# Resource Letter SS–1: The Spin-Statistics Connection ⊘

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# RESOURCE LETTER

# Resource Letter SS–1: The Spin-Statistics Connection

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This Resource Letter provides a guide to the literature on the spin-statistics connection and related issues such as the Pauli exclusion principle and particle indistinguishability. Journal articles and books are cited for the following topics: basic resources, general aspects of spin-statistics, the spin-statistics theorem, theories that go beyond the standard spin-statistics theorem, and experimental tests of the spin-statistics theorem and the symmetrization postulate. © *2012 American Association of Physics Teachers.*
[http://dx.doi.org/10.1119/1.4704899]

## I. INTRODUCTION

The spin-statistics connection is a fundamental principle of physics that is easy to state but difficult to prove. David Goodstein tells of Richard Feynman's thinking along these lines: "Once I asked him [Feynman] to explain to me, so that I could understand it, why spin-1/2 particles obey Fermi-Dirac statistics. Gauging his audience perfectly, he said, 'I'll prepare a freshman lecture on it.' But a few days later he came to me and said, 'You know, I couldn't do it. I couldn't reduce it to the freshman level. That means we really don't understand it.'" ["Richard P. Feynman, Teacher," David Goodstein, Phys. Today **42**(2), 70–75 (1989)]. Feynman put it in his own words like this: "It appears to be one of the few places in physics where there is a rule which can be stated very simply, but for which no one has found a simple and easy explanation. The explanation is deep down in relativistic quantum mechanics. This probably means that we do not have a complete understanding of the fundamental principle involved." [**Quantum Electrodynamics,** Richard Feynman (W. A. Benjamin, New York, 1961) p. 162]. A similar statement appears in **The Feynman Lectures on Physics**, Vol. III., R. P Feynman, R. B. Leighton, and M. Sands (Addison-Wesley, Reading, 1965), Chap. 4, p. 3.

When Wolfgang Pauli won the 1945 Nobel Prize for his discovery of the exclusion principle, he explained that despite his publication of a theoretical "proof" of the spin-statistics connection (the "spin-statistics theorem" cited below) in 1936 with an improved proof in 1940, he still considered funda-

mental understanding of the issue to be inadequate because of the need to make too many assumptions upon which the proof rests. In his Nobel address, he said, "This situation appeared to me as disappointing in an important respect. Already in my original paper, I stressed the circumstance that I was unable to give a logical reason for the exclusion principle or to deduce it from more general assumptions. I had always the feeling, and I still have it today, that this is a deficiency….The impression that the shadow of some incompleteness fell here on the bright light of success of the new quantum mechanics seems to be unavoidable." ["Exclusion Principle and Quantum Mechanics," Wolfgang Pauli in **Nobel Lectures, Physics 1942–1962** (Elsevier Publishing Company, Amsterdam, 1964). pp. 31–32] In his quantum mechanics textbook, Pauli wrote "The fact that quantum mechanics yields *more* states than actually occur in nature (and all of them equally possible), is still a puzzle." [**General Principles of Quantum Mechanics**, Wolfgang Pauli. First published as a chapter in the **Handbuch der Physik Vol. V** (Springer-Verlag, Berlin, 1958)]. In this Resource Letter, we provide the resources to help students and scientists who wish to explore this puzzle further.

We begin with a few statements of the basics. In its most straightforward form, the spin-statistics connection asserts that elementary particles with spin quantum numbers that are odd-integer multiples of $1/2$ have collective states described by Fermi-Dirac statistics, while particles with spin quantum numbers that are even-integer multiples of $1/2$ (including 0) have collective states described by Bose-Einstein statistics. The same principle can be rephrased in terms of particle

occupation of states: no more than one spin-$\frac{1}{2}$ particle can occupy a particular state. This formulation constitutes the Pauli exclusion principle, which was articulated (Ref. 16) several years before the connection with spin was recognized. For integer-spin particles, there is no limit on the number of particles in a particular state.

The spin-statistics connection has several strange aspects. First, why is spin (often viewed as a relativistic effect) connected with the collective behavior of quantum states even when the spin dynamical interactions are very small or completely negligible? Second, is the spin-statistics connection an independent principle or is it a consequence of other physical principles? Third, why does it apply only to collective systems of *identical* particles?

The last question raises the thorny issue of what we mean by "identical" particles. We take the point of view that a particle's properties can be sorted into two groups: (1) intrinsic (unchanging) properties (such as mass, electric charge, spin quantum number, magnetic dipole moment, parity, and so on) and (2) dynamical properties such as position, momentum, and energy. We say that two particles are identical if all of their intrinsic properties are the same. Consequently, the "statistics" in the spin-statistics connection refer to the properties of states of many identical particles. In classical mechanics, the notion of identical particles is different, because we can always (at least in principle) track a particle by its space-time trajectory and distinguish it from other particles. Not so in quantum mechanics where the notion of trajectory is problematic, because there is a fundamental uncertainty described by the nonzero product of the uncertainties associated with position and momentum. Hence, in many cases, we can no longer legitimately talk about having electron A in one state and electron B in another state. All we can say is that there is some probability for having one of the two electrons (and we can't tell which one) in one of the states and some other probability for having one of the electrons (and we can't tell which one) in the other state. We say that the particles are indistinguishable.

The above considerations are linked to surprisingly important physical effects from the smallest to the largest scales: the structure and stability of nuclei, the structure and stability of atoms, Bose-Einstein condensation, the formation and evolution of neutron stars, and so on. Again, one of the mysterious aspects of these phenomena is that spin does not seem to play an important dynamical role in these systems except for its link to the statistics of the collective states.

In quantum mechanics as established by Heisenberg, Schrödinger, and Dirac, there is a way of talking about the statistics of collective states that does not involve spin. Heisenberg recognized that when quantum mechanics is applied to systems of two identical particles (for example, the helium atom with two electrons), there are two classes of possible states, which today we would call symmetric states and antisymmetric states. ["Über die Spektra von Atomsystemen mit zwei Elecktronen," W. Heisenberg, Zeit. für Physik **39**, 499–518 (1926).] At the time, there seemed to be no way to decide between the two. In fact, the situation gets worse when one considers systems of three or more identical particles. For a system with $N$ identical particles, the number of possible states increases as $N!$. To end up with just a single state, we must, apparently, eliminate most of the possible collective states of the system. This restriction is implemented through what is called the symmetrization postulate,

which decrees that only two types of states are possible: those that are symmetric under the interchange of identical particle labels and those that are antisymmetric under that kind of interchange. It is then a question of experiment to determine which type of state applies to a particular type of particle. Particles with antisymmetric collective states are called fermions and we say they obey Fermi-Dirac statistics. Particles with symmetric collective states are called bosons and we say they obey Bose-Einstein statistics.

The symmetrization postulate makes no statement about a connection with the spin angular momentum of the particles and it is, from that point of view, an empirical matter that spin-1/2 particles are described by antisymmetric states and integer-spin particles are described by symmetric states. The mathematics of permutation symmetry allows for more complicated states—with so-called parastatistics—but nature does not seem to have made use of them.

It was quickly realized in the early days of quantum mechanics that the antisymmetric states were those to which the Pauli exclusion principle applied. An antisymmetric state for two identical particles can be written as

$$\Psi(1,2) = \frac{1}{\sqrt{2}}[\psi_a(1)\psi_b(2) - \psi_a(2)\psi_b(1)], \tag{1}$$

where 1 and 2 are labels for the two identical particles and the subscripts $a$ and $b$ stand for the quantum numbers that specify the single-particle states. The collective state is antisymmetric (changes sign) under the interchange of the identical-particle labels and obviously vanishes if the two sets of quantum numbers are the same, i.e., the state vector is zero if we try to put the two particles in the same single-particle state. Thus, antisymmetric states are the formal embodiment of the exclusion principle in ordinary quantum mechanics.

Yet another approach to the quantum mechanics of identical-particle systems makes use of creation and annihilation operators to generate the relevant states—the so-called second quantization approach. In this scheme, the algebraic relations between creation operators $a_i^\dagger$ and annihilation operators $a_j$ carry the burden of describing the statistics of multiparticles states. (The subscripts $i$ and $j$ are labels indicating the quantum state.) The creation and annihilation operators for bosons satisfy commutation relations, while the operators for fermions satisfy anti-commutator relations

$$a_j a_k^\dagger \pm a_k^\dagger a_j = \delta_{jk}, \tag{2}$$

where the $+$ sign applies to fermions and the $-$ sign to bosons. $\delta_{jk} = 1$ if $j = k$ and zero otherwise.

Modifications of the creation and annihilation operator formalism are particularly useful for describing violations of the symmetrization postulate and the spin-statistics connection. For example, Greenberg (Ref. 95) introduced a modified commutator called the q-mutator

$$a_i a_j^\dagger - q a_j^\dagger a_i = \delta_{ij}, \tag{3}$$

where $q$ is a parameter that is $-1$ for ordinary fermions and $+1$ for ordinary bosons but in principle can take on any value in the range $[-1, +1]$. Hypothetical particles with $-1 < q < 1$ have been called quons.

The spin-statistics theorem provides a link between other fundamental physical principles and the spin-statistics connection. It is now generally accepted that the spin-statistics

theorem requires the machinery of relativistic quantum field theory (Ref. 32). The key ingredients in the theorem seem to be (1) Lorentz invariance, (2) locality of the fields and inter-actions, (3) positive norms for states, and (4) positive energies for states. In addition, different proofs of the spin-statistics theorem bring in further assumptions, whose physical motivation is less apparent. If any of these principles were to be violated, the spin-statistics theorem would apparently fail. The current situation might best be summarized by saying that the spin-statistics connection is compatible with relativistic quantum field theory, but a fully satisfying proof of the spin-statistics theorem has yet to be demonstrated.

Given the lack of a "simple" proof of the spin-statistics theorem, and the existence of various theoretical ideas in string theory (see Sec. V K.), M-theory, and quantum gravity that bring into question one or more of the key ingredients of the theorem, it becomes a question for experiment to see if there are violations of the spin-statistics connection. For example, is there a (presumably small) probability of finding two identical integer (half-integer) spin particles in an anti-symmetric (symmetric) state? Is there a (presumably small) probability of finding three or more identical particles in a state that is neither symmetric nor antisymmetric? In reviewing the status of relevant atomic-physics experiments, where one might expect to be able to answer such questions with a high degree of confidence, noted theorist Lev Okun summarized the situation by writing: "The absence of quantitative experimental tests of the Pauli principle in atoms is like a blank spot on the map of experimental physics." [L. B. Okun, in **Festi-Val–Festschrift for Val Telegdi**, edited by K. Winter (Elsevier Science Publishers B.V., Amsterdam, 1988), p. 210] In 1987, Greenberg and Mohapatra (Ref. 92) made an even more sweeping statement with regard to all fields of physics: "No high precision tests of the Pauli principle have been made." This situation stimulated a number of new experiments in subsequent years. In Sec. VI, we list many such attempts to observe a violation of the spin-statistics connection. The q-mutator formalism allows experimental tests of the spin-statistics connection to be parameterized as a measure of the difference $1 - |q|$. (We expect $|q|$ to be close to one since we don't observe any large-scale deviations from the spin-statistics connection.) So far, no experiments have yielded results with $|q|$ different from 1, given the uncertainties in the experimental results.

The literature on the spin-statistics connection is vast, and given the limited space of this Resource Letter we have had to omit many references and topics. For example, there are intriguing issues in quantum gravity that affect our understanding of the spin-statistics connection, but many of the papers that discuss such issues are quite sophisticated mathematically. Consequently, we decided not to include them. We are preparing, however, a more complete collection of references dealing with the spin-statistics connection, which we will post on arXiv (www.arXiv.org).

Each of the references cited below is categorized as (E) elementary, (I) intermediate, or (A) advanced. For our purposes, (E) means that the material could be expected to be understandable by someone familiar with introductory physics at the undergraduate level, (I) material requires some knowledge of the formalism of quantum mechanics at the upper-undergraduate level, and (A) material requires familiarity with relativistic quantum mechanics, quantum field theory, and the related mathematics.

