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Optimality Conditions for a Hierarchical Control Problem Governed by a PDE

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Abstract— **We consider the approximate pointwise control of a linear parabolic system with multiple targets. Assuming a hierarchy among the objectives, we derive optimality conditions for a particular test problem and provide numerical results.**

Keywords-component; Optimization, Control, Partial Differential Equations

I. INTRODUCTION

Optimal control problems constitute an interesting case of PDE-based optimization problems. Instances of these types of problems abound in application and development of efficient numerical methods for the solution of these problems has also been the subject of significant recent research (see [3] and [4]).

Motivated by specific engineering applications, such as those arising in optimal well placement in reservoir engineering [9], we investigate a means of formulating a class of optimal control problems in which the targets can be partitioned into categories of increasing relative importance. This approach, based on the work of von Stackelberg [8] in an economic context, requires that the deviations from the least important targets, called the "follower" targets, be decreased only after the deviations from the most important targets, called the "leader" targets, satisfy prescribed bounds. This type of optimal control problem has been termed hierarchical control. One way of formulating this type of problem is in terms of nested optimization structure in which, in an "inner minimization", the follower targets are minimized subject to fixed values of certain of the control variables and then an "outer minimization" is performed over the remaining control variables to obtain optimal leader target satisfaction. The resulting accuracy on the follower targets is therefore determined by and is subordinate to the optimization over the leader targets.

II. MODEL FORMULATION

We are concerned with a class of optimal control problems in which there are multiple goals that are to be satisfied, i.e., a multicriteria control problem, and in which the underlying state variables are governed by the parabolic partial differential equation with mixed boundary conditions:

$$
\begin{gathered}
y_t - \mathbf{A} y = f(x,t) + V(x,t), \quad (x,t) \in Q \\
y(x,0) = b_0(x), \quad x \in \Omega \\
y(x,t) = b_1(x,t), \quad (x,t) \in \Gamma_1 \times (0,T) \\
\frac{dy}{d\eta}(x,t) = b_2(x,t), \quad (x,t) \in \Gamma_2 \times (0,T),
\end{gathered}
$$

where Ω is a bounded open subset of $\Re^2$, $T > 0$ is finite, $Q = \Omega \times (0,T)$ and $\Gamma_1 \cup \Gamma_2$ is the boundary of Ω. We assume that the functions in the model are well-behaved, i.e., $f(x,t) \in L^2(0,T;\Omega)$, $b_0(x) \in L^2(\Omega)$, and $b_j(x,t) \in L^2(0,T;\Omega)$, $j = 1,2$. Here $\mathbf{A}$ is a strongly elliptic operator and V represents the action of the controls on the system. In particular, we consider the case in which there are k pointwise controls $v_1(t),\ldots,v_k(t)$ located respectively at the points $a_1(t),\ldots,a_k(t)$ and that for a given choice of the controls,

$$V(x,t) = \sum_{j=1}^{k} v_j(t)\delta(x - a_j(t)).$$

Our goal is to formulate and solve an optimization problem that results in a selection of controls, including both time-dependent magnitudes and locations that force the solution to the above system at time, T to be ``close" to a set of targets, $Y_1,\ldots,Y_k$ where each $Y_j \in L^2(\Omega)$ while simultaneously minimizing a cost functional $C(v,a)$. In addition, a set of restrictions on the location of the sites, $a_1(t),\ldots,a_k(t)$, are possible.

Obviously, it is generally impossible to force all of the targets to be satisfied to within some pre-assigned tolerance (in fact, it is not always possible to satisfy one target exactly). To formulate an optimization problem that can be solved, some priority must be established among the set of targets. A variety of methods have been proposed for carrying out this task. One such formulation is obtained by assigning a set of weights to

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the targets and minimizing the weighted sum of deviations from the targets. This problem can be expressed in the form:

$$\min\ C(v,a)+\sum_{j=1}^{k}\frac{\gamma_j}{2}\int_{\Omega}(y(x,T)-Y_j(x))^2\,dx$$

subject to:

$$y_t-\mathbf{A}y=f(x,t)+V(x,t),\ (x,t)\in Q$$

$$\mathrm{y(x,0)}=b_0,\ x\in\Omega$$

$$\mathrm{y(x,t)}=\ b_1(\mathrm{x,t})\ ,\ (x,t)\in\Gamma_1\times(0,T)$$

$$\frac{dy}{d\eta}(x,t)=b_2(\mathrm{x,t}),\ (x,t)\in\Gamma_2\times(0,T)$$

where the γ_j are the respective weights associated with the different targets. A second approach is to assign acceptable deviations of the state variable from each of the targets and express these tolerances as constraints in the optimization problem. In this case the problem becomes equation (SD):

$$\min\ C(v,a)$$

subject to:

