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ABSTRACT

The nonadiabatic nature of low-speed combustion and fire, in which strongly exothermic reactions produce large temperature variations but only mild pressure variations, can cause difficulty when integrating zone models of enclosure fires. Examples of simple zone fire models are examined to illustrate the analytical nature of the problems encountered. These difficulties arise in the solution of the equations for the pressure in general enclosures because the pressure equilibrates much more rapidly than other dynamical variables. Singular perturbation methods and phase plane analyses, together with numerical integration of the nondimensionalized equations, are employed to study the stiff nature of the equations. We conclude that many of the difficulties associated with numerical integration of zone fire models may be circumvented by appropriate analysis of the zone fire model equations.

1. INTRODUCTION

Mathematical models of low-speed combustion and fire are confronted with a common dilemma: the rate at which the pressure equilibrates is very much more rapid than the rates at which the other phenomena evolve. The nonadiabatic nature of these phenomena, in which strongly exothermic reactions produce large temperature variations but only mild pressure variations, is the cause of the dilemma. It arises in both analytical studies, where the approximation of constant-pressure combustion is often introduced [1], [2], and in numerical calculations, where slight compressibility can cause inaccuracy and inefficiency [3]. In fire research, the dilemma is perhaps more acute because most mathematical modeling has been of fires in enclosures where the exothermicity produces a pressure that is nearly uniform throughout the enclosure, but increases slowly with time due to the heat introduced by the fire. In the field-modeling approach, where the model consists of governing partial differential equations, this dilemma was resolved some time ago by Rehm and Baum [4], who introduced an approximate set of equations for a thermally expandable gas in which the sound waves were analytically removed; other researchers have introduced different approaches to resolve the dilemma with varying degrees of success. In the zone-model approach in fire research, this dilemma has not been entirely resolved to date [5], [6].

There is a long history of analysis of the dynamical behavior of fires in buildings using mathematical models. The reason for development of the mathematical models and their use in practice has been reviewed in [7], [8]. The original mathematical model of a plume used in zone fire models was developed by Morton, Taylor and Turner [9]. Other early work contributing to the basic development of these models includes experimental [10] and theoretical [11] studies of the effects of flow through openings induced by fires in enclosures; analytical examples of the development of a stratified ceiling layer and the filling of an enclosure by the heated gases [5]; analytical examples of two layer modeling of the smoke movement in two-room structures [12]; and theoretical study of the flow of smoke and hot gases through vents [13].

In an early paper on zone–fire modeling, Zukoski [5] demonstrated that, for a fire in an enclosure, a substantial pressure over ambient could not in general be sustained. His work formed the basis for the quasi–steady approximation often invoked in zone–fire modeling; in this approximation, pressure is assumed to be spatially uniform at each time during the slow evolution of the other dependent variables. In Section 2, we derive the equations for a simple two–layer zone–fire model and show that these equations exhibit stiffness. Stiffness is a term used to describe the equations arising from a physical system that is governed by multiple time scales. The equations reduce to those obtained earlier by Zukoski when the pressure is assumed to equilibrate instantaneously. When, however, the pressure equilibrates more slowly, the nature of the stiffness can be seen.

In Section 3, an example is presented which illustrates the structure of the mathematical problem that arises when solving pressure equations in a two–room enclosure. In the first room, there is a fire and two vents, one to the outside, and the second vent to the other room. The difficulty encountered in attempting to integrate numerically the equations arising from this more general model is again the stiffness due to the pressure equations. Under very reasonable conditions, the pressure differential across the vent between the rooms remains very small relative to the average pressure in the two rooms during evolution of the fire, and this small differential causes stiffness.

Since these models are very simple, analytical techniques can be applied and insight gained regarding the nature of the stiffness. As with more general zone models, these models consist of ordinary differential equations coupled with algebraic equations. Singular perturbation methods and phase plane analyses, together with numerical integration of the appropriately non-dimensionalized equations, are employed to examine the stiff nature of the equations. In Section 4 we conclude that many of the difficulties as-sociated with numerical integration of zone fire models in general may be circumvented by appropriate analysis of the model equations.

2. SINGLE–ROOM ZONE–FIRE MODEL

Over a decade ago, Zukoski [5] demonstrated that, for a fire in an enclosure, a substantial pressure over ambient could not in general be sustained. Any leak in the enclosure would rapidly reduce the overpressure to near zero, or, alternately, the overpressure would cause structural failure, such as the breaking of a window, and the overpressure would again become small. Zukoski's observation is again simply the statement that the time required for the pressure to equilibrate spatially is much smaller than the time required for other variables, such as temperature or density, to evolve during an enclosure fire. Practically, this demonstration has been used in zone–fire modeling to invoke a quasi–steady approximation in which pressure is assumed to be spatially uniform at each time in the slow evolution of the other dependent variables in the enclosure fire.

Problems involving multiple time scales such as that described above, where one time scale is much shorter than the others, are said to be stiff mathematically and can be very difficult to calculate. Paradoxically, solutions to stiff problems often appear to evolve slowly, yet have enormous computational requirements when solved using standard or non–stiff solvers. In stiff systems, the short time–scale phenomena approach a quasi–steady state rapidly while the other phenomena evolve on a much longer time scale. Zone models have been known for some time to exhibit this stiff nature due to the rapid pressure equilibration. The numerical problems produced during the numerical integration of zone–fire models have been highlighted by Forney and Moss [6].