## II. BASIC RESOURCES

### A. Journals

Rapid publication journals with a significant number of papers devoted to spin-statistics:
*Europhysics Letters*
*Physics Letters A*
*Physical Review Letters*
There are many journals that have articles on the spin-statistics connection. Here is a selected list to get started:
*American Journal of Physics*
*Foundations of Physics*
*Physical Review A*
*Physical Review D*
*Journal of Mathematical Physics*

### B. Conference proceedings

**1.** **Spin-Statistics Connection and Commutation Relations: Experimental Tests and Theoretical Implications,** Robert C. Hilborn and Guglielmo M. Tino, eds. (American Institute of Physics, New York, 2000). Contains 33 conference papers dealing with the theoretical, experimental, and philosophical aspects of the spin-statistics connection and the symmetrization postulate. (I-A)

**2.** "Special Issue on Spin Statistics," Edoardo Milotti, C. Curceanu (Petrascu), Stephen L. Adler, and Michael Berry, Found. Phys., **40**(7), 681–683 (2010). An introduction to this issue of Found. Phys. devoted to spin-statistics issues. (I-A)

### C. Popularizations

**3.** "Fiber Bundles and Quantum Theory," H. J. Bernstein and A. V. Phillips, Sci. Am. **245**(1), 123–137 (1981). A discussion of how the mathematical notion of fiber bundles illuminates several issues in quantum theory including the behaviour of identical particles. (I)

**4.** "The Effects of Spin in Gases," F. Laloe and J. H. Freed, Sci. Am. **258**(4), 94–101 (1988). A survey of how quantum statistics affects the macroscopic properties of helium and spin-polarized atomic hydrogen, including properties such as thermal conductivity. (E)

**5.** "Roll over, Wolfgang? New experiments seek violations of the Pauli exclusion principle," June Kinoshita, Sci. Am. **258**, 25 (1988). A brief discussion of some of the first "modern" attempts to test the Pauli exclusion principle and the theories that might describe such violations. (E)

**6.** "Exclusion Principle Still Intact," Tony Sudbery, Nature **348** (6298), 193–194 (1990). A general exposition of experimental tests of the Pauli exclusion principle. (E)

**7.** "Testing the Pauli exclusion principle," Ivars Peterson, Science News **137**(18) 287 (1990). Reports on a special session held on the topic at a meeting of the American Physical Society in April 1990. (E)

**8.** "A testing time for bosons," O. W. Greenberg, Phys. World **9**(7), 27 (1996). An overview of experimental tests of the spin-statistics connection in light of two papers that describe new experimental tests for bosons. (E)

**9.** "Precision tests find no violation of bose statistics," B. Schwarzschild, Phys. Today **49**(11), 19–20 (1996). Another overview of the two experiments discussed in the previous reference. (E)

**10.** "Statistics given a spin," Jason Twamley, Nature **389**(6647), 128–129 (1997). A review of other recent tests of the spin-statistics connection. (E)

## D. Textbooks

Most textbooks for undergraduate and graduate quantum mechanics provide brief discussions of identical particles and the spin-statistics connection. The discussions generally fall into two categories: (1) the spin-statistics connection is observed experimentally and here is how we describe it, and (2) the physical properties of a system must be unchanged if identical particles are exchanged, with some additional argument to focus on just symmetric or antisymmetric states (leaving out other possibilities). The spin-statistics connection is then brought in by (generally inadequate) theoretical arguments.

Examples of texts that explore the fundamental issues more deeply, or at least provide warnings to the reader that the situation is considerably more subtle than most other texts suggest, include those listed below. (Identification of certain commercial textbooks in this Resource Letter is not intended to imply recommendation or endorsement by the National Institute of Standards and Technology, nor is it intended to imply that the texts identified are necessarily the best available for the purpose.)

**11. The Principles of Quantum Mechanics**, P. A. M. Dirac, 4th ed. (Oxford U.P., Oxford, 1967). "[It] may be that for a particular kind of particle only symmetrical states occur in nature, or only anti-symmetrical states occur in nature. If either of these possibilities held, it would lead to certain special phenomena for the particles in question."(p. 209) "Other more complicated kinds of symmetry are possible mathematically, but do not apply to any known particles." (p. 211) (I)

**12. Quantum Mechanics**, A. M. L. Messiah (John Wiley and Sons, New York, 1976). The entire chapter XIV is devoted to identical particles and the Pauli exclusion principle. Messiah adopts the position that symmetrization is a *postulate* of quantum mechanics, rather than something to be proved. "In Classical Mechanics…indistinguishability of identical particles plays a secondary role; in Quantum Mechanics it raises a serious problem." (p. 582) "*Experiment shows* [italics in the original] that the elementary particles of spin $1/2$ occurring in nature (electrons, protons, neutrons, etc.,) are fermions, while those of integral spin (photons, $\pi$ mesons, etc.) are bosons." (p. 595) (I)

**13. Lectures on Quantum Mechanics,** Gordon Baym (Westview Press, Boulder, CO, 1974). Includes a chapter devoted entirely to spin and identical particles. The author notes that in systems of more than two identical particles, "other possible symmetry schemes [beyond the usual symmetric and antisymmetric ones] cannot be ruled out on logical grounds alone,…" (p. 392). (I)

**14. Quantum Mechanics**, C. Cohen-Tannoudji, B. Diu, and F. Laloe (Wiley, New York, 1977) Vol. II, Chapter XIV. Provides an excellent introduction to the quantum mechanics of systems of identical particles. (I)

**15. PCT, Spin, Statistics, and All That**, R. F. Streater and A. S. Wightman (W. A. Benjamin, New York, Amsterdam, 1964). An advanced treatment of the spin-statistics theorem in quantum field theory. (A)

## III. GENERAL ASPECTS OF IDENTICAL PARTICLES AND THE SPIN-STATISTICS CONNECTION

### A. Historically important papers

**16.** "Über den Zusammenhang des Abschlusses der Electronengruppen im Atom mit der Komplexstruktur der Spektren," W. Pauli, Zeit. für Physik **31**, 765–783 (1925). Pauli's introduction of his exclusion principle. (I)

**17.** "On the Theory of Quantum Mechanics," P. A. M. Dirac, Proc. Roy. Soc. [A] **112**, 661–667 (1926). The first coherent theory of identical particles in quantum mechanics. (I)

**18.** "Über das Paulische Äquivalenzverbot," P. Jordan and E. Wigner, Zeit. für Physik **47**(9-10), 631–651 (1928). The authors show that if creation and annihilation operators obey anticommutation relations, the resulting particle states satisfy the Pauli exclusion principle. (I)

**19.** "Gehörchen die Stickstoffkerne der Böseschen Statistik?" W. Heitler and G. Herzberg, Naturwiss. **17**, 673–674 (1929). This paper noted the "nitrogen problem": In the late 1920s, everyone thought (incorrectly, as we now know) that the $^{14}$N nucleus contains 14 protons and 7 electrons, by which accounting it should be a fermion. However, molecular spectroscopy experiments on $N_2$ indicated the $^{14}$N nucleus is a boson. (I)

**20.** "The Statistics of Composite Systems According to the New Quantum Mechanics," E. P. Wigner, Anz. Ung. Ak. Wiss. **46**, 576–579 (1929). A system with an odd number of fermions is itself a fermion. All other systems are bosons. This paper is available in A. S. Wightman, J. Mehra, B. R. Judd, and G. W. Mackey, **The Collected Works of Eugene Paul Wigner** (Springer-Verlag, Berlin/Heidelberg/New York, 1993). (I)

**21.** "Über die Rotations-Ramanspektren von Stickstoff und Sauerstoff," F. Rasetti, Zeit. für Physik **61**(9-10), 598–601 (1930). Molecular spectroscopy of diatomic nitrogen shows without a doubt that $^{14}$N is a boson. (I)

**22.** "Note on the Statistics of Nuclei," P. Ehrenfest and J. R. Oppenheimer, Phys. Rev. **37**(4), 333–338 (1931). Derives the result developed independently by Wigner (Ref. 20) for composite systems. Applies those rules to nuclei now that the spin-statistics connection had been recognized and resolves a number of "paradoxes" such as the $^{14}$N problem. (I)

**23.** "Discussion of Probability Relations Between Separated Systems," E. Schrödinger, Proc. Cambridge Philos. Soc. **31**, 555–563 (1935). The introduction of the notion of quantum entanglement, a feature that plays a critical role in the quantum mechanics of identical particles. (I)

Pauli was one of the first to attempt a formal proof of the spin-statistics connection. His first papers are given in the following two references:

**24.** "Relativistic Quantized Theory of Particles Obeying Bose-Einstein Statistics," W. Pauli, Ann. Inst. Henri Poincaré **6**, 137–152 (1936). (A)

**25.** "The Connection Between Spin and Statistics," W. Pauli, Phys. Rev. **58**, 716–722 (1940). (A)

### B. History of spin-statistics

**26.** "Remarks on the History of the Exclusion Principle," Wolfgang Pauli, Science **103**, 213–215 (1946). Pauli's

recollection of the history of the development of the exclusion principle. (E)

27. "The Turning Point," R. Kronig, in **Theoretical Physics in the Twentieth Century**, edited by M. Fierz and V. F. Weisskopf (Interscience, New York, 1960), pp. 5–39. This book also contains several other articles about the spin-statistics connection and its history. (I)-(A).

28. "Statistical symmetries in physics," H. S. Green, Aust. J. Phys. **47**(2), 109–122 (1994). Surveys the history of "statistical" symmetries in quantum mechanics from Bose up to the early 1990s. Includes some technical discussion of symmetries other than Bose-Einstein and Fermi-Dirac using "deformed" commutators. (I-A)

29. **Pauli and the Spin-Statistics Connection**, Ian Duck and E. C. G. Sudarshan (World Scientific, River Edge, NJ, 1997). A detailed history of the spin-statistics theorem with excerpts from historically important papers. The emphasis is on theoretical developments. (I)

30. "Review of Pauli and the Spin Statistics Theorem by Ian Duck and E. C. G. Sudarshan." Arthur S. Wightman, Am. J. Phys. **67**(8), 742–746 (1999). Book review that contains a fine discussion of spin-statistics issues. (I)

31. **The Story of Spin**, Sin-itiro Tomonaga (The University of Chicago Press, Chicago, 1997). A historical account of spin in quantum mechanics and the relationship between spin and particle statistics. (E-I)

32. "Toward an understanding of the spin-statistics connection," I. Duck and E. C. G. Sudarshan, Am. J. Phys. **66**(4), 284–303 (1998). Detailed historical survey of various theoretical approaches to the spin-statistics connection and a survey of attempts to provide elementary proofs of the spin-statistics connection. (I)

## C. Philosophical issues

There is a large literature dealing with the philosophical issues surrounding identical and indistinguishable particles in quantum mechanics. This literature links to the philosophical arguments about individuality and distinguishability that go back (at least) to Leibniz. Here, we give just a few references that are germane to the spin-statistics connection.

33. **An Interpretative Introduction to Quantum Field Theory**, Paul Teller (Princeton U. P., Princeton, NJ, 1995). Detailed analysis of the machinery of quantum field theory, including how it handles identical particles. The discussion on pages 83–85 analyzes the role of microcausality in establishing the spin-statistics connection. (I)

34. **Pauli's Exclusion Principle: The Origin and Validation of a Scientific Principle**, Michela Massimi (Cambridge U.P., Cambridge, 2005). Detailed philosophical and historical analysis of Pauli's exclusion principle. (I)

35. "Putting a New Spin on Particle Identity," Steven French, in Ref. 1, pp. 305–318. Lays out the philosophical consequences of the forms of quantum statistics different from the standard Fermi-Dirac and Bose-Einstein statistics. (I)

36. "Identical Particles in Quantum Mechanics Revisited," Robert C. Hilborn and Candice Yuca, Br. J. Philos. Sci.
**53**, 355–389 (2002). Re-examination of the issues of identity and distinguishability in quantum mechanics in light of possible violations of the spin-statistics connection. (I)

## D. Identical particles in nonrelativistic quantum mechanics

The literature on identical particles in quantum mechanics is vast. Here, we cite a few references most relevant to the spin-statistics connection.

