

RESEARCH ARTICLE

Antiderivatives and integral representations of associated Legendre functions with $\mu = \pm\nu$

Howard S. Cohl^{a*}

^a*Applied and Computational Mathematics Division, National Institute of Standards and Technology, Gaithersburg, MD 20899-8910, U.S.A.*

(v3.5 released October 2008)

We obtain antiderivatives and complex integral representations for associated Legendre functions and Ferrers functions (associated Legendre functions on-the-cut) of the first and second kind with degree and order equal to within a sign.

Keywords: Associated Legendre functions; Ferrers functions; Integral representations; Gauss hypergeometric function

AMS Subject Classification: 30E20; 35C05; 26A36

1. Introduction

Using analysis for fundamental solutions of Laplace’s equation on Riemannian manifolds of constant curvature, we have previously obtained antiderivatives and integral representations for associated Legendre and Ferrers functions of the second kind with degree and order equal to within a sign. For instance in Cohl (2011) [2, Theorem 1], we derived using the d -dimensional hypersphere with $d = 2, 3, 4, \dots$, an antiderivative and an integral representation for the Ferrers function of the second kind with order equal to the negative degree. In Cohl & Kalnins (2012) [3, Theorem 3.1], we derived using the d -dimensional hyperboloid model of hyperbolic geometry with $d = 2, 3, 4, \dots$, an antiderivative and an integral representation for the associated Legendre function of the second kind with degree and order equal to each other. In [2], [3], the antiderivatives and integral representations were restricted to values of the degree and order ν such that 2ν is an integer.

The purpose of this paper is to generalize the results presented in [2], [3] for associated Legendre and Ferrers functions of the first and second kind, and to extend them such that the degree and order are no longer subject to the above restriction. Our extremely simple integral representations are consistent with known special values for associated Legendre and Ferrers functions of the first kind when the order is equal to the negative degree.

Throughout this paper we rely on the following definitions. Let $a_1, a_2, a_3, \dots \in \mathbf{C}$, with \mathbf{C} being the set of complex numbers. If $i, j \in \mathbf{Z}$ and $j < i$, then $\prod_{n=i}^j a_n = 1$. The set of natural numbers is given by $\mathbf{N} := \{1, 2, 3, \dots\}$, the set $\mathbf{N}_0 := \{0, 1, 2, \dots\} = \mathbf{N} \cup \{0\}$, and $\mathbf{Z} := \{0, \pm 1, \pm 2, \dots\}$.

*Corresponding author. Email: howard.cohl@nist.gov

arXiv:1301.3556v1 [math.CA] 16 Jan 2013

2. Associated Legendre functions of the first and second kind

As is the common convention for associated Legendre functions [1, (8.1.1)], for any expression of the form $(z^2 - 1)^\alpha$, read this as

$$(z^2 - 1)^\alpha := (z + 1)^\alpha (z - 1)^\alpha,$$

for any fixed $\alpha \in \mathbf{C}$ and $z \in \mathbf{C} \setminus (-\infty, 1]$.

THEOREM 2.1 *Let $z \in \mathbf{C} \setminus (-\infty, 1]$, $\nu \in \mathbf{C} \setminus \{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots\}$, and C be a constant. Then we have the following antiderivative*

$$\begin{aligned} \int \frac{dz}{(z^2 - 1)^{\nu+1}} &= \frac{-1}{(2\nu + 1)z^{2\nu+1}} {}_2F_1 \left(\begin{matrix} \nu + \frac{1}{2}, \nu + 1 \\ \nu + \frac{3}{2} \end{matrix}; \frac{1}{z^2} \right) + C \\ &= \frac{-2^{-\nu} e^{-i\pi\nu}}{\Gamma(\nu + 1)(z^2 - 1)^{\nu/2}} Q_\nu^\nu(z) + C. \end{aligned} \quad (1)$$

In the above expression, the Gauss hypergeometric function ${}_2F_1 : \mathbf{C}^2 \times (\mathbf{C} \setminus -\mathbf{N}_0) \times \{z \in \mathbf{C} : |z| < 1\} \rightarrow \mathbf{C}$ can be defined in terms of the following infinite series

$${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix}; z \right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