$$y_t-\mathbf{A}y=f(x,t)+V(x,t),\ (x,t)\in Q$$

$$\mathrm{y(x,0)}=b_0,\ x\in\Omega$$

$$\mathrm{y(x,t)}=\ b_1(\mathrm{x,t})\ ,\ (x,t)\in\Gamma_1\times(0,T)$$

$$\frac{dy}{d\eta}(x,t)=b_2(\mathrm{x,t}),\ (x,t)\in\Gamma_2\times(0,T)$$

$$\int_{\Omega}(y(x,T)-Y_j(x))^2\,dx\le\beta_j,\quad j=1,\ldots,k.$$

In these formulations, additional constraints on the controls could be included. Each of these formulations has certain drawbacks; in the first case a choice of weights is necessary without any *a priori* indication of how this choice will affect the solution; in the latter case it is difficult to specify the small tolerances in such a way as to avoid infeasibilities.

In this paper we follow the work of von Stackelberg (see [11]) and Lions (see [5]) and formulate the problem as a *hierarchical control problem*. This means that we prioritize the goals, i.e., specify a hierarchy of targets. The leading target is taken to be the one of the highest priority and the overriding task of the control problem is to have the state variable approximate this target as accurately as possible at time, $t=T$. Given this highest priority, the deviation from the target of next highest priority is minimized subject to the satisfaction of this primary goal. Then the deviation from the target of the third highest priority is minimized subject to the condition that the higher targets are satisfactorily approximated, and so on. This hierarchical structure requires a partition of the controls and control locations into corresponding hierarchies. In some problems there may be a natural correspondence but in other cases some flexibility in choosing the controls is available. For this preliminary study we presume that there is a single leader target, denoted $Y_{L(x)}$ and a single target of lower priority called the follower target and denoted $Y_{F(x)}$. We also assume that there are two controls that we arbitrarily partition into leader and follower controls, $(v_L(t),a_L(t))$ and $(v_F(t),a_F(t))$, respectively. Additional follower targets and controls can be added without fundamentally affecting the nature of the model. The control problem we consider is the nested optimization problems, denoted by (IP1):

$$\min_{v_F,y}\ C(v,a)+\frac{\gamma_F}{2}\int_{\Omega}(y(x,T)-Y_F(x))^2\,dx$$

subject to:

$$y_t-\mathbf{A}y=f(x,t)+V(x,t),\quad (x,t)\in Q,$$

$$y(x,0)=b_0(x),x\in\Omega,$$

$$y(x,t)=b_1(x,t),(x,t)\in\Gamma_1\times(0,\,T),$$

$$\frac{dy}{d\eta}(x,t)=b_2(x,t),(x,t)\in\Gamma_2\times(0,T),$$

and (IP2):

$$\min_{v_F,y}\ C(v,a)+\frac{\gamma_F}{2}\int_{\Omega}(y(x,T)-Y_F(x))^2\,dx$$

subject to:

$$\int_{\Omega}(y(x,T)-Y_L(x))^2\,dx\le\beta \qquad \text{(IP1)}$$

and (IP):

$$\min_{a_L,a_F}\ C(v,a)$$

subject to:

$$g(a)\le 0 \qquad \text{(IP2)}$$

where γ_F and β are fixed positive constants, $C(v,a)$ represents a general convex cost function depending on the controls, and the last inequalities involving $g:\Re^4\to\Re^m$ represent constraints on the locations of the controls. These inequalities may be nonlinear and nonconvex; for example, $a_L(t)$ and $a_F(t)$ might be constrained to be a certain minimal distance apart. This problem is interpreted in the following manner. The control variables a_L, a_F and v_L are held fixed and the inner problem (IP1) is solved to determine the optimal choices for v_F and y, thus theoretically determining optimality functions $v_F^*(a_L,a_F,v_L)$ and $y^*(a_L,a_F,v_L)$. It is well known that the problem (IP1) has a unique solution for fixed a_L, a_F, and v_L. Next, these optimality functions are substituted into the objective function and the target constraint for the second inner problem, (IP2). Then this problem is solved with a_L and a_F held fixed determining another optimality function $v_L^*(a_L,a_F)$. Finally, the outer problem, (OP), now having the form

$$min_{a_L,a_F} C(v_L^*(a_L,a_F),v_F^*(a_L,a_F),v_L^*(a_L,a_F),a_L,a_F)$$

subject to: $g(a_L,a_F)\le 0$, is solved.

Note that the cost function can be thought of as a regularization term in the inner problems, i.e., a term that is used to guarantee the existence of a solution. However, it also has a role as a general objective function to be minimized to the extent possible. In this model, we have optimized the variables (a_L,a_F) outside the optimization with respect to the other control variables and the state variables in order to facilitate the solution of the problem. As noted above, in applications the constraints on these variables can be nonlinear and nonconvex and if included in the inner optimization problems would make these problems difficult to solve and negate the advantages of the hierarchical structure.