37. "Interference of Nonidentical Particles," V. L. Lyuboshitz and M. I. Podgoretskii, Sov. Phys. JETP **28**(3), 469–475 (1969). Describes the quantum mechanics of "almost identical" particles and shows how quantum interference wipes out exchange interactions if the particles are not identical. (I)

38. "The Question of Identity of Elementary Particles," V. L. Lyuboshitz and M. I. Podgoretskii, Sov. Phys. JETP **33**(1), 5–10 (1971). Shows how a notion of "variable" identity of particles can be formulated when the internal states of the particles are described by superposition states, the mixture of which can be continuously varied. (I)

39. "On the Theory of Identical Particles," J. M. Leinaas and J. Myrheim, Nuovo Cimento **37B**(1), 1–23 (1977). Formulates the quantum theory of identical particles by defining the configuration space to take into account the indistinguishability of the particles by excluding those points when particle coordinates are the same. Symmetric and antisymmetric states are claimed to emerge in a natural way, without invoking the symmetrization postulate. However, there are some mathematical subtleties in quantizing the theory on the restructured configuration space. The authors note the peculiarities of a two-spatial-dimensional system, which later was rediscovered as the type of system in which anyons can occur. See Ref. 142. (I)

40. "Spin and statistics in nonrelativistic quantum mechanics: The spin-zero case," Murray Peshkin, Phys. Rev. A **67**, 042102 (2003). Uses a variation on the modified configuration-space approach along with an assumption of continuity of the wave function as a function of the coordinates in configuration space to prove that spin-zero particles are described by symmetric states. The following reference replies to a comment on this paper. (I)

41. "Reply to "Comment on 'Spin and statistics in nonrelativistic quantum mechanics: the spin-zero case'"," M. Peshkin, Phys. Rev. A **68**(4), 46102 (2003). (I)

## E. Symmetrization postulate

The symmetrization postulate does not connect spin and statistics; it merely claims that all quantum states must be either symmetric or antisymmetric under exchange of identical particles. It postulates that states with more complicated symmetries (see Sec. V A. on Parastatistics below, for example) do not exist in nature.

42. "Symmetrization Postulate and Its Experimental Foundation," A. M. L. Messiah and O. W. Greenberg, Phys. Rev. **136**(1B), B248–B267 (1964). A seminal and comprehensive review of the symmetrization postulate, including its application in systems with a variable

number of particles. Shows that particles not obeying the symmetrization postulate can be accommodated in quantum mechanics if the usual one-dimensional rays are replaced by many-dimensional rays. Proves that there is an absolute selection (superselection) rule that prohibits transitions between states that obey the symmetrization postulate and those that violate it. Also reviews experiments that test the symmetrization postulate for elementary particles. (I)

43. "Permutation Symmetry of Many-Particle Wave Functions," M. D. Girardeau, Phys. Rev. **139**, B500–B508 (1965). Detailed analysis of the symmetrization postulate. Shows that the symmetrization postulate is not demanded by the usual physical interpretation of the mathematical formalism of quantum mechanics unless further physical properties of real systems are invoked. (I)
44. "Symmetrization Postulate of Quantum Mechanics," Michael Flicker and Harvey S. Leff, Phys. Rev. **163**(5), 1353–1357 (1967). Uses "reasonable physical assumptions" to prove that for systems of identical spinless particles, the permissible nonrelativistic Schrödinger wave functions are either all symmetric or all antisymmetric. (I)

## F. Permutation group

The permutation group (also called the symmetric group) can be used to classify states of multiparticle systems of identical particles. There are two one-dimensional representations corresponding to symmetric and antisymmetric states, but for systems with three or more identical particles there are higher-dimensional representations, apparently not made use of by Nature.

45. **Atoms and Molecules**, Mitchel Weissbluth (Academic Press, New York, 1978). Chapter 5 has an excellent introduction to the permutation group and representations other than the symmetric and antisymmetric ones. (I)

## G. Geometry and identical particles

Topological arguments can be used to provide support for the spin-statistics connection.

46. "The Reason for Antiparticles," R. P. Feynman, in **Elementary Particles and the Laws of Physics. The 1986 Dirac Memorial Lectures**, by R. P. Feynman and S. Weinberg (Cambridge U.P., New York, 1987) pp. 1–61. Addendum (pp. 56–59) contains Feynman's "belt argument," a simple analogy (but only an analogy) of the spin-statistics connection. (I)
47. "A Topological Spin-Statistics Theorem or a Use of the Antiparticle," A. P. Balachandran, A. Daughton, Z. C. Gu, G. Marmo, R. D. Sorkin, and A. M. Srivastava, Mod. Phys. Lett. A **5**(20), 1575–1585 (1990). Analysis of using the permutation group and braid group to prove a version of the spin-statistics connection. (I)-(A)

## H. Identical particles, entanglement, and quantum information

Schrödinger (Ref. 23) identified entanglement as "the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought." Entanglement plays a crucial role in quantum computation and in the formal description of identical-particle states.

48. "Generic Entanglement Generation, Quantum Statistics, and Complementarity," S. Bose and D. Home, Phys. Rev. Lett. **88**(5), 050401 (2002). Particles do not need to interact directly to become entangled. Individual interactions with beam splitters and detectors combined with their indistinguishability can be used to produce entangled states. (I)
49. "General criterion for the entanglement of two indistinguishable particles," GianCarlo Ghirardi and Luca Marinatto, Phys. Rev. A **70**(1), 012109 (2004). More discussion of entanglement for identical particles. (I)
50. **Quantum Systems, Processes, and Information**, Benjamin Schumacher and Michael Westmoreland (Cambridge U.P., New York, 2010). Intended as a textbook for an undergraduate course in quantum mechanics, this book emphasizes the roles of entanglement, information, and computation (along with identical particles) in quantum theory. (I)
51. "Interacting electrons, spin statistics, and information theory," L. M. Ghiringhelli, I. P. Hamilton, and L. Delle Site, J. Chem. Phys. **132**(1), 014106 (2010). When the fermionic character of a gas of interacting electrons is included via a statistical spin approach, correlation terms have the form of the Shannon entropy. The results strengthen the connection between Shannon entropy and electron correlation and, more generally, between information theory and quantum mechanics. (A)

## I. Applications where spin-statistics provides unexpected results

52. "Using the Pauli Exclusion-Principle to Exhibit Local-Realism Violations in Overlapping Interferometers," B. Yurke and D. Stoler, Phys. Rev. A **47**(3), 1704–1707 (1993). The issue of local realism (apparently violated by quantum mechanics) can be explored by experiments with fermions. (I)
53. "Spin-statistics connexion, neutrinos, and big bang nucleosynthesis," L. Cucurull, J. A. Grifols, and R. Toldra, Astropart. Phys. **4**(4), 391–395 (1996). Argues that $^4$He abundance in the early universe can be used to show that neutrinos are distributed according to Fermi-Dirac statistics. (I)
54. "Giant suppression of shot noise as signature of coherent transport in double barrier resonant diodes," V. Ya Aleshkin, L. Reggiani, N. V. Alkeev *et al*., Semicond. Sci. Technol. **19**, S161–S163 (2004). The Pauli exclusion principle can lead to substantial suppression of shot noise in a double-barrier tunnel diode at low temperatures. (I)
55. "Quantum interference in electron collision," R. C. Liu, B. Odom, Y. Yamamoto, and S. Tarucha, Nature **391**, 263–265 (1998). Collisions of electrons at a beam splitter show "anti-bunching," a manifestation of the Pauli exclusion principle. (I)
56. "Spectral simulations of polar diatomic molecules immersed in He clusters: application to the ICl(X) molecule," P. Villarreal, M. P. de Lara-Castells, R. Prosmiti *et al*., Phys. Scr. **76**, C96–C103 (2007). The infrared spectrum of a polar molecule trapped in a solvent cluster

depends on the bosonic or fermionic character of the solvent. (I)

57. "Quantum gases: The cold reality of exclusion," Zoran Hadzibabic, Nat. Phys. **6**(9), 643–644 (2010). Provides a nice overview of recent experiments that demonstrate the importance of the Pauli exclusion principle in ultracold atom physics, including spatial statistical fluctuations reduced compared to those expectedly classically. (I)

58. "Quantum-State Controlled Chemical Reactions of Ultracold Potassium-Rubidium Molecules," S. Ospelkaus, K.-K Ni, D. Wang *et al.*, Science **327**(5967), 853–857 (2010). Reports on experiments showing that the fermionic statistics of a polar molecule have a strong influence on the collision rate with identical molecules. If the two molecules are in the same internal state, then the relative translational state must be antisymmetric and hence p-wave amplitudes dominate. If the internal states are mixed, then the relative translational state can be symmetric with a drastically different collision rate. (I)

## IV. SPIN-STATISTICS THEOREM

### A. Proofs of the spin-statistics theorem

Developing a proof of the spin-statistics theorem (SST) has a long and convoluted history with many different and sometimes conflicting approaches. The history of these efforts is described in Ref. 29. Here, we list several papers that illustrate the range of methods used. See also Refs. 24 and 25.

59. **Introduction to the theory of quantized fields**, N. N. Bogoliubov and D. V. Shirkov, Authorized English ed., rev. and enl. (Interscience Publishers, New York, 1959). An early textbook on relativistic quantum field theory. Gives an introduction to the spin-statistics theorem. (A)

60. **Relativistic quantum fields**, James D. Bjorken and Sidney D. Drell (McGraw-Hill, New York, 1965). A textbook account of the spin-statistics theorem. (A)

61. "Causality, Antiparticles, and the Spin-Statistics Connection in Higher Dimensions," S. Weinberg, Phys. Lett. B **143**(1-3), 97–102 (1984). What happens to spin-statistics when the theory has more than three spatial dimensions. (A)

62. "Can the Pauli Principle Be Deduced With Local Quantum-Field Theory," A. B. Govorkov, Phys. Lett. A **137**(1-2), 7–10 (1989). Shows that the local quantum theory of free fields allows only the generalizations of the conventional (Fermi-Dirac and Bose-Einstein) quantization that correspond to para-Fermi and para-Bose statistics (see Sec. V A. ) Argues that these generalizations are equivalent to the inclusion of inner degrees of freedom and hence concludes that the Pauli principle is a consequence of general assumptions of QFT, and "small" violations of it are not admissible. (I)

63. "A Cyclic Symmetry Principle in Physics," H. S. Green, Aust. J. Phys. **47**(1), 25–43 (1994). Links permutation symmetry to other fundamental symmetries. (A)

64. "A New Approach to Spin and Statistics," B. Kuckert, Lett. Math. Phys. **35**(4), 319–331 (1995). An algebraic proof of the spin-statistics connection for para-bosonic and para-fermionic quantum topological charges of a theory of local observables (see Sec. V A for parastatistics). (A)

65. "CPT Violation Implies Violation of Lorentz Invariance," O. W. Greenberg, Phys. Rev. Lett. **89**(23), 231602 (2002). Proves that if CPT symmetry is violated by a theory, then Lorentz invariance is also violated. Both of these symmetries are critical for most proofs of the spin-statistics theorem. (A)

66. **Quantum theory of fields**, Steven Weinberg (Cambridge U.P., Cambridge and New York, 1995). A definitive account of quantum field theory by one of its masters. Chapters 4 and 5 of Volume I have sections that discuss the spin-statistics connection. On page 198, Weinberg notes that the usual argument of commutation (or anticommutation) of field operators at space-like separation as a way of implementing microcausality is problematic. However, the space-like commutivity is required for Lorentz invariance of the S-matrix, and thus we really don't need the interpretation of "causality." (A)

67. "Spin-statistics, spin-locality, and TCP: three distinct theorems," O. W. Greenberg, Phys. Lett. B **416**(1-2), 144–149 (1998). Distinguishes between the spin-statistics theorem for particles and the spin-locality theorem for quantum fields. Also discusses the weak conditions under which the TCP (time-reversal, charge-conjugation, and parity-reversal) theorem holds. (A)

68. "Spin and Statistics and First Principles," Sergio Doplicher, Found. Phys. **40**(7), 719–732 (2010). Reviews methods of connecting spin and statistics in quantum field theory, assuming locality among observables and indicates where work remains to be done to extend the proof to theories with massless particles and theories such as quantum gravity that are expected to violate locality. (A)

69. "Symmetries and the connection between spin and statistics in rigorous quantum field theory," E. H. Wichman, in **Art and Symmetry in Experimental Physics**, edited by D. Budker, P. H. Buchsbaum, and S. J. Freedman (American Institute of Physics, New York, 2001), pp. 201–231. Detailed analysis of the role of symmetries in proving the spin-statistics theorem. (A)

### B. Attempts to prove the spin-statistics theorem and the symmetrization postulate in nonrelativistic quantum mechanics

Many papers have been devoted to methods of establishing the spin-statistics connection and the symmetrization postulate without invoking the machinery of relativistic quantum field theory. In our opinion, none of these "proofs" is entirely satisfactory and many suffer from serious defects, some of which are discussed in Refs. 32 and 72. Many of the proofs make use of the equation $|\psi(\alpha_1, \alpha_2, ...)|^2 = |\hat{P}\psi(\alpha_1, \alpha_2, ...)|^2$ where $\hat{P}$ is a permutation operator and the $\alpha$s are the relevant dynamical variables for the particles that make up the system. Pauli noted that the probability density on the left-hand-side of that equation is not an observable for a system of identical particles because it claims to give the probability of finding particle 1 at $r_1$, particle 2 at $r_2$, and so on. ("Die Allgemeinen Prinzipien der Wellenmechanik," **Handbuch der Physik Vol. V**, Springer-Verlag, Berlin, 1958) That labeling is equivalent to distinguishing among the identical particles and is inconsistent, in general, with the assumption of their identity. Some proofs are just analogies See Ref. 46, for example.