In this section, we present a derivation of a set of differential equations which govern all of the dependent variables (room pressure, layer height, layer temperatures and densities, etc.) for

a simple two–layer zone–fire model. This model is obtained by examining the equations for conservation of mass and energy, integrated over the upper and the lower layers, the two control volumes for the zone model. These equations are supplemented by the equation of state for an ideal gas, and definitions of density, volume, mass, etc. in each layer. In Figure 1, a schematic
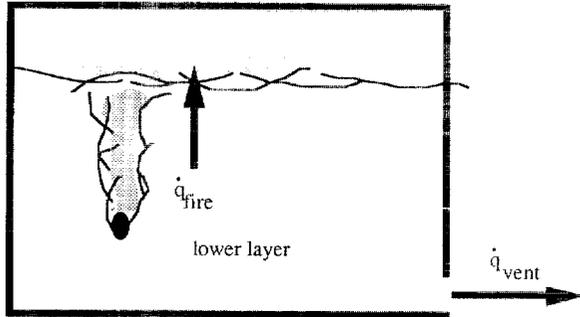

Figure 1. One Room Two Layer Schematic Diagram

diagram of the zone model is shown; the compartment is divided into two control volumes, an upper layer of hot gases and smoke, and a lower layer of air. The fire produces a plume which acts as a pump to transfer mass from the lower to the upper layer, adding energy to the transferred fluid. The basic assumption of a zone fire model is that properties such as temperatures can be approximated as uniform throughout the zone. The two layer model is quite adequate for many applications because upper and lower layers as described are often observed experimentally in room fires. A careful derivation of the equations used in zone fire models has been given in [8], while the advantages and disadvantages of zone models have been discussed in [7]. It is remarkable that this assumption seems to hold for as few as two gas layers, which is the model considered in this paper.

The model demonstrates that the equations exhibit stiffness even in this simple case. The time scale for pressure equilibration is determined by the size of the enclosure divided by the speed of sound, whereas the time for evolution of other variables, temperature, density, layer height, etc., is determined by the size of the enclosure and the buoyant velocity induced in the enclosure. The stiffness arises because of the small Mach number of the flow induced during buoyant convection, or because of the thermal expandability of the gas in a fire. The equations are shown to reduce to those obtained earlier by Zukoski when the pressure is assumed to equilibrate instantaneously.

The gas in each layer has attributes of mass, internal energy, density, temperature, and volume denoted respectively by $\tilde{m}_i$, $\tilde{E}_i$, $\tilde{\rho}_i$, $\tilde{T}_i$, and $\tilde{V}_i$ where $i = L$ for the lower layer and $i = U$ for the upper layer. The compartment as a whole has the attribute of pressure $\tilde{p}$. Here, tildes have been used to denote dimensional quantities. Many differential equation formulations based upon these assumptions can be derived. One formulation can be converted into another using definitions of density, internal energy and the ideal gas law. Discussions of different formulations of a zone fire model are given in [6] and [14]. These eleven variables are related by means of the following seven constraints

$$\tilde{\rho}_i = \frac{\tilde{m}_i}{\tilde{V}_i} \ (\text{density}), i = L, U \tag{1}$$

$$\tilde{E}_i = c_v \tilde{m}_i \tilde{T}_i (\text{internal energy}), i = L, U \tag{2}$$

$$\tilde{p} = R \tilde{\rho}_i \tilde{T}_i (\text{ideal gas law}), i = L, U \tag{3}$$

$$\tilde{V} = \tilde{V}_L + \tilde{V}_U (\text{total volume}). \tag{4}$$

The specific heats at constant volume and at constant pressure, c_v and c_p, the universal gas constant, R, and the ratio of specific heats, γ, are related by $\gamma = c_p / c_u$, $R = c_p - c_u$.

The first law of thermodynamics states that the rate of increase of layer internal energy plus the rate at which the layer does work by expansion is equal to the rate at which enthalpy is added to the gas (where we consider the enthalpy added as that from any sources minus losses to the walls). In differential equation form, the energy equations for the two layers are

$$\frac{dE_U}{dt} + \bar{p}\frac{d\bar{V}_U}{dt} = \dot{q}_{fire} + \dot{q}_{plume} - \dot{q}_{walls} \qquad (5)$$

$$\frac{d\bar{E}_L}{dt} + \bar{p}\frac{d\bar{V}_L}{dt} = - \dot{q}_{vent} - \dot{q}_{plume} \qquad (6)$$

Similarly, the mass equations are

$$\frac{dm_U}{dt} = \dot{m}_{fire} + \dot{m}_{plume} \qquad (7)$$

$$\frac{d\bar{m}_L}{dt} = - \dot{m}_{vent} - \dot{m}_{plume} \qquad (8)$$

The mass and enthalpy flow rates are denoted $\dot{m}$, and $\dot{q}$ with appropriate subscripts. These flow rates represent the net exchange of mass or energy between zones due to physical phenomena or sub-models such as fire plumes, natural and forced vents, convective, radiative heat transfer, etc. For example, a vent exchanges mass and energy between zones in connected rooms, a fire plume typically adds heat to the upper layer and transfers entrained mass and energy from the lower to the upper layer, and convection and conduction transfer energy from the gas layers to the surrounding walls. Here, we use the following definitions and submodels:

$$\bar{V}_0 = \bar{V}_L + \bar{V}_U = \text{constant} \qquad (9)$$

$$\dot{q}_{vent} = C_p \bar{T}_L(\bar{t}) \dot{m}_{vent} \qquad (10)$$

$$\dot{m}_{vent} = C_{vent} A_{vent} \sqrt{2\bar{\rho}_L(\bar{p} - \bar{p}_0)} \qquad (11)$$

$$\dot{q}_{plume} = C_p \bar{T}_L(\bar{t}) \dot{m}_{plume} \qquad (12)$$

$$\dot{m}_{plume} = \rho_0 \sqrt{gH} \, H^2 (Q^*)^{1/3} \tilde{\alpha}(\bar{z}/H) \qquad (13)$$

$$\dot{q}_{walls} = [S + 2(L + W)(H - \bar{z})] \qquad (14)$$
$$K(\bar{T}_U - \bar{T}_0)$$

For simplicity, $\dot{q}_{fire}$ is assumed to be a constant fire (heat) source, $\dot{q}_{vent}$ is the enthalpy loss through the vent, $\dot{q}_{plume}$ is the enthalpy pumped from the lower layer into the upper layer by the plume, and $\dot{q}_{walls}$ is the heat transfer rate to the walls. $\dot{m}_{fire}$ is the mass added by the fire, $\dot{m}_{vent}$ is the mass loss through the vent and $\dot{m}_{plume}$ is the mass pumped by the fire plume from the lower

layer into the upper layer. Q^* is the dimensionless fire input parameter defined by Zukoski [5]

$$Q^* = \frac{\dot{q}_{fire}}{\rho_0 C_p T_0 \sqrt{gH} \, H^2} \qquad (15)$$

and ρ_0, T_0 are a reference density and temperature, g is the acceleration of gravity and H is the height of the enclosure. $S = LW$ is the floor area of the enclosure, where L is the length of the enclosure and W is its width. K is a heat transfer coefficient for heat transfer from the gas to the enclosure walls. Then the volume of the enclosure is $\bar{V}_0 = LWH$.

