\ddot{\phi}^{\rm inc}_{nm}\,.\end{aligned} \tag{129}$$

The last equation, which enforces the Lorenz gauge, is obtained by taking the divergence of (126). Clearly, both potentials $\ddot{\boldsymbol{A}}^{\rm inc}_{mn}$ and $\ddot{\phi}^{\rm inc}_{mn}$ are of the order $O(1)$.

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