Other proofs either implicitly or explicitly assume that the quantum state is represented by a single ray in Hilbert space, which is equivalent to requiring that the states transform under a one-dimensional representation of the permutation group, and there are only two of these: the symmetric one and the antisymmetric one. So assuming that a single ray is required is equivalent to assuming the result (namely, the symmetrization postulate).

70. "Spin and statistics," Arthur Broyles, Am. J. Phys. **44**(4), 340–343 (1976). An attempt to provide a proof of the spin-statistics connection within the context of ordinary (nonrelativistic) quantum mechanics. (I)
71. Dwight E. Neuenschwander, "Question #7. The spin-statistics theorem," Am. J. Phys. **62**(11), 972 (1994). A series of articles appeared in this journal in response to this question about the possibility of a "simple" proof of the spin-statistics theorem. See also Ref. 32. (E)
72. Robert C. Hilborn, "Answer to Question #7," Am. J. Phys. **63** (4), 298–299 (1995). This article describes flaws in many of the purported proofs of the spin-statistics connection in nonrelativistic quantum mechanics. (I)
73. "Indistinguishability for quantum particles: Spin, statistics and the geometric phase," M. V. Berry and J. M. Robbins, Proc. R. Soc. Lond. [A] **453**, 1771–1790 (1997). Uses a novel transported spin basis and relates the spin-statistics connection to the single-valuedness of the wave function and a geometric phase in quantum mechanics. (I)
74. "The spin-statistics connection: Some pedagogical remarks in response to Neuenschwander's question," A. S. Wightman, Electron. J. Differ. Equations Conf. **04**, 207–213 (2000). Argues that using nonrelativistic quantum mechanics and the transformation law of wave functions under translations and rotations cannot lead to the spin-statistics connection. Various proofs using relativistic field theory are also analysed. (A)
75. "Spin and statistics in Galilean covariant field theory," C. R. Hagen, Phys. Rev. A **70**(1), 012101 (2004). What happens to the spin-statistics connection if a field theory is not Lorentz invariant? Shows that in a Galilean covariant field theory there is no spin-statistics connection. (A)
76. "Connecting Spin and Statistics in Quantum Mechanics," A. Jabs, Found. Phys. **40**, 776–792 (2010). Provides a proof of the spin-statistics theorem based on postulates concerning the behaviour of the spin azimuthal angle, which defines the orientation of the spin part of a single-particle spin-component eigenfunction. (I)
77. "Addendum to: Connecting Spin and Statistics in Quantum Mechanics," A. Jabs, Found. Phys. **40**, 793–794 (2010). Generalizes some of the arguments in Ref. 76. (I)

## C. Composite systems

As mentioned previously, the Ehrenfest-Oppenheimer theorem (Refs. 20 and 22) states that composite systems consisting of an even number of identical fermions (and any number of bosons) behave like a boson. If the number of fermions is odd, then the composite behaves like a fermion. The Ehrenfest-Oppenheimer theorem holds as long as the composite system remains in its ground state, which condition holds when the relevant interaction energies with the environment are small compared to the energies between the ground state and excited states of the composite.

78. "Formulation of the many-body problem for composite particles," M. Girardeau, J. Math. Phys. **4**(8), 1096–115 (1963). Detailed treatment of the many-body theory of a system of composite particles in terms of creation and annihilation operators. One application is the Bose-Einstein condensation of fermion pairs. (I)
79. "Superficial Violation of the Pauli Principle Due to the Possible Substructure of Electrons," K. Akama, H. Terazawa, and M. Yasue, Phys. Rev. Lett. **68**(12), 1826–1829 (1992). Argues that if the electron is a composite system, the behavior of the composite system need not be described by the Pauli exclusion principle. Violations are expressed in terms of the size of the electron. (I)

## V. BEYOND THE STANDARD SPIN-STATISTCS THEOREM AND SYMMETRIZATION POSTULATE

### A. Parastatistics

The permutation group applied to systems of identical particles has representations (when the system contains three or more identical particles) that are neither symmetric nor antisymmetric under the interchange of identical-particle coordinates and spin. Such states are said to obey parastatistics. Several textbooks point out this possibility. For example, in the section on permutation symmetry of Ref. 14, p. 1387, questionable assumptions upon which the spin-statistics theorem depend are noted ("these hypothesis may not all be correct"), and the reader is warned that, "It is not inconceivable that, for certain particles, the physical kets might have more complex symmetry properties than those envisaged here."

There are two types of parastatistics—finite and infinite. In the finite case, the occupation number of individual states is limited. For example, if at most $p$ particles can occupy a symmetric state, the particles are called $p$-order parafermions. In quantum field theory, a so-called Klein transformation transforms fields that have parastatistics properties into fields that are bosonic (for parabosons) and fermionic (for parafermions). Infinite statistics has an unbounded occupation number. It is pointed out in several papers that quantum mechanics can accommodate such states (usually focusing on finite-order parastatistics) but Nature apparently does not make use of them. In the early days of quark theory, Greenberg (Ref. 83) suggested that quarks are para-fermions of order 3, which generated much interest in parastatistics.

It is often argued (Refs. 88 and 154) that parastatistics is equivalent to having a new degree of freedom. However, this equivalence holds only if other assumptions are invoked. Messiah and Greenberg noted that "…neutrons and protons could be treated either as two different species of particles or as two different states of the same species with an internal degree of freedom (Ref. 42, p. B256). As is well-known (Ref. 12, Chap. XIV), the two treatments are rigorously equivalent when the symmetrization postulate is made. However, this is no longer true when the symmetry types are not restricted to that extent."

80. "A Generalized Method of Field Quantization," H. S. Green, Phys. Rev. **90**, 270–273 (1952). The original paper on parastatistics. Demonstrates that with noncanonical

tri-linear commutation relations (more general than those in ordinary quantum mechanics), quantum states with parastatistics can be described. (A)

81. "Do Strange Particles Obey Parastatistics?" H. Feshbach, Phys. Lett. **3**(7), 317–319 (1963). Claims that the conservation of strangeness and the phenomena of associated production could be explained as "conservation of statistics" if the strange particles are described by parastatistics. (A)

82. "The Existence and Significance of Parastatistics," Max Dresden, in **Brandeis Summer Institute in Theoretical Physics: Lectures on Astrophysics and Weak Interactions**, edited by S. Hayakawa, H.-Y Chiu, G. Feinberg, and M. Dresden (Brandeis University, Waltham, MA, 1964), pp. 347–469. A very nice introduction to the issues of identical particles, the permutation group, and parastatistics. (I)

83. "Spin And Unitary-Spin Independence in a Paraquark Model of Baryons and Mesons," O. W. Greenberg, Phys. Rev. Lett. **13**(20), 598–602 (1964). The original paper that suggested that quarks are "parafermions" (fermions described by parastatistics) of order 3 (at most three quarks in a given quantum state). Later it was shown that this analysis is equivalent to having quarks with a three-valued internal quantum number ("color"). (I)

84. "Selection Rules for Parafields and the Absence of Paraparticles in Nature," O. W. Greenberg and A. M. L. Messiah, Phys. Rev. **138**, B1155–B1167 (1965). Thoughtful, thorough discussion of various theoretical constructs, particularly Green's deformed commutation relations and parastatistics that describe violations of the symmetrization postulate. (I)

85. "On parastatistics," K. Drühl, R. Haag, and J. E. Roberts, Commun. Math. Phys **18**, 204–226 (1970). Shows that parastatistics of order $p$ ($p$ being the maximum occupation number) is associated with the exact internal symmetry group $SU(p)$ or $SO(p)$. The result fails for infinite statistics but the following paper argues that infinite statistics can be ruled out. (A)

86. "On the Existence of Antiparticles," K. Fredenhagen, Commun. Math. Phys. **79**, 141–151 (1981). Shows that infinite statistics is ruled out by the requirement of local observables. (A)

87. **Quantum Field Theory and Parastatistics**, Y. Ohnuki and S. Kamefuchi (Springer-Verlag, Berlin/Heidelberg/New York, 1982). Monograph showing how quantum field theory can incorporate parastatistics. (A)

88. "Parastatistics and internal symmetries," A. B. Govorkov, Sov. J. Part. Nucl. **14**(5), 520–537 (1983). Discusses the equivalence of parastatistics and internal degrees of freedom, which holds for finite-order parastatistics. The equivalence fails for infinite statistics, in which the occupation number can be infinite. (A)

## B. Negative probabilities

Most proofs of the spin-statistics theorem and the symmetrization postulate are based on the requirement of positive norms for physical states. Negative norms are usually ruled out because they represent "negative probabilities." Many authors have speculated about how to give physical meaning to negative probabilities. Here is a selection:

89. "The physical interpretation of quantum mechanics," P. A. M. Dirac, Prof. Roy. Soc. [A] **180**, 1–40 (1942).

90. "The search for unity: Notes for a history of quantum field theory," S. Weinberg, Daedalus **2**, 17–35 (1977). In passing, discusses the possible role of negative probabilities. (I)

91. "Negative Probability," R. P. Feynman, in **Quantum Implications: Essays in honour of David Bohm**, edited by B. J. Hiley and F. David Peat (Routledge, London and New York, 1987) pp. 235–248. Describes how negative probabilities might be interpreted. (I)

## C. Q-mutators and quons and q-deformed commutations

As noted in the introduction, O. W. Greenberg (Ref. 95) introduced a modified commutation relation for creation and annihilation operators that interpolates between the usual commutator for bosons and the anti-commutator for fermions. The modified commutator is a specific example of deformed commutation relations, which appear in so-called quantum groups—noncommutative algebras with additional structure that interpolate between the algebras associated with standard Lie groups. Besides the Greenberg q-mutator, other modified commutation relations are possible. For example, many authors have studied a $q$-bosonic oscillator algebra with the commutation relation $aa^{\dagger} - qa^{\dagger}a = q^{-N}$, where $N$ is the number operator $a^{\dagger}a$. Fermionic $q$-oscillators are described by similar operator relations with the minus sign replaced by plus. So far, no experiments have detected particles with $q \neq \pm 1$ within experimental uncertainties. See Sec. V I.