Three differential equations plus the relations between the dependent variables can be used to define the zone model; these equations are chosen to be an equation for the pressure, one for the upper-layer temperature and the lower-layer mass equation. The equation for the pressure is found by adding the energy equations for the two layers, taking account of the equation of state for the ideal gas.

$$\frac{d\bar{p}}{dt} = \frac{\gamma - 1}{V_0}(\dot{q}_{fire} - \dot{q}_{vent} - \dot{q}_{walls})$$

Here $\bar{p}$ is the pressure in the room relative to the ambient pressure outside the room $\bar{p} = \bar{p}_{enc} - \bar{p}_{atm}$, and $\bar{t}$ is the time. The upper-layer energy equation and the upper-layer mass equation combine to give an equation for the upper-layer temperature.

$$C_p \bar{\rho}_U \bar{V}_U \frac{d}{dt}(\bar{T}_U) = \bar{V}_U \frac{d\bar{p}}{dt} + \dot{q}_{fire} - \dot{q}_{walls}$$
$$- C_p(\bar{T}_U - \bar{T}_L)\dot{m}_{plume} - C_p \bar{T}_U \dot{m}_{fire}$$

The equation for the mass in the lower layer is simply:

$$\frac{d\bar{m}_L}{dt} = \frac{d\bar{\rho}_L \bar{V}_L}{dt} = - \dot{m}_{plume} - \dot{m}_{vent}$$

We use the same scales as used by Zukoski [5] to define most dimensionless quantities. Heights are made dimensionless with respect to the enclosure height, H, volumes with respect to the enclosure volume V_0 and densities and tem-

peratures relative to the reference quantities defined above. The time is made dimensionless with respect to the time scale defined by Zukoski:

$$t_Z \equiv \sqrt{H/g} \; (S/H^2)$$

There is, however, another time scale characteristic in the problem, and it is the time associated with a pressure rise in the room due to heating, and this time gives rise to the stiffness in the problem:

$$\tau = \frac{V_0}{(\gamma - 1)} \; \frac{\dot{q}_{fire}}{(C_p \, T_L \, c_{vent} \, A_{vent} \sqrt{2\rho_L})^2} \quad (16)$$

Similarly, there are two characteristic pressures which can be defined. One obviously is defined in terms of the reference temperature and density, $p_0 = R\bar{\rho}_0 T_0$, and this pressure is chosen as the reference pressure. The second characteristic pressure is the overpressure which arises due to heat addition in the enclosure:

$$p_\infty = \left(\frac{\dot{q}_{fire}}{C_p T_L c_{vent} A_{vent} \sqrt{2\rho_L}}\right)^2 \quad (17)$$

If we define the total pressure in the enclosure as the reference pressure plus the overpressure $\bar{p} = \bar{p}_0 + \Delta\bar{p}$, then the dimensionless pressure is

$$p \equiv \bar{p}/\bar{p}_0 = 1 + \Delta\bar{p}/\bar{p}_0$$
$$= 1 + (\bar{p}_\infty/\bar{p}_0)\Delta p = 1 + \varepsilon\Delta p$$

where $\varepsilon \equiv (\bar{p}_\infty/\bar{p}_0)$.

The mass and heat sources and sinks can be made dimensionless and written in terms of a few dimensionless parameters. These are

$$M = \frac{\dot{m}_{fire}}{\rho_0 \sqrt{gH} \, H^2} \quad (18)$$

$$Q^* = \frac{\dot{q}_{fire}}{\rho_0 C_p T_0 \sqrt{gH} \, H^2} \quad (19)$$

$$Q_V = \frac{c_{vent} A_{vent} \sqrt{2\rho_0 p_0}}{\bar{\rho}_0 \sqrt{gH} \, H^2} \quad (20)$$

$$Q_W = \frac{SK}{C_p \bar{\rho}_0 \sqrt{gH} \, H^2} \quad (21)$$

Here M is the dimensionless rate of mass addition by the fire, Q^* is the dimensionless rate of heat addition by the fire (Zukoski's parameter defined earlier), Q_V is the dimensionless enthalpy lost through the vent, and Q_W is the dimensionless enthalpy lost into the walls.

The equations for the pressure, the temperature in the upper layer and the mass in the lower layer in dimensionless form then become:

$$\varepsilon\frac{d\Delta p}{dt} = \gamma\bigg(Q^* - Q_V\sqrt{\varepsilon}\, T_L \sqrt{\rho_L\Delta p}$$
$$- Q_W[1 + 2\frac{(L + W)H}{S}(1 - z)](T_U - 1)\bigg)$$

$$\rho_U V_U\frac{dT_U}{dt} = \varepsilon\frac{\gamma - 1}{\gamma}V_U\frac{d\Delta p}{dt} + Q^* - Q_W$$
$$[1 + 2\frac{(L + W)H}{S}(1 - z)](T_U - 1)$$
$$- (Q^*)^{1/3}\tilde{\alpha}Z^{5/3}(T_U - T_L) - MT_U$$

$$\frac{d\rho_L V_L}{dt} = - (Q^*)^{1/3}\tilde{\alpha}Z^{5/3} - Q_V\sqrt{\varepsilon}\sqrt{\rho_L\Delta p}$$

where all quantities are now regarded as dimensionless. For more details concerning the non-dimensionization and the derivation of the equations, see [14] and [5].

Zukoski [5] has presented numerical estimates of the magnitudes of the various parameters which appear in these equations. He has demonstrated that leaks generally are large enough in most enclosure fire scenarios that the overpressure which can develop is rather small. (In fact, Zukoski uses this fact to ignore any overpressure and make a quasi-steady approximation for the pressure, assuming that it equilibrates instantaneously to the pressure outside the enclosure, during the enclosure-filling process. When the leak is large enough to sustain only small values of overpressure, then $\varepsilon \equiv \bar{p}_\infty/\bar{p}_0 \ll 1$ is a small parameter. Using equation (17) with $\dot{q}_{fire} = 100,000W$, $A_{vent} = 1m^2$, $c_{vent} = .68$ and ambient density and temperature, $\varepsilon \approx 10^{-6}$. Since this small parameter multiplies the time derivative of the overpressure, the system of equations is stiff, and the culprit is the overpressure equation.