(see (2.1.5) in Andrews, Askey & Roy 1999) and elsewhere on $z \in \mathbf{C} \setminus (1, \infty)$ by analytic continuation. The Pochhammer symbol (rising factorial) $(\cdot)_n : \mathbf{C} \rightarrow \mathbf{C}$ is defined by

$$(z)_n := \prod_{i=1}^n (z + i - 1),$$

where $n \in \mathbf{N}_0$. The associated Legendre function of the second kind $Q_\nu^\mu : \mathbf{C} \setminus (-\infty, 1] \rightarrow \mathbf{C}$ can be defined in terms of the Gauss hypergeometric function for $\nu + \mu \notin -\mathbf{N}$, as (Olver *et al.* (2010) [7, (14.3.7) and section 14.21])

$$Q_\nu^\mu(z) := \frac{\sqrt{\pi} e^{i\pi\mu} \Gamma(\nu + \mu + 1) (z^2 - 1)^{\mu/2}}{2^{\nu+1} \Gamma(\nu + \frac{3}{2}) z^{\nu+\mu+1}} {}_2F_1 \left(\begin{matrix} \frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2} \\ \nu + \frac{3}{2} \end{matrix}; \frac{1}{z^2} \right), \quad (2)$$

for $|z| > 1$ and by analytic continuation of the Gauss hypergeometric function elsewhere on $z \in \mathbf{C} \setminus (-\infty, 1]$.

Proof. The antiderivative (1) is verified as follows. By using

$$\frac{d}{dz} {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix}; z \right) = \frac{ab}{c} {}_2F_1 \left(\begin{matrix} a + 1, b + 1 \\ c + 1 \end{matrix}; z \right) \quad (3)$$

(see (15.5.1) in Olver *et al.* (2010) [7]), and the chain rule, it follows that

$$\begin{aligned} \frac{1}{(2\nu+1)} \frac{d}{dz} \frac{1}{z^{2\nu+1}} {}_2F_1\left(\begin{matrix} \nu + \frac{1}{2}, \nu + 1 \\ \nu + \frac{3}{2} \end{matrix}; \frac{1}{z^2}\right) &= -\frac{1}{z^{2\nu+2}} {}_2F_1\left(\begin{matrix} \nu + \frac{1}{2}, \nu + 1 \\ \nu + \frac{3}{2} \end{matrix}; \frac{1}{z^2}\right) \\ &\quad - \frac{2(\nu+1)}{(2\nu+3)z^{2\nu+4}} {}_2F_1\left(\begin{matrix} \nu + \frac{3}{2}, \nu + 2 \\ \nu + \frac{5}{2} \end{matrix}; \frac{1}{z^2}\right). \end{aligned} \quad (4)$$

The second hypergeometric function on the right-hand side can be simplified using Gauss' relations for contiguous hypergeometric functions, namely

$$z {}_2F_1\left(\begin{matrix} a+1, b+1 \\ c+1 \end{matrix}; z\right) = \frac{c}{a-b} \left[{}_2F_1\left(\begin{matrix} a, b+1 \\ c \end{matrix}; z\right) - {}_2F_1\left(\begin{matrix} a+1, b \\ c \end{matrix}; z\right) \right] \quad (5)$$

(see p. 58 in Erdélyi *et al.* (1981) [4]), and

$${}_2F_1\left(\begin{matrix} a, b+1 \\ c \end{matrix}; z\right) = \frac{b-a}{b} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) + \frac{a}{b} {}_2F_1\left(\begin{matrix} a+1, b \\ c \end{matrix}; z\right) \quad (6)$$

(see (15.5.12) in Olver *et al.* (2010) [7]). After simplification and utilization of

$${}_2F_1\left(\begin{matrix} a, b \\ b \end{matrix}; z\right) = (1-z)^{-a}$$

(see (15.4.6) in Olver *et al.* (2010) [7]), the right-hand side of (4) simplifies to $1/(z^2 - 1)^{\nu+1}$. The equality of the Gauss hypergeometric function in the antiderivative (1), in terms of the associated Legendre function Q_ν^ν , follows directly from (2). ■