92. "Local Quantum-Field Theory of Possible Violation of the Pauli Principle," O. W. Greenberg and R. N. Mohapatra, Phys. Rev. Lett. **59**(22), 2507–2510 (1987). Highly influential paper that stimulated several new experiments; first in a series by these authors. Deformed tri-linear commutators are used to describe a theory with possible "small" violations of the Pauli exclusion principle. See Refs. 93 and 94. (I)

93. "Correction," O. W. Greenberg, Phys. Rev. Lett. **61**(12), 1432 (1988). Indicates that the method suggested in Ref. 92 leads to difficulties (negative norms) in states with four or more identical particles. (I)

94. "Difficulties With a Local Quantum-Field Theory of Possible Violation of the Pauli Principle," O. W. Greenberg and R. N. Mohapatra, Phys. Rev. Lett. **62**(7), 712–714 (1989). Detailed explanation of the difficulties recognized in the formulation in Ref. 93. (I)

95. "Example of Infinite Statistics," O. W. Greenberg, Phys. Rev. Lett. **64**, 705–708 (1990). Influenced much of the work on the spin-statistics connection in the years to follow. Shows that generalized commutation relations $a_j a_k^{\dagger} - qa_k^{\dagger}a_j = \delta_{jk}$ interpolate smoothly between commutators and anticommutators. The particles described by such operators became known as quons. The $q = 0$ case gives rise to "infinite" statistics in which all possible representations of the permutation group are present and there is no limit to the occupation number for any state. (I)

96. "Infinite Statistics and a Possible Small Violation of the Pauli Principle," R. N. Mohapatra, Phys. Lett. B **242**(3,4), 407–411 (1990). Develops a matrix representation of the creation and annihilation operators satisfying q-deformed commutation relations and shows how

they can be interpreted as describing "small" (i.e., low probability) violations of the Pauli principle. (I)

97. "Interpolation Between Fermi and Bose Statistics Using Generalized Commutators," D. I. Fivel, Phys. Rev. Lett. **65**(27), 3361–3364 (1990). Establishes that the states generated by creation operators described by q-mutators all have positive norms if $q$ is in the range $-1 \leq q \leq 1$. See Phys. Rev. Lett. **69**, 2020 (1992) for Errata. (A)

98. "Particles With Small Violations of Fermi or Bose Statistics," O. W. Greenberg, Phys. Rev. D **43**(12), 4111–4120 (1991). Discusses representations of the quon algebra, the TCP theorem, clustering, locality properties, and experimental constraints of particles described by q-mutators. (I)

99. "Interactions of particles having small violations of statistics," O. W. Greenberg, Physica A **180**, 419–427 (1992). Discusses the interactions of particles with small violations of Fermi or Bose statistics. (I)

100. "New Statistics for Mixing System of Bosons and Fermions," L. A. Wu, Z. Y. Wu, and J. Sun, Phys. Lett. A **170**(4), 280–282 (1992). Introduces a new version of the q-mutator with the numerical parameter $q$ replaced by an operator q with eigenvalues $\pm 1$. The thermodynamic behavior of such a system is evaluated. (I)

101. "Realizability of a Model in Infinite Statistics," D. Zagier, Commun. Math. Phys. **147**, 199–210 (1992). A tour de force general proof of the positive norm for quon states. (A)

102. "The Energy Operator for Infinite Statistics," Sonia Stanciu, Commun. Math. Phys. **147**, 211–216 (1992). Constructs the energy operator for quons. (A)

103. "Generalized statistics of macroscopic fields," R. Speicher. Lett. Math. Phys. **27**, 97–104 (1993). Proves that a Fock space representation exists for particles described by q-mutators. (A)

104. "2nd Quantization in a Quon-Algebra," J. S. Moller, J. Phys. A-Math. Gen. **26**(18), 4643–4652 (1993). Constructs normal-ordered operator expressions for quon fields. These are more complex than the quadratic forms that emerge for operators described by the ordinary commutation or anticommutation relations. (I)

105. "Black Hole Statistics," A. Strominger, Phys. Rev. Lett. **71**(21), 3397–3400 (1993). Extremal black holes (those with the minimum mass allowed for a given charge and angular momentum) are described by neither Bose nor Fermi statistics but rather by quon statistics. (I)

106. "Quons, An Interpolation Between Bose and Fermi Oscillators," O. W. Greenberg, in **Workshop on Harmonic Oscillators**, edited by D. Han, Y. S. Kim, and W. W. Zachary (National Aeronautics and Space Administration, Greenbelt, MD, 1993), pp. 5–18. This conference paper discusses how to formulate interactions for quons with external fields. This formulation requires a quon generalization of Grassmann (anticommuting) numbers and leads to failure of local commutivity for relativistic quon fields. (A)

107. "The Canonical Partition-Function for Quons," J. W. Goodison and D. J. Toms, Phys. Lett. A **195**(1), 38–42 (1994). Shows how the partition function is changed if the particles are described by a quon algebra. (A)

108. "Interpolation between the wave and particle properties of bosons and fermions," Richard A. Campos, Phys. Lett. A **184**, 173–178 (1994). Examines how two quon fields mix at beam splitters and interferometers. (I)

109. "The Existence of Antiparticles Seems to Forbid Violations of Statistics," A. B. Govorkov, Physica A **203**(3-4), 655–670 (1994). Argues that relativistic quantum field theory plus antiparticles rules out everything except the ordinary spin-statistics connection. (A)

110. "(Para)Bosons, (Para)Fermions, Quons, and Other Beasts in the Menagerie of Particle Statistics," O. W. Greenberg, D. M. Greenberger, and T. V. Greenbergest, in **Quantum Coherence and Reality**, edited by J. S. Anandan and J. L. Safko (World Scientific, Singapore, 1994), pp. 301–312. General survey of the issues of parastatistics and quons. (I)

111. "Classical-Model of Intermediate Statistics," G. Kaniadakis, Phys. Rev. E **49**(6), 5111–5116 (1994). Proposes a classical model that produces Fermi-Dirac, Bose-Einstein, and quon statistics in various limits. (I)

112. "Conservation of Statistics and Generalized Grassmann Numbers," O. W. Greenberg, Phys. Lett. A **209**(3-4), 137–142 (1995). Conservation of statistics in standard quantum field theory requires that fermions be coupled to external sources described by non-commuting (Grassmann) numbers. For parabosons, parafermions, and quons, the external sources must be described by generalizations of Grassmann numbers. (A)

113. "Superfluidity of Repulsing "Quasifermions"," I. Lutzenko and A. Zhedanov, Phys. Rev. Lett. **74**(18), 3507–3510 (1995). Uses a q-mutator relationship with $q > 1$. For a gas of such particles, the behaviour is like that of a Fermi gas but it may exhibit a type of superfluidity. (A)

114. "Small Violations of Statistics," O. W. Greenberg, in **Orbis Scientiae 1998 Confluence of Cosmology, Massive Neutrinos, Elementary Particles, and Graviation**, edited by B. Kursunoglu, S. Mintz and A. Perlmutter (Plenum Press, New York, 1999), pp. 173–194. Surveys experiments designed to test the spin-statistics connection and then describes various theoretical formalisms that describe violations of spin-statistics. (I)

115. "Quons as su(2) irreducible tensor operators," S. S. Avancini, F. F. D. Cruz, J. R. Marinelli, and D. P. Menezes, Phys. Lett. A **267**(2-3), 109–112 (2000). The creation and annihilation operators obeying the q-mutator formalism can be treated as tensor operators. (A)

116. "Theories of Violation of Statistics," O. W. Greenberg, in Ref. 1, pp. 113–127. Excellent and authoritative overview of theories that go beyond the standard spin-statistics connection. (I)

117. "Quons in Relativistic Theories Must be Bosons or Fermions," Chi-Keng Chow and O. W. Greenberg, Phys. Lett. A **283**, 20–24 (2001). Argues that the combination of relativity and the existence of antiparticles rules out everything except standard bosons and fermions. (A)

118. "Construction of bosons and fermions out of quons," O. W. Greenberg and J. D. Delgado, Phys. Lett. A **288**(3), 139–144 (2001). Shows how to construct boson and fermion creation and annihilation operators from quon operators. (I)

## D. Quons and composite systems

119. "Quon Statistics for Composite Systems and a Limit on the Violation of the Pauli Principle for Nucleons and Quarks," O. W. Greenberg and Robert C. Hilborn,

Phys. Rev. Lett. **83**(22), 4460–4463 (1999). For a system of $N$ identical particles described by the quon algebra, the parameter $q_c$ for the composite is related to the parameter $q$ for the constituents by the relationship $q_c = q^{N^2}$. This relationship can be used to find the limits on the violations of the spin-statistics connection by the constituents from measurements made on the composite system. (I)

120. "Applications of the quon algebra: 3-D harmonic oscillator and the rotor model," S. S. Avancini, J. R. Marinelli, and C. E. de Rodrigues, Phys. Lett. A **297**(3), 137–145 (2002). The quon algebra is applied to composite systems. (I)

121. "Compositeness effects in the Bose-Einstein condensation," S. S. Avancini, J. R. Marinelli, and G. Krein, J. Phys. A **36**(34), 9045–9052 (2003). The quon algebra is applied to composite systems restricted to symmetric state spaces with applications to Bose-Einstein Condensation. (I)

122. "Quons restricted to the antisymmetric subspace: Formalism and applications," S. S. Avancini, F. F. de Souza Cruz, J. R. Marinelli, D. P. Menezes, and M. M. Watanabe Morales, Phys. Lett. A **307**(4), 202–208 (2003). The quon algebra is applied to composite systems restricted to anti-symmetric state spaces with applications to fermion pairing. (I)

## E. Statistical mechanics and nonstandard particle statistics

Quantum statistical mechanics—the application of Fermi-Dirac and Bose-Einstein statistics to systems of many particles–plays a crucial role in our understanding of the structure of matter. The standard treatments are well covered in the many books on statistical mechanics. Here, we mention only a few papers that discuss what happens in statistical mechanics when the spin-statistics connection is brought into question.

123. "Osservazioni sopra le Statistiche Intermedie," G. Gentile, Il Nuovo Cimento **17**, 493–497 (1940). An attempt to formulate the statistical mechanics of particles intermediate between fermions and bosons by limiting the number of particles occupying a given state. (I)

124. "Gentile's intermediate statistics," D. ter Haar, Physica **XVIII**(3), 199–200 (1952). Argues that Gentile's approach at formulating "intermediate statistics" fails and that it reduces to Bose-Einstein statistics. (I)

125. "Thermodynamical Implications of a Violation of the Pauli Principle," V. Rahal and A. Campa, Phys. Rev. A **38**(7), 3728–3731 (1988). Presents a method for introducing a small violation of the Pauli exclusion principle and working out the grand partition function for the corresponding free-electron gas in a metal. (I)

126. "Thermodynamics of Parabosonic and Parafermionic Systems of Order Two," Peter Suranyi, Phys. Rev. Lett. **65**(19), 2329–2330 (1990). The thermal properties of a gas of order-two parabosons or parafermions (at most two particles per quantum state of the "wrong" symmetry) is equivalent to a mixture of two gases of ordinary bosons or fermions. (I)

127. "Statistics of q-oscillators, quons, and relations to fractional statistics," M. Chaichian, R. Gonzalez Felipe, and C. Montonen, J. Phys. A **26**, 4017–4034 (1993).

Discussion of the relationships of the statistical mechanics of particles described by nonstandard statistics of various kinds. (A)

128. "Quon gas with the boson, fermion, and near-classical limits," A. Inomata, Phys. Rev. A **52**(2), 932–935 (1995). A quon gas model interpolates between a boson gas and a fermion gas. (A)

129. "Classical-Model of Boson and Fermions," G. Kaniadakis and P. Quarati, Phys. Rev. E **49**(6), 5103–5110 (1994). Proposes a classical kinetics model that interpolates between the statistical mechanics of bosons and that of fermions. (I)

130. "Spectral distribution of photons admitting anomalous statistics," C. Wolf, Nuovo Cim. B–Gen. Phys. Relativ. Astron. Math. Phys. Methods **110**(12), 1481–1485 (1995). Calculates the thermal-radiation spectrum for photons whose statistics differ by a small amount from Bose-Einstein statistics. There may be some possibility of seeing a difference in the measurements of the Cosmic Microwave Background. (I)

131. "Properties of Particles Obeying Ambiguous Statistics," M. V. Medvedev, Phys. Rev. Lett. **78**(22), 4147–4150 (1997). Describes a model in which particles may exhibit both Bose and Fermi statistics with different probabilities. The probabilities are related to the $q$ parameter of the q-mutator. The partition function is derived and applied to an ideal gas and to extremal black holes. (A)

## F. Geometric phase and spin-statistics

132. "Quantum Indistinguishability: Spin-Statistics without Relativity or Field Theory?" Michael Berry and Jonathan Robbins, in Ref. 1, pp. 3–15. An elaboration of the geometric phase approach to connecting spin and statistics. See also Ref. 73. (I)

## G. Supersymmetry and spin-statistics

Supersymmetry links fermions and bosons. Under supersymmetry for each fermion, there is a bosonic partner (with a different mass), and vice versa for bosons.