Other observations can be made from this dimensionless system of equations. However, we will note only one. The state equation is $p = 1 + p_\infty \Delta p / \bar{p}_0 = \rho_L T_L = \rho_U T_U$. When $\varepsilon = \bar{p}_\infty / \bar{p}_0 \ll 1$, then $p \approx 1$ and $\rho_L T_L \approx 1, \rho_U T_U \approx 1$ (with $\rho_L \approx 1$ and $T_L \approx 1$)

2.1 Pressure Equation

A major approximation made in the model described above is that the heated gases never exit through the vent (i.e. the vent is near the floor, while the heated gas layer stays above the leak). In addition, we assume that there is no heat transfer to the walls, i.e. the enclosure has adiabatic boundaries. The mass loss through the vent equals a constant times the squareroot of the pressure difference across the vent, and the enthalpy loss through the vent is a constant times the mass loss. The pressure equation in dimensional form is'

$$\frac{d\bar{p}}{d\bar{t}} = \frac{\bar{r} - 1}{V}(\dot{q}_{fire} - \dot{q}_{vent})$$

The initial conditions are that $\bar{p} = 0$ at $\bar{t} = 0$. The solution to this equation forms the basis for analysis of more complicated cases.

If, rather than using the reference time scale t_Z and pressure $\bar{p}_0$ defined in the formulation above and used by Zukoski, we use the alternative ones $\bar{r}$ and $\bar{p}_\infty$ associated with the pressure rise in the enclosure, the equation for the pressure becomes

$$\frac{dp}{dt} = 1 - \text{sgn}(p)\sqrt{|p|} \qquad (22)$$

The solution to this equation with the initial conditions that $p = 0$ at $t = 0$ was given earlier in [6], [5] and [4].

Figure 2 shows solution plots to Eq. (22) for three values of the initial conditions, $p(0) = 2.0$, 0 and -2.0. These plots were generated using the software package Mathematica[15], particularly the command NDSolve. The initial condition that $p = 0$ for $t = 0$ is the base calculation performed in [4], [5] and in [6]. For all initial
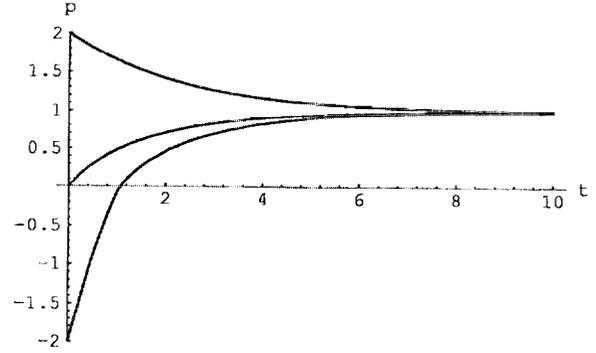

Figure 2. Solution of a Non – Dimensionalized Pressure Differential Equation for Initial Conditinons: p(0) $= -2$, p(0) $= 0$ and p(0) $= 2$

conditions shown, the solutions converge at long time to $p(t \to \infty) = 1$, the stable equilibrium solution to the problem.

The phase plane for the one–room model is just the line, but it is instructive to consider it. As defined, the dimensionless pressure can be either positive or negative. Let the initial conditions be $\bar{p}_{enc}(\bar{t}_0) = \bar{p}_{enc0}$ at time $\bar{t} = \bar{t}_0 \equiv 0$, where we retain the symbol $\bar{t}_0$ for use later. Then $\bar{p}_0 = \bar{p}_{enc} - \bar{p}_{atm}$ and $p_0 \equiv \bar{p}_0 / \bar{p}_\infty = (\bar{p}_{enc} - \bar{p}_{atm}) / \bar{p}_\infty$. Now, since we can have $\bar{p}_0 < 0$ as well as $\bar{p}_0 > 0$, and since $\bar{p}_\infty$ can be arbitrarily small, $-\infty < p_0 < \infty$. Hence the phase space for the one–room model is the whole line. Whether p_0 is positive or negative, the solutions pass, after long time, to the stable equilibrium value of $p = 1$, which is called the fixed point of the equation.

For $0 < p_0 < \infty$, the solution is

$$t - t_0 = 2\ln\frac{|1 - \sqrt{p_0}|}{|1 - \sqrt{p}|} + 2(\sqrt{p_0} - \sqrt{p})$$

and the phase diagram for either $p_0 > 1$ or for $p_0 < 1$ is the directed line segment from p_0 to unity.

When p_0 is negative initially, the solution for the pressure is

$$t = 2\ln\frac{1 + \sqrt{|p|}}{1 + \sqrt{|p_0|}} + 2(\sqrt{|p_0|} - \sqrt{|p|})$$

which rises to zero at t = t_c,

$$t_c = 2\ln\frac{1}{1 + \sqrt{|p_0|}} + 2\sqrt{|p_0|}$$

For t > t_c, p > 0, the solution is

$$t - t_c = 2\ln\frac{1}{|1 - \sqrt{p}|} - 2\sqrt{p}$$

In this case, the phase diagram is a directed line segment from p_0 < 0 to unity.