A straightforward consequence of the antiderivative (1) is the following integral representation for the associated Legendre function with degree and order equal, namely

$$Q_\nu^\nu(z) = 2^\nu \Gamma(\nu+1) e^{i\nu\pi} (z^2 - 1)^{\nu/2} \int_z^\infty \frac{dw}{(w^2 - 1)^{\nu+1}}, \quad (7)$$

where $\operatorname{Re} \nu > -\frac{1}{2}$. Using the negative order relation for associated Legendre functions of the second kind [7, (14.9.14)]

$$Q_\nu^{-\mu}(z) = e^{-2i\pi\mu} \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} Q_\nu^\mu(z),$$

we derive

$$Q_\nu^{-\nu}(z) = \frac{\sqrt{\pi} e^{-i\nu\pi} (z^2 - 1)^{\nu/2}}{2^\nu \Gamma(\nu + \frac{1}{2})} \int_z^\infty \frac{dw}{(w^2 - 1)^{\nu+1}}.$$

Using the Whipple relation for associated Legendre functions (cf. [7, (14.9.16)])

$$Q_\nu^\nu(z) = \sqrt{\frac{\pi}{2}} \Gamma(2\nu+1) (z^2 - 1)^{-1/4} e^{i\nu\pi} P_{-\nu-1/2}^{-\nu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right),$$

where the associated Legendre function of the first kind $P_\nu^\mu : \mathbf{C} \setminus (-\infty, 1] \rightarrow \mathbf{C}$ is defined in [7, (14.3.6)], we obtain

$$P_\nu^\nu(z) = -\frac{2^{\nu+1}}{\pi} \Gamma(\nu+1) \sin(\nu\pi) (z^2-1)^{\nu/2} \int_{z/\sqrt{z^2-1}}^{\infty} \frac{dw}{(w^2-1)^{-\nu+1/2}}.$$

The above expression is equivalent to

$$P_\nu^\nu(z) = \frac{2^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}} (z^2-1)^{\nu/2} + \frac{2^{\nu+1} \Gamma(\nu+1) (z^2-1)^{\nu/2}}{\pi} \sin(\nu\pi) \int_z^\infty \frac{dw}{(w^2-1)^{\nu+1}}, \quad (8)$$

where $\operatorname{Re} \nu > -\frac{1}{2}$. An interesting definite integral follows from the behavior of the above integral representation near the singularity at $z = 1$, namely

$$\int_1^\infty (w^2-1)^{\nu-1} dw = \frac{\Gamma(\nu) \Gamma(\frac{1}{2}-\nu)}{2\sqrt{\pi}},$$

for $0 < \operatorname{Re} \nu < \frac{1}{2}$. Using the negative order relation for associated Legendre functions of the first kind (cf. [7, (14.9.12)]), one has

$$P_\nu^{-\nu}(z) = \frac{1}{\Gamma(2\nu+1)} \left[P_\nu^\nu(z) - \frac{2}{\pi} e^{-i\nu\pi} \sin(\nu\pi) Q_\nu^\nu(z) \right]. \quad (9)$$

After replacement of (7) and (8) in (9) we obtain

$$P_\nu^{-\nu}(z) = -\frac{\sin(\nu\pi) (z^2-1)^{\nu/2}}{\sqrt{\pi} 2^{\nu-1} \Gamma(\nu + \frac{1}{2})} \left[\int_{z/\sqrt{z^2-1}}^\infty \frac{dw}{(w^2-1)^{-\nu+1/2}} + \int_z^\infty \frac{dw}{(w^2-1)^{\nu+1}} \right],$$

which reduces to the special value [7, (14.5.19)]

$$P_\nu^{-\nu}(z) = \frac{(z^2-1)^{\nu/2}}{2^\nu \Gamma(\nu+1)}.$$

3. Ferrers functions of the first and second kind

THEOREM 3.1 *Let $x \in (-1, 1)$, $\nu \in \mathbf{C}$, and C be a constant. Then we have the following antiderivative*

$$\int \frac{dx}{(1-x^2)^{\nu+1}} = x {}_2F_1 \left(\begin{matrix} \frac{1}{2}, \nu+1 \\ \frac{3}{2} \end{matrix}; x^2 \right) + C = \frac{2^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} (1-x^2)^{\nu/2}} Q_\nu^{-\nu}(x) + C.$$

In the above expression, the Ferrers function of the second kind (associated Legendre function of the second kind on-the-cut) $Q_\nu^\mu : (-1, 1) \rightarrow \mathbf{C}$ is defined in [7,