133. "Spin-statistics connection and supersymmetry," F. Iachello. Ref. 1, pp. 179–189. Describes how supersymmetry can be used to understand states in nuclei, in which pairs of fermions act like bosons. The paper also discusses the spin-statistics connection implications for supersymmetry.

## H. Spin-statistics and additional dimensions

Adding spatial dimensions to our standard three has a long history in attempts to generalize theories to include gravity with other fundamental forces. Several papers explore the role of the spin-statistics connection in these higher-dimensional theories.

134. "Causal Fields and Spin-Statistics Connection for Massless Particles in Higher Dimensions," N. Ohta, Phys. Rev. D **31**(2), 442–445 (1985). See also Ref. 61. (A)

135. "Relativistic spin-statistics connection and Kaluza-Klein space-time," J. Anandan, Phys. Lett. A **248**(2-4),

124–130 (1998). The formalism of Berry and Robbins (Ref. 73) for the Spin-statistics connection is made relativistic and extended to larger number spacetime dimensions. (A)

**136.** "The Spin-Statistics Theorem in Arbitrary Dimensions," Luis J. Boya and E. C. G. Sudarshan, Int. J. Theor. Phys. **46**(12), 3285–3293 (2007). Examines the spin-statistics connections in theories in various space dimensions and finds the usual spin-statistics connection only for spatial dimension $= 8n + 3$, $8n + 4$, and $8n + 5$, where $n = 0, 1, 2, \ldots$. (A).

## I. Noncommutative space-time

Another approach to generalized theories makes space and time coordinates into noncommuting quantities. The essential idea is to generalize the usual d-dimensional space-time, in which time and space coordinates all commute, to a space in which the space-time coordinates are represented by operators that no longer commute. The space-time coordinates satisfy $[\hat{X}_\mu, \hat{X}_\nu] = i\theta_{\mu\nu} = ic_{\mu\nu}/\Lambda_{NC}^2$, where $\Lambda_{NC}$ sets the scale at which noncommutative effects become important. $\theta_{\mu\nu}$ is a constant matrix. Products of operators that are functions of the space-time coordinates must be expressed in terms of so-called *-products. Because these noncommutative field theories violate Lorentz invariance at high energies, they are expected to have an effect on the spin-statistics connection.

**137.** "Noncommutative quantum mechanics," H. O. Girotti, Am. J. Phys. **72**(5), 608–612 (2004). Provides a nice introduction to noncommutative spacetime in nonrelativistic quantum mechanics. (A)

**138.** "On a Lorentz-invariant interpretation of noncommutative space–time and its implications on noncommutative QFT," M. Chaichian, P. P. Kulish, K. Nishijima, and A. Tureanu, Phys. Lett. B **604**(1), 98–102 (2004). Introduces the concept of twisted Poincaré symmetry and its use in formulating noncommutative quantum field theories. (A)

**139.** "New concept of relativistic invariance in noncommutative space-time: twisted Poincaré symmetry and its implications," M. Chaichian, P. Presnajder, and A. Tureanu, Phys. Rev. Lett. **94**(15), 151602 (2005). If Poincaré symmetry is replaced by a "twisted" Poincaré symmetry, noncommutative quantum field theories allow for the classification of particles according to representations of the twisted Poincaré symmetry with a corresponding spin-statistics theorem. (A)

**140.** "Non-Pauli transitions from spacetime noncommutativity," A. P. Balachandran, A. Joseph, and P. Padmanabhan, Phys. Rev. Lett. **105**(5), 051601 (2010). A "twist" in noncommutative space-time owing to Earth's rotation or cosmic motion can induce a boson-like component in a fermionic state and allow "non-Pauli" atomic or nuclear transitions. (A)

## J. Anyons and related issues

When particles are confined to two spatial dimensions, the permutation-group symmetry for identical-particle systems is replaced with the so-called braid group in which topologically inequivalent trajectories leading to the same permutation of particles (or particle labels) are distinguished. Since the particles can have any statistics, they are called anyons. This concept (without the name "anyon") was noted in Ref. 39.

**141.** Quantum statistics of ideal gases in two dimensions," Robert M. May, Phys. Rev. **135**, A1515–A1518 (1964). Shows that the specific heats for ideal Fermi gases and ideal Bose gases are in the same in two dimensions. (A)

**142.** "Quantum mechanics of fractional-spin particles," F. Wilczek, Phys. Rev. Lett. **49**(14), 957–959 (1982). One of the seminal papers that brought anyons to the attention of the physics community. (I)

**143.** "Linking numbers, spin, and statistics of solitons," F. Wilczek and A. Zee, Phys. Rev. Lett. **51**(25), 2250–2252 (1983). Further development of the idea of anyons. (I)

**144.** "Anyons," F. Wilczek, Sci. Am. **264**(5), 58–65 (1991). Introduction to anyons within the context of the general question of quantum statistics. (E)

**145.** "'Fractional statistics' in arbitrary dimensions: a generalization of the Pauli principle," F. D. M. Haldane, Phys. Rev. Lett. **67**(8), 937–940 (1991). A different way of thinking about generalized ("fractional") statistics. Shows how to generalize the Pauli exclusion principle for condensed matter quasi-particles in a method that applies to any number of spatial dimensions. (A)

**146.** "Ideal quantum gases in two dimensions," S. Viefers, F. Ravndal, and T. Haugset, Am. J. Phys. **63**(4), 369–376 (1995). A nice introduction to the thermodynamics properties of nonrelativistic bosons and fermions in two spatial dimensions and to the free anyon gas. (I)

## K. String theory and spin-statistics

String theory in its various forms might lead to violations of the standard spin-statistics connection. For example, in higher dimensional space-times, the exclusion principle might be valid, but it may appear to be violated when viewed from our traditional four-dimensional space-time because we do not "see" the degrees of freedom associated with the additional dimensions. Also, string theory replaces the usual point interactions of quantum field theory with extended objects (and interactions), thus altering the locality principle, which seems to be required for most quantum field theory "proofs" of the spin-statistics connection.

**147.** "Expectation values, Lorentz invariance, and CPT in the open bosonic string," V. A. Kostelecký and R. Potting, Phys. Lett. B **381**, 89–96 (1996). Further evidence that string theories may violate Lorentz invariance and CPT conservation, leading to possible spin-statistics violations. (A)

**148.** "Spin-statistics violations from heterotic string worldsheet instantons," M. G. Jackson, Phys. Rev. D **77**(12), 127901 (2008). Describes how one version of string theory leads to violations of the spin-statistics connection. (A)

**149.** "Spin-Statistics Violations in Superstring Theory," Mark G. Jackson. Phys. Rev. D **78**, 126009 (2008). Describes how superstring theory may violate the spin-statistics connection in an observable way. (A)

## VI. EXPERIMENTAL TESTS OF THE SPIN-STATISTICS CONNECTION

### A. General issues on experimental tests

Testing the spin-statistics connection turns out to be more subtle than one might think. In particular, there are restrictions

on the types of transitions allowed in quantum mechanics for identical-particle systems whether or not the spin-statistics connection holds. For example, as long as the general framework of quantum mechanics holds true, transitions from a symmetric state (in a system of identical particles) to an antisymmetric state (or vice versa) is absolutely forbidden (as long as the number of identical particles does not change). Dirac notes this fact in his book on quantum mechanics: "*..a state which is initially symmetrical* [under permutation of identical particles] *always remains symmetrical and a state which is initial antisymmetrical always remains antisymmetrical.*" (italics in the original). (Ref. 11, p. 209) Messiah and Greenberg (Ref. 42) explore the formulation of the symmetrization postulate and note that that there were at the time (1964) very few high precision tests of it. They argue that violations of the symmetrisation postulate only reflect new quantum numbers.

150. "Comments on testing the Pauli principle," R. D. Amado and H. Primakoff, Phys. Rev. C **22**(3), 1338–1340 (1980). Points out that many experiments that purport to test the Pauli exclusion principle in fact do not because they look for transitions from states of one permutation symmetry (usually antisymmetric) to states of different permutation symmetry. Asserts that at best the experiments are tests of electron or nucleon stability. (I)

151. "Some Possible Tests of the Inapplicability of Pauli Exclusion-Principle," Y. F. Chang, Hadronic J. **7**(6), 1469–1473 (1984). Lists and briefly discusses six situations in which the author believes the Pauli exclusion principle might be violated at high energies, including in highly excited states of atoms, in atoms under strong fields, and in black holes. (I)

152. "Possible Violation of the Pauli principle in atoms," L. B. Okun, JETP Lett. **46**, 529–532 (1987). Presents a model of violations of the Pauli principle, but the model has difficulties when including the electromagnetic interaction; the author calls for experiments in atoms. (I)

153. "Phenomenology of small violations of Fermi and Bose statistics," O. W. Greenberg and R. N. Mohapatra, Phys. Rev. D **39**(7), 2032–2038 (1989). Provides an excellent overview of the tests of the spin-statistics connection and symmetrization postulate up to the late 1980s. Also discusses the phenomenology of violations of the spin-statistics connection as a guide for deciding among possible future experiments. (I)

154. "A Remark On Possible Violations of the Pauli Principle," L. C. Biedenharn, P. Truini, and H. Van Dam, J. Phys. A **22**(3), L67–L71 (1989). Argues that any apparent violation of the Pauli principle must be interpreted as an indication of new degrees of freedom. (A)

155. "Comments on testing charge conservation and the Pauli exclusion principle," L. B. Okun, Comments Nucl. Part. Phys. **19**(3), 99–116 (1989). Reviews the literature of the past 30 yr on tests of charge conservation and the exclusion principle. Points out the difficulty of formulating a theory of those violations. (I)

156. "A Review of the Experimental Tests of Quantum-Mechanics," S. K. Lamoreaux, Int. J. Mod. Phys. A **7**(27), 6691–6762 (1992). An overview of a wide range of experiments testing various aspects of quantum mechanics. The Pauli exclusion principle and the symmetrization postulate tests are discussed in Section 5.10 (pp. 6737–6740). (I)

157. "Note on Testing Charge Conservation and the Pauli Exclusion Principle," L. B. Okun, Phys. Rev. D **45**, VI.10–14 (1992). Condensed and edited from the author's review in Ref. 152. A survey of experiments designed to test the Pauli principle. (I)

158. "The Relation of Constraints on Particle Statistics for Different Species of Particles," O. W. Greenberg and Robert C. Hilborn, Found. Phys. **29**, 397–407 (1999). Shows how experimental limits on the violation of spin-statistics for one type of particle can be interpreted as setting limits on the violation of spin-statistics for other particles with which the first type interacts. (I-A)

159. "Connecting q-mutator Theory to Experimental Tests of the Spin-Statistics Connection," Robert C. Hilborn, in Ref. 1, pp. 128–141. Shows how the q-mutator formalism provides a framework for comparing the sensitivity of different experimental tests of the spin-statistics connection. (I)

160. "High precision experimental tests of the symmetrization postulate for fermions," J. D. Gillaspy in Ref. 1, pp. 241–252. Good survey of various experiments testing the symmetrization postulate for fermions and includes a previously unpublished limit based on the decay of spin-polarized atomic hydrogen in strong magnetic fields. (I)