3. PRESSURE EQUATION TWO – ROOM MODEL

The second example, which also illustrates the structure of the mathematical problem in solving pressure equations, is shown schematically in Figure 3. In this example, there are two rooms. In the first room, denoted by subscript 1, there is a fire and two vents, one to the outside, which is denoted vent 1, and the second vent, denoted vent 2, to the second room. Again, the walls are assumed to be adiabatic. The dimensional equations for this example are:

$$\frac{d\tilde{p}_1}{dt} = \frac{\gamma - 1}{V_1}(\dot{q}_{fire} - \dot{q}_{vent1} - \dot{q}_{vent2})$$

$$\frac{d\tilde{p}_2}{dt} = \frac{\gamma - 1}{V_2}(\dot{q}_{vent2}) \qquad (23)$$

where

$$\dot{q}_{vent1} = C_p T_{vent1} C_{vent1} A_{vent1} \ sgn(\tilde{p}_1)\sqrt{2\rho|\tilde{p}_1|}$$

$$\dot{q}_{vent2} = C_p T_{vent2} C_{vent2} A_{vent2}$$
$$sgn(\tilde{p}_1 - \tilde{p}_2)\sqrt{2\rho|\tilde{p}_1 - \tilde{p}_2|}$$

We define the following scaling parameters:

$$p_{\infty 1} = \left(\frac{\dot{q}_{fire}}{C_p T_{vent1} C_{vent1} A_{vent1}\sqrt{2\rho}}\right)^2$$

$$\tilde{p}_{\infty 2} = \left(\frac{\dot{q}_{fire}}{C_p T_{vent2} C_{vent2} A_{vent2}\sqrt{2\rho}}\right)^2$$

$$\tau_1 = \frac{V_1 \tilde{p}_{\infty 1}}{(\gamma - 1)\dot{q}_{fire}}$$

$$= \frac{V_1}{(\gamma - 1)}\frac{\dot{q}_{fire}}{(C_p T_{vent1} C_{vent1} A_{vent1}\sqrt{2\rho})^2}$$

$$\tau_2 = \frac{V_2 \tilde{p}_{\infty 2}}{(\gamma - 1)\dot{q}_{fire}}$$

$$= \frac{V_2}{(\gamma - 1)}\frac{\dot{q}_{fire}}{(C_p T_{vent2} C_{vent2} A_{vent2}\sqrt{2\rho})^2}$$
$$(24)$$

We use the pressure scale and the time scale associated with vent one as the basic scaling parameters with which to make the equations dimensionless. Then, the ratio τ_1/τ_2 appears in the dimensionless equations, which we can write as follows

$$\frac{dx}{dt} = 1 - sgn(x)\sqrt{|x|} - a\ sgn(y)\sqrt{|y|}$$

$$\frac{dx}{dt} = 1 - sgn(x)\sqrt{|x|} - ab\ sgn(y)\sqrt{|y|}$$

$$(25)$$

where x = p_1, y = Δp = $p_1 - p_2$,

$$a = \sqrt{\tau_1 V_2/\tau_2 V_1},\ b = (V_1 + V_2)/V_2.$$

These equations are autonomous, see [16] and [17] for example, and therefore can be reduced to a single first–order nonlinear ODE by eliminating time. A phase plane analysis formally starts from this single first–order equation; however, when we numerically integrated in the phase plane, we used the parametric form of the equations, Eqs. (25), and then plotted the results in the phase plane.

In general, the fixed point of the system for (x(t), y(t)) is determined by setting the right hand sides of Eqs. (25) to zero; it is given by x_0 = 1, y_0 = 0. We note an important difference, between these equations and the usual ones encountered in phase–plane analysis [16] and [17]; these equations are not analytic around the fixed point. The fixed point is a stable one as determined by the numerical integrations described below.

If the two room volumes are identical and the conditions in the lower layer are the same for the example illustrated in Figure 3 then the parameters a and b are given by a = A_{vent2}/A_{vent1} and b = 2. We consider this special case for simplicity. Large a then implies that the vent
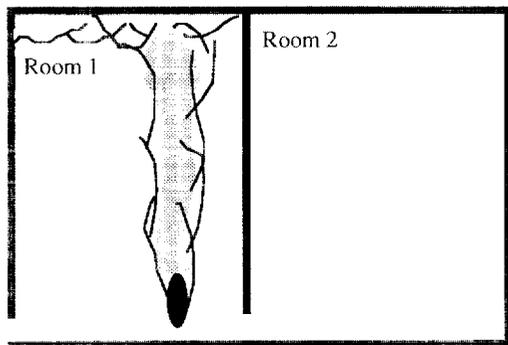
Figure 3. Two Room Schematic Diagram

connecting the two rooms is large compared to the vent connecting the first room to the outside.

Equations (25) have been integrated using the software package Mathematica. Once again, as in the one–room example, we have used the command NDSolve for this first–order nonlinear ODE system. Figure 4 shows the pressure p_1 and the pressure difference $p_1 - p_2$ for a $= 1$, b $= 2$ with initial conditions (I.C.) $x(0) = y(0) = 0$. The

dimensionless pressure in room 1, $x(t)$, starts at zero and increases monotonically to unity over of order ten dimensionless time units. The pressure difference between room 1 and room 2, $y(t)$, starts at zero, increases to a maximum of about 0.1 at about one dimensionless time unit and then decreases to zero again. The solutions displayed are well behaved, and the numerical calculation of them encounters no particular difficulty.

Figure 5 shows $x(t)$ and $y(t)$ for a $= 4$, b $= 2$ with initial conditions (I.C.) $x(0) = y(0) = 0$. The primary difference between these plots and those of Fig. 4 is that the solution for y rises more rapidly as a function of t from its initial condition to a smaller maximum of about 0.01 and then decays to zero. Large a is a condition on the ratio of time scales and volumes of the two rooms. As we see below, large a implies that the equations are stiff and a singular perturbation analysis of the problem is applicable. We emphasize that this problem, with a relatively large value of a, would most probably cause difficulty using a nu-
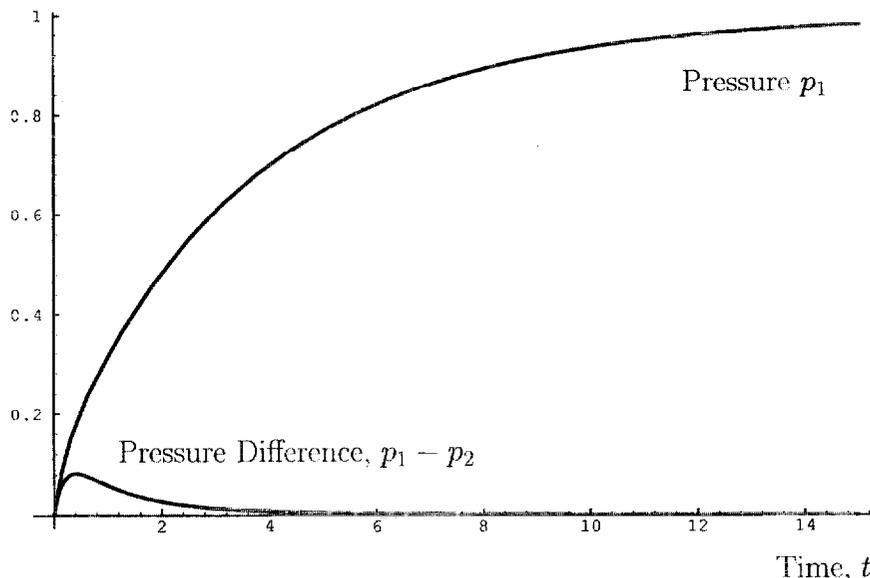