(14.3.12)], and for $\mu = -\nu$ we have

$$Q_{\nu}^{-\nu}(x) = \frac{\sqrt{\pi} x(1-x^2)^{\nu/2}}{2^{\nu} \Gamma(\nu + \frac{1}{2})} {}_2F_1\left(\frac{1}{2}, \nu + 1; \frac{3}{2}; x^2\right). \quad (10)$$

Proof. The Gauss hypergeometric function in the antiderivative follows using (3), (5), (6), as in the proof of Theorem 2.1, with the Ferrers function following directly using (10). ■

The following integral representation for the Ferrers function of the second kind is an obvious consequence of Theorem 3.1, namely

$$Q_{\nu}^{-\nu}(x) = \frac{\sqrt{\pi}(1-x^2)^{\nu/2}}{2^{\nu} \Gamma(\nu + \frac{1}{2})} \int_0^x \frac{dw}{(1-w^2)^{\nu+1}}. \quad (11)$$

A definite-integral result near the singularity at $x = 1$ follows using (10), (11), and Gauss's sum [7, (15.4.20)]

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

for $\operatorname{Re}(c-a-b) > 0$, namely

$$\int_0^1 (1-w^2)^{\nu-1} dw = \frac{\sqrt{\pi} \Gamma(\nu)}{2\Gamma(\nu + \frac{1}{2})},$$

for $\operatorname{Re} \nu > 0$. The well-known special value (see [7, (14.5.18)])

$$P_{\nu}^{-\nu}(x) = \frac{(1-x^2)^{\nu/2}}{2^{\nu} \Gamma(\nu + 1)},$$

where $P_{\nu}^{\mu} : (-1, 1) \rightarrow \mathbf{C}$ is the Ferrers function of the first kind (associated Legendre function of the first kind on-the-cut) defined in [7, (14.3.1)], in conjunction with [5, (8.737.1)], yields the following integral representation

$$P_{\nu}^{\nu}(x) = \frac{2^{\nu}(1-x^2)^{\nu/2}}{\sqrt{\pi}} \left[\Gamma(\nu + \frac{1}{2}) \cos(\nu\pi) + \frac{2\Gamma(\nu + 1)}{\sqrt{\pi}} \sin(\nu\pi) \int_0^x \frac{dw}{(1-w^2)^{\nu+1}} \right].$$

Finally using the negative order relation for the Ferrers functions of the second kind [6, p. 170], namely

$$Q_{\nu}^{-\mu}(x) = \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \left[\cos(\mu\pi) Q_{\nu}^{\mu}(x) + \frac{\pi}{2} \sin(\mu\pi) P_{\nu}^{\mu}(x) \right],$$

we have the following integral representation of the Ferrers function of the second kind with degree and order equal

$$Q_{\nu}^{\nu}(x) = -2^{\nu-1} \sqrt{\pi} \Gamma(\nu + \frac{1}{2}) \sin(\nu\pi) (1-x^2)^{\nu/2} \\ + 2^{\nu} \Gamma(\nu + 1) \cos(\nu\pi) (1-x^2)^{\nu/2} \int_0^x \frac{dw}{(1-w^2)^{\nu+1}}.$$

Acknowledgements

Part of this work was conducted while H. S. Cohl was a National Research Council Research Postdoctoral Associate in the Applied and Computational Mathematics Division at the National Institute of Standards and Technology, Gaithersburg, Maryland, U.S.A.

References

- [1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. U.S. Government Printing Office, Washington, D.C., 1972.
- [2] H. S. Cohl. Fundamental solution of Laplace's equation in hyperspherical geometry. *Symmetry, Integrability and Geometry: Methods and Applications*, 7(108), 2011.
- [3] H. S. Cohl and E. G. Kalnins. Fourier and Gegenbauer expansions for a fundamental solution of the Laplacian in the hyperboloid model of hyperbolic geometry. *Journal of Physics A: Mathematical and Theoretical*, 45(14):145206, 2012.
- [4] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Higher transcendental functions. Vol. I*. Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1981.
- [5] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, seventh edition, 2007.
- [6] W. Magnus, F. Oberhettinger, and R. P. Soni. *Formulas and theorems for the special functions of mathematical physics*. Third enlarged edition. Die Grundlehren der mathematischen Wissenschaften, Band 52. Springer-Verlag New York, Inc., New York, 1966.
- [7] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST handbook of mathematical functions*. Cambridge University Press, Cambridge, 2010.