## B. Electrons (bound and free)

161. "Identification of Beta-Rays with Atomic Electrons," M. Goldhaber and G. Scharff-Goldhaber, Phys. Rev. **73**, 1472–1473 (1948). An experiment to answer the question: Are beta-decay electrons identical to (in the quantum-mechanical sense) ordinary atomic electrons in bulk matter? The experiment looked for X rays that would be emitted if the beta-ray electrons fell into already occupied states in a solid. (I)

162. "Experimental Limits on the Existence of Anomalous Electrons," E. Fischbach, T. Kirsten, and O. A. Schaeffer, Phys. Rev. Lett. **20**(18), 1012 (1968). Reports a search for Be atoms with two ordinary electrons and two "false" electrons, all in the $K$ shell. Alternatively, it can be interpreted as setting a limit on the violation of the exclusion principle. (I)

163. "Test of the Pauli Exclusion Principle for Atomic Electrons," F. Reines and W. H. Sobel, Phys. Rev. Lett. **32**(17), 954 (1974). Uses the data from Moe and Reines in 1965 to infer that the ratio of the strength of a Pauli-principle-violating transition to the strength of an electromagnetic dipole transition to the $K$ shell of an atom is $< 3 \times 10^{-44}$. However, their analysis looks for a transition from an original antisymmetric state to one that violates the Pauli exclusion principle. Such a transition is absolutely forbidden for quantum mechanics as we know it. See Ref. 150. (I)

164. "Search for small violation of the Pauli principle," A. Gavrin, A. Yu. Ignatiev, and V. A. Kuzmin, Phys. Lett. B206, 343–345 (1988). Suggests extracting and counting atoms with wrong (excessive) filling of the electron shells as a test of the Pauli exclusion principle. (I)

165. "Predicted energy shifts for "paronic" helium," G. W. F. Drake, Phys. Rev. A **39**, 897–899 (1989). A calculation of electron energy levels in atomic helium if the two

electrons are in a permutation symmetric state. The results guided an experiment to search for symmetric states in helium (Ref. 172). The symmetric-state transition is shifted about 500 MHz from the normal $2^3S_1$ to $3^3P_1$ transition. (I)

166. "Experimental test of a possible violation of the Pauli principle," V. M. Novikov and A. A. Pomanskii, Pis'ma Zh. Eksp. Teor. Fix. **49**(2), 81–83 (1989). A proposal to test the Pauli exclusion principle for electrons by looking for "anomalous" carbon atoms (with three electrons in the $K$ shell) in a large sample of boron. (I)

167. "Ultra-High Sensitive Study of the Identity of Beta-Rays With Atomic Electrons," X. W. Tang, Y. G. Wu, S. Zhang, Q. W. Chen, and X. Q. Xie, Phys. Lett. B **252**(2), 301302 (1990). Used a high-intensity beta-ray source and a sensitive X ray detector to set a limit of $10^{-13}$ as the fraction of captured beta rays that lead to transitions into the $K$ shell and associated emission of X rays. (I)

168. "Test of the Pauli Exclusion-Principle for Atomic Electrons," V. M. Novikov, A. A. Pomansky, T. Faestermann et al., Phys. Lett. B **240**(1-2), 227–231 (1990). Accelerator mass spectrometry was used to look for anomalous $^{20}$Ne atoms in a sample of fluorine. The ratio of anomalous Ne to normal Ne was found to be $< 2 \times 10^{-21}$. (I)

169. "Test of the Pauli Principle," V. M. Novikov, E. Nolte, and A. A. Pomanskii, JETP Lett. **51**(1), 1–3 (1990). Another report on the experiment described in Ref. 168. (I)

170. "Experimental Limit on a Small Violation of the Pauli Principle," E. Ramberg and G. A. Snow, Phys. Lett. B **238**, 438–441 (1990). A high electrical current running through solid copper fails to yield X rays from "new" electrons falling into already occupied states. The probability that a "new" electron entering the copper would form a mixed symmetry state with respect to the electrons already in the copper is $< 1.7 \times 10^{-26}$. (I)

171. "Test of the Pauli Exclusion-Principle for Nucleons and Atomic Electrons By Accelerator Mass-Spectrometry," E. Nolte, T. Faestermann, A. Gillitzer et al., Z. Phys. A:Hadrons Nucl. **340**(4), 411–413 (1991). A proposal to look for atoms that show accumulation of particles in low-energy states exceeding the limit indicated by the exclusion principle. These atoms would have energies (and masses) different from "ordinary" atoms. (I)

172. "Search for Small Violations of the Symmetrization Postulate in an Excited State of Helium," K. Deilamian, J. D. Gillaspy, and D. E. Kelleher, Phys. Rev. Lett. **74**(24), 4787–4790 (1995). A search for transitions between symmetric states (violating the Pauli exclusion principle) in atomic helium. (I)

173. "Search for non-Paulian transitions in Na-23 and I-127," R. Bernabei, P. Belli, F. Montecchia et al, Phys. Lett. B **408**(1-4), 439–444 (1997). This paper describes experiments setting limits on the rates of Pauli exclusion-principle-violating transitions (nucleon transitions into already occupied states) in sodium and iodine nuclei. (I)

174. "Testing the Pauli exclusion principle with the NEMO-2 detector," R. Arnold, C. Augier, J. Baker et al., Eur. Phys. J. A **6**, 361–366 (1999). A search for gamma rays emitted from Pauli-forbidden nuclear transitions in 120 kg of carbon in 50 scintillators over the course of

more than a year. The 45 coauthors are from 11 different institutions in 5 countries. (I)

175. "New experimental test of the Pauli exclusion principle using accelerator mass spectrometry," D. Javorsek II, M. Bourgeois, D. Elmore et al., Phys. Rev. Lett. **85**(13), 2701–2704 (2000). A search for Be atoms with all four electrons in the $K$ shell sets an upper limit of "anomalous" Be to normal Be of about $10^{-11}$. (I)

176. "Testing the atomic structure of beryllium with AMS," D. Javorsek II, D. Elmore, E. Fischbach et al., Nucl. Instrum. Methods Phys. Res. Sec. B **194**(1), 78–89 (2002). The Purdue Rare Isotope Measurement Laboratory was modified and deployed to search for beryllium gas with four electrons in the 1s ground state. Samples studied included air, helium obtained from a natural gas field, and the gas liberated from dissolved Be metal and beryl (ore). The measurement assumed that all of the exotic Be was converted into ordinary Be by collisions with $Cs^+$ ions in the apparatus and was sensitive to concentrations at the part-per-billion level. (I)

177. "New experimental limit on the Pauli exclusion principle violation by electrons," S. Bartalucci, S. Bertolucci, M. Bragadireanu et al., Phys. Lett. B **641**(1), 18–22 (2006). An improved version of the kind of experiment described in Ref. 170 shows that the probability of an electron making a transition into an already-filled $K$ shell is $< 4.5 \times 10^{-28}$. (I)

178. "X rays test the Pauli exclusion principle," A. Yu Ignatiev, Radiat. Phys. Chem. **75**(11), 2090–2096 (2006). Outlines the theoretical ideas lying behind tests of the Pauli exclusion principle and provides a survey of the different kinds of experiments used in these tests. (I)

179. "New search for processes violating the Pauli exclusion principle in sodium and in iodine," R. Bernabei, P. Belli, F. Cappella et al., Eur. Phys. J. C **62**(2), 327–332 (2009). A search for Pauli-forbidden nuclear and atomic transitions in 250 kg of highly radiopure NaI(Tl) in 25 scintillators at Gran Sasso National Laboratory (the world's largest underground particle-physics laboratory). Twenty-four detectors were used to search for nuclear decays for a period of 24 days. Results for searching for atomic transitions over much longer times were also interpreted as an upper bound on the size of the electron (energy scale $>3.5$ TeV). (I)

180. "VIP Experiment: New Experimental Limit on Pauli Exclusion Principle Violation by Electrons," D. Pietreanu, S. Bartalucci, S. Bertolucci et al., Int. J. Mod. Phys. A **24**(2), 506–510 (2009). A result of the VIP experiment (Ref. 177), obtained by running the setup at the Gran Sasso underground laboratory, shows that the probability of an electron making a transition in copper into an already-filled $K$ shell is $<6 \times 10^{-29}$. (I)

181. "The VIP Experimental Limit on the Pauli Exclusion Principle Violation by Electrons," S. Bartalucci, S. Bertolucci, M. Bragadireanu et al., Found. Phys. **40**(7), 765–775 (2010). The limit of $<6 \times 10^{-29}$ (Ref. 180) is discussed in the framework of quon-theory. A future major upgrade of the experiment is presented. (I)

182. "New Experimental Limit on the Pauli Exclusion Principle Violation by Electrons-The VIP Experiment," C. Curceanu (Petrascu), S. Bartalucci, S. Bertolucci et al., Found. Phys. **41**(3), 282–287 (2011). An improved limit of the VIP experiment with respect to the one

reported in Ref. 180 is given, together with a discussion of some conceptual and interpretational problems involved. (I)

183. "An Improved Limit on Pauli-Exclusion-Principle Forbidden Atomic Transitions," S. R. Elliott, B. H. LaRoque, V. M. Gehman, M. F. Kidd, and M. Chen, Found. Phys. (accepted for publication 2012, available at arXiv:1107.3118v2 [nucl-ex]. Provides new results a detailed analysis and comparison of experiments that look for X rays emitted from conductors with electrical current flowing through them—generalizations of the experiment described in Ref. 170. (I)

## C. Nuclei and nucleons

184. "Validity of the Pauli Exclusion Principle for Nucleons," B. A. Logan and A. Liubicic, Phys. Rev. C **20**, 1957–1958 (1979). The mean lifetime for p-shell nucleons in $^{12}C$ to fall into the fully occupied $1s_{1/2}$ shell is $>6 \times 10^{27}$ s. However, this experiment looks for transitions between states of different permutation symmetry, which are absolutely forbidden by quantum mechanics. See Ref. 150. (I)

185. "Violations of the Pauli Principle and the Interior of the Sun," R. Plaga, Z. Phys. A: Hadrons Nucl. **333**(4), 397–403 (1989). Uses the hydrogen burning rate in the Sun to set a limit on finding two protons in a symmetric state. (I)

186. "An upper limit to violations of the Pauli exclusion principle," D. Kekez, A. Ljubicic, and B. A. Logan, Nature **348**, 224 (1990). An interpretation of beta-decay data in $^{12}C$ that gives the ratio of Pauli-violating transitions to normal transitions as $<6.5 \times 10^{-34}$. (I)

187. "Search for Violation of the Pauli Principle in Nuclear Decays," D. Kekez, A. Ljubicic, S. Kaucic, and B. A. Logan, Nuovo Cimento Soc. Ital. Fis., A **104**(4), 607–609 (1991). A 10-day search for gamma rays from forbidden nuclear decays in $^{71}Ge$ (half-life 11 days); also searched for forbidden nuclear decay in NaI detector by looking for proton-neutron emission (events detected, but attributed to background). (I)

188. "Limits on small violations of the Pauli exclusion principle in the primordial nucleosynthesis," M. H. Thoma and E. Nolte, Phys. Lett. B **291**, 484–487 (1992). Uses limits on $^5Li$ with three protons in the lowest nuclear level from primordial nucleosynthesis to set a limit on the probability of finding two protons in the same state, $<2 \times 10^{-28}$. (I)

189. "Search for Violation of the Pauli Principle Through Spontaneous Neutron Emission From Lead," T. Kishimoto, T. Shibata, M. Imamura, S. Shibata, and Y. Uwamino, J. Phys. G: Nucl. Part. Phys. **18**(2), 443–448 (1992). A search for spontaneously produced holes in nuclei that may occur because of either a violation of the Pauli exclusion principle or the exotic decay of a nucleon. (I)

190. "Test of the Symmetrization Postulate for Spin-0 Particles," M. de Angelis, G. Gagliardi, L. Gianfrani, and M. Tino, Phys. Rev. Lett. **76**(16), 2840–2843 (1996). This and the following paper use high-sensitivity laser spectroscopy of diatomic oxygen to look for states violating the spin-statistics connection for $^{16}O$ nuclei. The experimental limit is $<5 \times 10^{-7}$. (I)