Figure 4. Solution plot of Equations 25 for a $= 1$, b $= 2$. Here, the parameters a and b are defined following that equation. The pressures in each room are ambient initially.
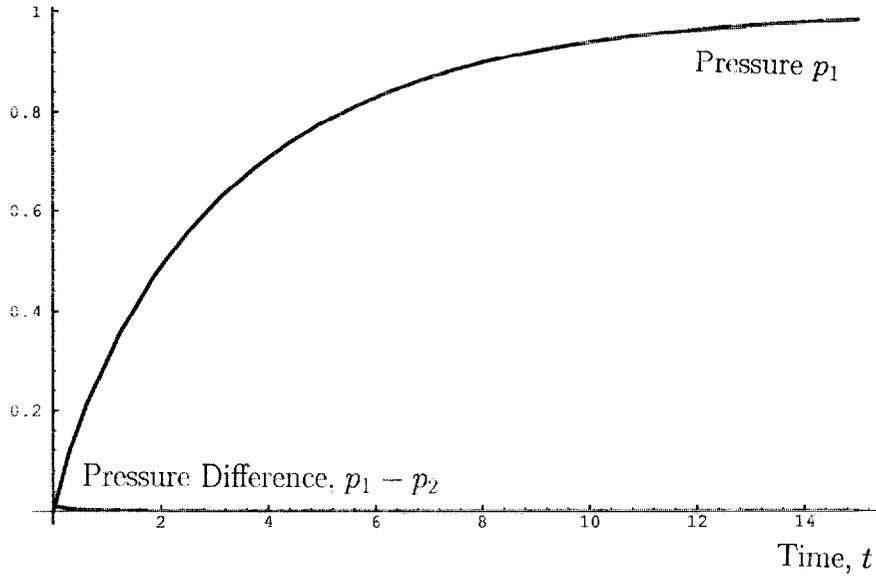
Figure 5. Solution plot of Equations 25 for a = 4, b = 2. Here, x is the normalized pressure in room 1, y is the normalized pressure difference between room 1 and room 2, and the parameters a and b are defined following that equation. The pressures in each room are ambient initially.
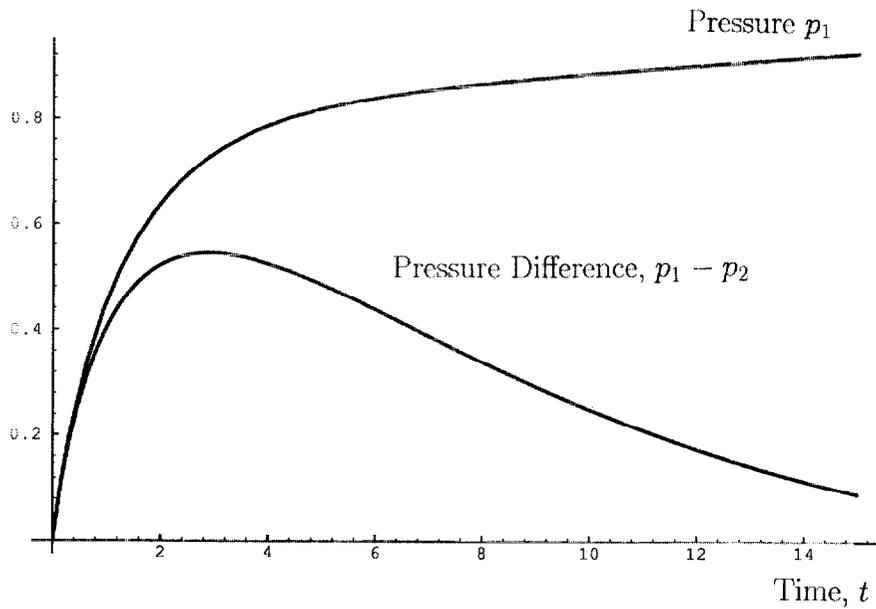

Figure 6. Solution plot of Equations 25 for a = 0.1, b = 2 Here, x is the normalized pressure in room 1, y is the normalized pressure difference between room 1 and room 2, and the parameters a and b are defined following that equation. The pressures in each room are ambient initially.

merical solver that did not account for stiffness of the equations.

Figure 6 shows x(t) and y(t) for a = 0.1, b = 2 with initial conditions (I.C.) x(0) = y(0) = 0. The curves show a much more gentle time dependence than that displayed in Fig. 5.

Figure 7 shows a phase plane plot of the solution to Eqs. (25) with a = 1, b = 2, where these parameters are defined following the equation. This phase plane plot demonstrates that x = 1, y = 0 is a stable fixed point of the solution since all solutions progress toward this point as time increases. For each initial condition, i.e. dimensionless pressure in room 1 (x) and dimensionless pressure difference between room 1 and room 2 (y), the solution of the equations starts at a position in the phase plane and progresses along the curve through that point down to the x–axis and then toward the fixed point. This is known as the trajectory of the solution in phase space. Since time is not shown in these plots, the time history cannot be inferred from these plots

alone. However, by comparison with the plot shown in Figure 4, we can see that the solution moves rather quickly to the x–axis, and then more slowly along the axis toward z = 1. The plot was prepared by integrating Eqs. (25) for thirteen different initial conditions and then plotting each curve parametrically. All solutions reach the stable fixed point.

Figure 8 shows a phase plane plot of the solution to Eqs. (25) with a = 4, b = 2. This figure shows that the trajectories of the solution have become much more angular with nearly 45–degree lines joined to sections of the z–axis. This very abrupt behavior is an indication that the Eqs. (25) are becoming stiff for the parameters chosen. Comparison of this figure with the time plot for the same parameters a, b shown in Figure 5 demonstrates that the solution progresses rapidly along its trajectory down to the x–axis and then very much more slowly along the axis to the fixed point at x = 1. This behavior makes numerical integration much more difficult
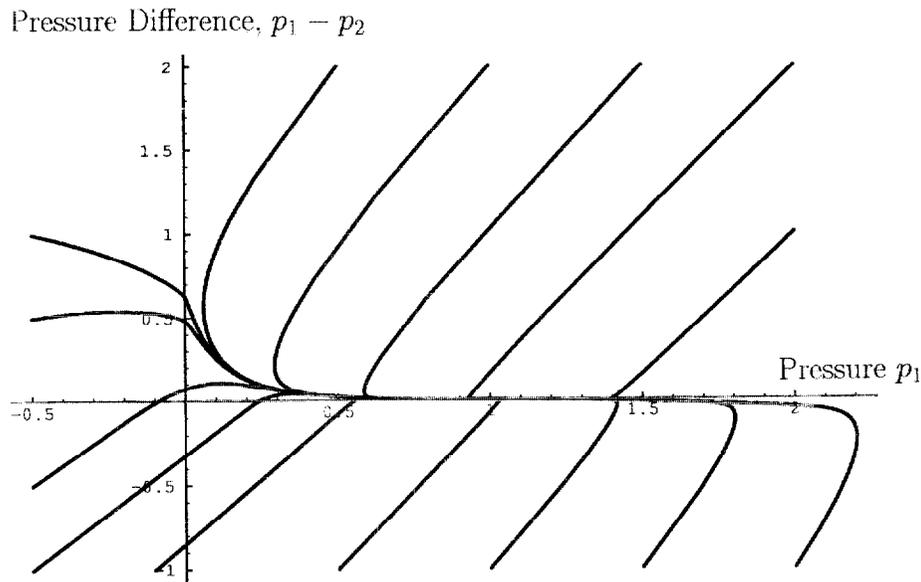

Figure 7. Phase plane plot of Equations 25 for a = 1, b = 2. Here, x is the normalized pressure in room 1, y is the normaized pressure difference between room 1 and room 2, and the parameters a and b are defined following that equation.

Pressure Difference, $p_1 - p_2$

Figure 8. Phase plane plot of Equations 25 for a = 4, b = 2. Here, x is the normalized pressure in room 1, y is the normalized pressure difference between room 1 and room 2, and the parameters a and b are defined following that equation.
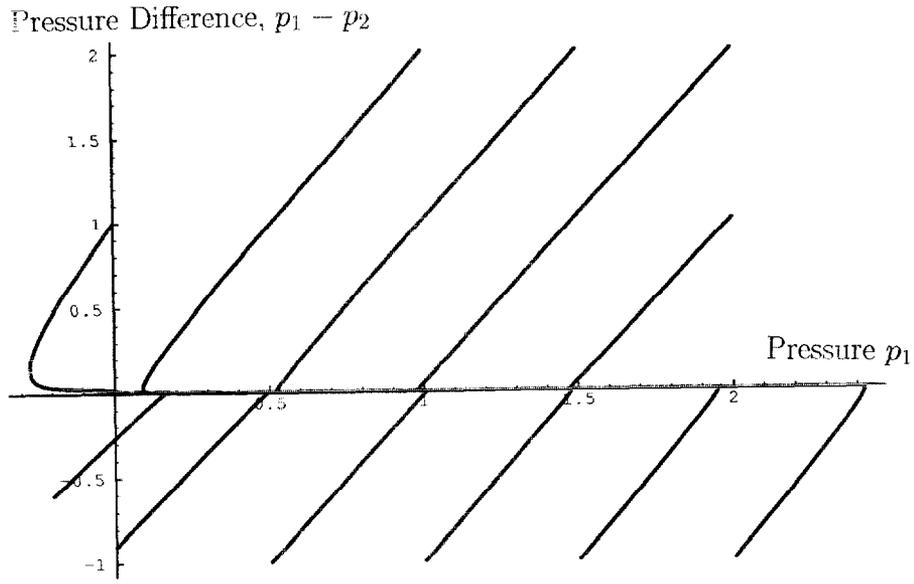
Pressure Difference, $p_1 - p_2$

Figure 9. Phase plane plot of Equations 25 for a = 0.1, b = 2. Here, x is the normalized pressure in room 1, y is the normalized pressure difference between room 1 and room 2, and the parameters a and b are defined following that equation.
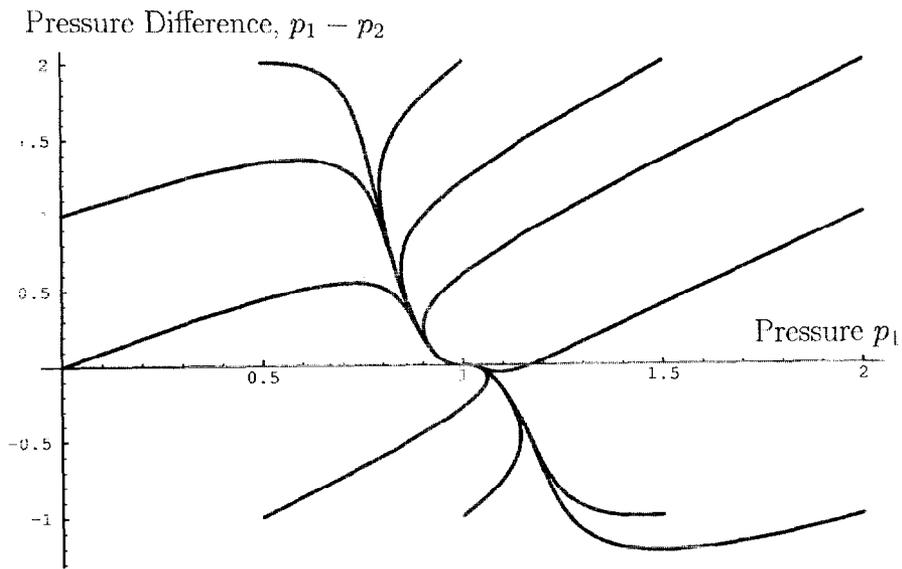
(unless a stiff solver is used) since the computation is limited by the smallest time scale, in this case the time required for the solution to reach the x-axis.