191. "Spectroscopic Test of the Symmetrization Postulate for Spin-0 Nuclei," Robert C. Hilborn and Candice L. Yuca, Phys. Rev. Lett. **76**(16), 2844–2847 (1996). (I)

192. "$b^1\Sigma_g^+ - X^3\Sigma_g$ (0,0) band of oxygen isotopomers in relation to tests of the symmetrization postulate in $^{16}O_2$," H. Naus, A. de Lange, and W. Ubachs, Phys. Rev. A **56**(6), 4755–4763 (1997). A precision spectroscopy experiment on the transitions in molecular oxygen used to test the symmetrization postulate for $^{16}O$ nuclei. (I)

193. "Search for anomalous carbon atoms—evidence of violation of the Pauli principle during the period of nucleosynthesis," A. S. Barabash, V. N. Kornoukhov, Yu M. Tsipenyuk, and B. A. Chapyzhnikov, JETP Lett. **68**(2), 112–116 (1998). Search for $^{12}C$ atoms with three $K$ shell electrons. The ratio of anomalous $^{12}C$ atoms to normal $^{12}C$ atoms is $<2.5 \times 10^{-12}$. (I)

194. "Experimental test of the possible violation of the Pauli exclusion principle by photoactivation analysis of carbon content in pure boron," Y. M. Tsipenyuk, A. S. Barabash, V. N. Kornoukhov, and B. A. Chapyzhnikov, Radiat. Phys. Chem. **51**(4-6), 507–511 (1998). Radiochemical realization of the proposal of Novikov and Pomansky in 1988 to search for anomalous atoms with carbon nuclei but boron-like outer electron shells. (I)

195. "Search for Small Violations of the Symmetrization Postulate for Spin-0 Particles," G. Modugno, M. Inguscio, and G. M. Tino, Phys. Rev. Lett. **81**(22), 4790–4793 (1998). Molecular spectroscopy of $CO_2$ yields a low limit on the violations of the spin-statistics connection for oxygen nuclei. The probability of finding the two $^{16}O$ nuclei in an antisymmetric state is $<2.1 \times 10^{-9}$. (I)

196. "Cavity-Enhanced Absorption Spectroscopy of Molecular Oxygen," L. Gianfrani, R. W. Fox, and L. Hollberg, J. Opt. Soc. Am. B **16**(12), 2247–2254 (1999). Describes a number of experimental enhancements in the optical spectroscopy of $O_2$ that lead to measurements that indicate that the probability of finding the molecule in a permutation antisymmetric state for the two $^{16}O$ nuclei is $<5 \times 10^{-8}$. (I)

197. "Limits on Pauli Principle Violation by Nucleons," E. Baron, R. N. Mohapatra, and V. L. Teplitz, Phys. Rev. D **59**, 036003 (1999). Proposal to look for anomalous iron isotopes and anomalous cobalt isotopes to set limits on the violation of the Pauli principle for neutrons and protons, respectively. (I)

198. "Testing the Pauli exclusion principle with the NEMO-2 detector," R. Arnold, C. Augier, J. Baker *et al.*, Eur. Phys. J. A **6**, 361–366 (1999). An experiment describing a search for Pauli exclusion principle-violating transitions in $^{12}C$ in the underground NEMO-2 detector. The half-life for transitions to the fully occupied $1s_{1/2}$ shell is greater than $4.2 \times 10^{24}$ y. (I)

199. "Testing the symmetrization postulate of quantum mechanics and the spin-statistics connection," G. M. Tino, Fortschr. Physik **48**, 537–543 (2000). Proposes using polyatomic molecules with identical nuclei to look for violations of the symmetrization postulate, that is, looking for states that are neither symmetric nor antisymmetric. (I).

200. "The Pauli principle and ultrahigh resolution spectroscopy of polyatomic molecules," C. J. Bordé and C. Chardonnet, Ref. 1, pp. 274–280. Describes preliminary

experimental results for tests of the symmetrization postulate using polyatomic molecules. (I)

201. "Spectroscopic Tests of the Symmetrization Postulate and of the Statistics for Nuclei in Molecules," G. Modugno, D. Mazzotti, M. Modugno *et al.*, Ref. 1, pp. 295–301. Reports on refined versions of the $O_2$ and $CO_2$ molecular spectroscopy tests of the spin-statistics connection. (I)

202. "Search for Exchange-Antisymmetric States for Spin-0 Particles at the $10^{-11}$ Level," D. Mazzotti, P. Cancio, G. Giusfrede, M. Inguscio, and P. De Natale, Phys. Rev. Lett. **86**(10), 1919–1922 (2001). Molecular spectroscopy of the strong 4.25 $\mu$m rotational-vibrational band in $CO_2$ yields an upper limit of $1.7 \times 10^{-11}$ for the probability of finding the molecule in a state of permutation antisymmetry for the two $^{16}O$ nuclei. (I)

203. "Searches for processes violating the Pauli Exclusion Principle in sodium and iodine," R. Bernabei, P. Belli, F. Cappella *et al.*, J. Phys.: Conf. Ser., **202**(1), 012039 (2010). Searches for nucleon transitions into already occupied states. (I)

204. "Non-paulian Nuclear Processes in Highly Radiopure NaI(Tl): Status and Perspectives," R. Bernabei, P. Belli, F. Cappella *et al.*, Found. Phys. **40**(7), 807–813 (2010). Another report on the experiments described in Ref. 203. (I)

205. "Experimental test of the Pauli Exclusion Principle," A. S. Barabash, Found. Phys. **40**(7), 703–718 (2010). Summary of three recent searches for violations of the exclusion principle: (1) anomalous carbon atoms, (2) p-shell nucleons in $^{12}C$, and (3) neutrinos in double beta decay. (I)

206. "New experimental limits on the Pauli forbidden transitions in C-12 nuclei obtained with 485 days Borexino data," G. Bellini *et al.*, Phys. Rev. C **81**, 034317 (2010). Review of several tests of the Pauli exclusion principle: a search for anomalous carbon atoms, Pauli-principle-violating transitions in carbon nuclei, and bosonic neutrinos that would drastically affect double beta-decay rates. (I)

## D. Neutrinos

207. "Neutrino statistics and big bang nucleosynthesis," A. D. Dolgov, S. H. Hansen, and A. Yu. Smirnov, J. Cosmology, Astropart. Phys. **2005**(06), 004 (2005). Argues that relative abundances in big bang nucleosynthesis can be used to determine the statistical character of neutrinos. However, current inconsistencies between $^4$He and $^2$H abundances do not rule out even pure bosonic statistics for neutrinos. See also Ref. 53. (I)

## E. Photons

Photons are spin-1 particles and hence should behave as bosons, and their states should be symmetric if the spin-statistics connection and the symmetrization postulate apply to them.

208. "Selection Rules for the Dematerialization of a Particle into Two Photons," C. N. Yang, Phys. Rev. **77**(2), 242–245 (1950). Proves a famous theorem in particle physics: a spin-1 particle cannot decay into two identi-

cal photons if the photons are pure bosons. The result is sometimes called the Landau-Yang theorem: See also L. D. Landau, Dokl. Akad. Nauk **60**, 207–209 (1948). (I)

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210. "Experimental Limit On the Blue-Shift of the Frequency of Light Implied By a Q-Nonlinearity," V. I. Man'ko and G. M. Tino, Phys. Lett. A **202**(1), 24–27 (1995). Shows that laser-output frequency is independent of intensity, thereby setting a limit on a proposed nonlinearity that would violate the spin-statistics connection. (I)

211. "The search for the decay of the Z boson into two photons as a test of Bose statistics," A. Yu Ignatiev, G. C. Joshi, and M. Matsuda, Mod. Phys. Lett. A **11**, 871–876 (1996). A decay of a spin-1 particle into two identical bosons is forbidden by the famous Yang-Mills theorem if the decay products are "pure" bosons. (I)

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213. "A proposal for testing symmetrization postulate for photons via Raman-atom-field interaction," Shi Bao Zheng, Opt. Commun. **155**(4-6), 288–291 (1998). Proposal to use cavity QED to test the symmetrization postulate, an extension of the proposal of Gerry and Hilborn (Ref. 212). (I)

214. "Search for exchange-antisymmetric two-photon states," D. DeMille, D. Budker, N. Derr, and E. Deveney, Phys. Rev. Lett. **83**(20), 3978–3981 (1999). An experiment looking for an atomic transition that is forbidden if photons are in purely symmetric states. The atomic equivalent of the Yang theorem (Ref. 208). The probability that photons are in exchange-symmetric states is $<1.2 \times 10^{-7}$. (I)

215. "A simplified method to test the symmetrization postulate for photons," Shi-Biao Zheng, Opt. Commun. **168**, 363–367 (1999). Proposes testing the symmetrization postulate for photons by measuring a two-level atom interacting dispersively with photons in a cavity. (I)

216. "Analysis of an atomic $J = 0$ to $J = 1$ two-photon transition as a test of the spin-statistics connection for photons," Robert C. Hilborn, Phys. Rev. A **65**, 032104 (2002). Shows how a forbidden two-photon atomic transition depends on the parameter $q$ using Greenberg's q-mutator formalism. (I)

217. "Photon spin and Bose-Einstein statistics," B. Ritchie, Opt. Commun. **262**, 229–233 (2006). Shows that the spin-statistics theorem holds for canonical quantization of the Dirac field for a zero-mass particle and that Maxwell's equations can be inferred from the pair of Dirac equations for zero-mass, zero-charge particles. (I)

218. "Scheme for proving the bosonic commutation relation using single-photon interference," M. S. Kim, H. Jeong,

A. Zavatta, V. Parigi, and M. Bellini, Phys. Rev. Lett. **101**(26), 260401 (2008). Proposes using a single-photon interferometer to check the commutation relations among creation and annihilation operators for photons as a test of the spin-statistics connection. (I)

219. "Symmetry-suppressed two-photon transitions induced by hyperfine interactions and magnetic fields," M. G. Kozlov, D. English, and D. Budker, Phys. Rev. A **80**(4), 042504 (2009). Analyzes hyperfine-interaction and magnetic-field-induced two-photon transitions that can mimic the spin-statistics-connection-violating two-photon transitions. It is important to take these effects into account for experiments such the one described in the following reference. (I)

220. "Spectroscopic Test of Bose-Einstein Statistics for Photons," D. English, V. V. Yashchuk, and D. Budker, Phys. Rev. Lett. **104**(25), 253604 (2010). The results of a more-refined implementation of the two-photon-transition experiment in atomic barium (Ref. 214) are used to test the spin-statistics connection for photons. The probability of finding two photons in a spin-statistics-connection-violating state compared to the standard symmetric state is $< 4.0 \times 10^{-11}$. (I)
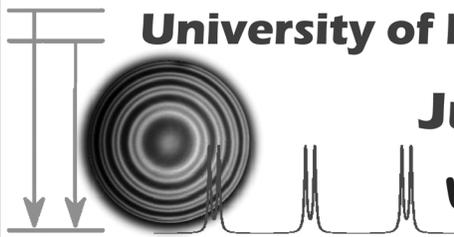
### F. Phonons

221. "Test of the Symmetrization Postulate for Phonons in an Ion Trap," S.-B Zheng and X.-W Zhu, Chin. Phys. Lett. **17**(6), 391–392 (2000). Phonons are spin-0 quasi-particles for vibrational excitations in condensed matter. Hence, they should be described as bosons with only symmetric states. Proposes a method for testing this statement. (I)
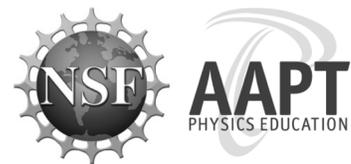
The Conference on Laboratory Instruction Beyond the First Year of College will be held immediately preceding the 2012 AAPT Summer meeting in Philadelphia. This conference will provide a unique opportunity to gain *hands-on* exposure to contemporary instructional labs appropriate to Modern Physics, Electronics, Optics, and Advanced Labs, as well as instructional labs covering Statistical Physics, Condensed Matter and Materials Physics, and Quantum Mechanics. The conference will serve as an opportunity to discuss a range of curricular models that allow for enhancement of the undergraduate physics major.