Figure 9 shows a phase plane plot of the solution to Eqs. (25) with a = 0.1, b = 2. This figure shows that the trajectories of the solution have become much smoother than those shown in either Figs. 7 or 8, and the solution is not stiff for the parameters chosen.

When the ratio of time scales τ_1/τ_2 becomes either large or small in Eqs. (25), situations of actual practical interest, they are stiff. Then, a singular perturbation analysis can be performed to obtain the analytical behavior; here, however, we will not perform a formal analysis, but only show how the zero-order behavior of the system can be determined under these conditions.

First, consider the case when $\tau_1/\tau_2 \gg 1$. This will occur, for example, when the area of vent two, that connecting the two rooms, is moderately large, while vent one, the vent to the outside, is a small leak. When this is the case, then, since τ_1/τ_2 multiplies the difference in room pressures, $p_1 - p_2 \to 0$. If we eliminate (τ_1/τ_2) $(p_1 - p_2)$ between the dimensionless form of the two Eqs. (23), we get

$$\frac{dp_1}{dt} = 1 - \text{sgn}(p_1) \sqrt{|p_1|} - \frac{V_2}{V_1} \frac{dp_2}{dt}$$

If we now say $p_2 \approx p_1$, then

$$\frac{dp_1}{dt} = \frac{V_1}{V_1 + V_2} [1 - \text{sgn}(p_1) \sqrt{|p_1|}]$$

If we choose the proper pressure and time scales, this becomes the same as Eq. (22) for the single room with a fire and a leak, but now with a volume $V_1 + V_2$, the volume of the two rooms.

Similarly, when $\tau_1/\tau_2 \ll 1$, we have the case where the vent area between the two rooms is small relative to the vent area to the outside for example. In this case, we concentrate on the first of Eqs. (23) and note that the term proportional to τ_1/τ_2, the term representing the effect of the second vent, is negligible. Then, this equation

becomes

$$\frac{dp_1}{dt} = 1 - \text{sgn}(p_1) \sqrt{|p_1|}$$

the equation for the single-room case again.

4. CONCLUSIONS

The simple problems examined in this paper illustrate the nature of the difficulties long encountered when numerically integrating zone fire models. The pressure equations equilibrate very rapidly compared to the equations governing the other dependent variables in the zone fire models. When equations of this nature are encountered, they are referred to as stiff. The simple problems analyzed here illustrate the nature of the stiffness and demonstrate that proper nondimensionalization together with singular perturbation analysis can provide insight into the behavior of the system for parameters of interest.

The methods can be used to examine much more general problems. For example, two rooms connected with each other and with the outside in different fashion can be analyzed similarly to the two-room example presented here. In addition, some multiroom enclosures have also been analyzed using the nondimensionalization and singular perturbation methods described herein. In the limit of various leak sizes between rooms (or time scales determined by the heat source, room volume and leak rate), the equations can be shown to reduce to the one-room equation with redefined leak rates and room volumes, as was done in the two-room case illustrated above. The methods should provide an opportunity to analyze difficulties with stiffness which are encountered in more general zone fire models.

REFERENCES

1. Williams, Forman A., Combustion Theory, The Benjamin/Cummings Publishing Co., Menlo Park, CA, second edition, 1985, Section 1.2.
2. Buckmaster, J.D. and Ludford, G.S.S., Theory of Laminar Flames, Cambridge Univer-

sity Press, Cambridge, England, 1982, Section 5.

3. Oran, Elaine S. and Boris, Jay P., Numerical Simulation of Reactive Flow, Elsevier, New York, 1987, Chapter 9.

4. Rehm, R.G. and Baum, H.R., "The Equations of Motion for Thermally Driven, Buoyant Flows", Journal of Research of the NBS, Vol. 83, pp 297–308, May–June 1978.

5. Zukoski, E.E., "Development of a Stratified Ceiling Layer in the Early Stages of a Closed–Room Fire," Fire and Materials, 2 No. 2, pp. 54–62, 1978.

6. Forney, Glenn P. and Moss, William F.,"Analyzing and Exploiting Numerical Characteristics of Zone Fire Models," National Institute of Standards and Technology Report 4763, March 1992.

7. Mitler, Henri E., "Mathematical Modeling of Enclosure Fires," Chapter 23, pp. 711–753, Numerical Approaches to Combustion Modeling, Elaine S. Oran and Jay P. Boris, eds., Vol. 135, Progress in Aeronautics and Astronautics; also, National Institute of Standards and Technology Report NISTIR 90–4294, May 1991.

8. Quintiere, James G., "Fundamentals of Enclosure Fire 'Zone' Models," J. of Fire Prot. Engr., 1(3), pp. 99–119, 1989.

9. Morton, B.R., Taylor, G.I., and Turner, J.S., "Turbulent Gravitational Convection from Maintained and Instantaneous Sources," Proc. Roy. Soc. London A, 234 pp. 1–23, 1956.

10. Prahl, J. and Emmons, H.W., "Fire Induced Flow Through an Opening," Combustion and Flame, 25 pp. 369–385, 1975.

11. Rockett, J.A., "Fire Induced Gas Flow in an Enclosure," Combustion Science and Technology, 12 pp. 165–175, 1976.

12. Zukoski, E.E. and Kubota, T., "Two–layer Modeling of Smoke Movement in Building Fires," Fire and Materials, 4 No. 1, pp. 17–27, 1980.

13. Emmons, H.W., "The Flow of Gases Through Vents," Home Fire Project Technical Report No. 75, Division of Applied Sciences, Harvard University, March 16, 1987.

14. Rehm, R.G. and Forney, G.P., "A Note on the Pressure Equations Used in Zone Fire Modeling," National Institute of Standards and Technology Report NISTIR 4906, August 1992.

15. Wolfram, Stephen, Mathematica, A System for Doing Mathematics by Computer, Addison–Wesley, New York, 1991.

16. Percival, I. and Richards, D., Introduction to Dynamics, Cambridge University Press, New York, 1982.

17. Jordan, D.W. and Smith P., Nonlinear Ordinary Differential Equations, Oxford Applied Mathematics and Computing Science Series, Clarendon Press, Oxford, 1979.