The theory underlying the hierarchical control problem defined by the pair of problems (IP1) and (IP2) has been studied by Lions [5], albeit for a different underlying PDE and with *boundary controls*. Lions shows that a solution exists *for every positive* β although in general the target cannot be met exactly ($\beta=0$); i.e., the problem is approximately controllable. These existence proofs for the solutions to the inner pair of optimization problems given by Lions are not constructive and hence provide no blueprint as to how to obtain numerical solutions. One natural approach is to use a variational method to obtain the optimality conditions for the innermost problem (IP1) and use these equations as constraints when solving (IP2). In the following we establish the optimality conditions for solving (IP1) and then discuss how to approach (IP2).

We assume that

$$C(v,a)=\frac{1}{2}\int_0^T (v_L^2(t)+v_F^2(t))dt, \tag{1}$$

that $\mathbf{A}$ is the Laplacian operator Δ, and that the boundary conditions are of the Dirichlet type. Extensions to more general parabolic systems are straightforward in concept (but may require significantly more effort to obtain numerical solutions). Thus our PDE has the form

$$y_t-\mathbf{\Delta} y=f(x,t)+v_L(t)\delta(x-a_L(t))+v_F(t)\delta(x-a_F(t)), \quad (x,t)\in Q \tag{2}$$

$$y(x,0) \;=\; b_0(x), \quad x\in\Omega, \tag{3}$$

$$y(x,t) \;=\; b_1(x,t), \quad (x,t)\in \Gamma\times(0,T), \tag{4}$$

where Γ is the boundary of Ω.

Proposition 1.

Let a_L,a_F and v_L be fixed. If v_F and y are optimal for (IP1) then there exists a dual function $p(x,t)\in L^2(0,T;\bar{\Omega})$ satisfying the PDE

$$p_t+\Delta p \;=\; 0, \quad (x,t)\in Q \tag{5}$$

$$p(x,T) \;=\; -\gamma_F(y(x,T)-Y_F(x)), \quad x\in\Omega \tag{6}$$

$$p(x,t)=0, \quad (x,t)\in\Gamma\times(0,T), \tag{7}$$

and v_F is given by

$$v_F(t)= p(a_F(t),t). \tag{8}$$

Proof: If v_F and y are optimal for (IP1) then the variational equality for the objective function is

$$\int_0^T v_F(t)\widehat{v_F}(t)dt+\gamma_F\int_\Omega (y(x,T)-y_F(x))\hat{z}(x,T)dx=0 \tag{9}$$

for all admissible $\widehat{v_F}\in L^2(0,T)$ and $\hat{z}\in L^2(0,T;\Omega)$. If $\widehat{v_F}$ and $\hat{z}$ satisfy

$$\hat{z}_t-\Delta\hat{z}=\widehat{v_F}(t)\delta(x-a_F(t)), \quad (x,t)\in Q \tag{10}$$

$$\hat{z}(x,t)=0, \quad x\in\Omega, \tag{11}$$

$$\hat{z}(x,t)=0, \quad (x,t)\in\Gamma\times(0,T) \tag{12}$$

then they are admissible.

Multiplying (10) by $p(x,t)$, integrating over Q and applying Green's theorem gives

$$\int_Q (p_t+\Delta p)\hat{z}(t)dxdt+\int_\Omega (p(x,T)\hat{z}(x,T)$$
$$-p(x,0)\hat{z}(x,0))dx$$
$$+\int_{\Gamma\times(0,T)}(p(x,t)\frac{d\hat{z}}{d\eta}(x,t)-\frac{dp}{d\eta}(x,t)\hat{z}(x,t))dxdt$$

$$= \int_Q \widehat{v_F}(t)\delta(x - a_F(t))p(x,t)dxdt \qquad (13)$$

where $\frac{d}{d\eta}$ represents the normal derivative. Using (5)-(7) and (10)-(12) this equation becomes

$$-\gamma_F \int_\Omega (y(x,T) - Y_F(x))\,\hat{z}(x,T)dx$$

$$= \widehat{v_F}(t)\,p(a_F(t),t)dt. \qquad (14)$$

Equation (8) follows immediately from the last equation and the Euler equation, (9). Using these necessary conditions the second inner problem (IP2) can now be written

$$\min_{v_L,y,p} \int_0^T (v_L^2(t) + p(a_F(t),t)^2)dt$$

subject to:

$$y_t - \mathbf{\Delta} y = f(x,t) + v_L(t)\delta(x - a_L(t))$$

$$+ \; p(x,t))\delta(x - a_F(t)), \quad (x,t) \in Q$$

$$y(x,0) = b_0(x), \quad x \in \mathbf{\Omega},$$

$$y(x,t) = b_1(x,t), \quad (x,t) \in \Gamma \times (0,T),$$

$$p_t + \Delta p = 0, \quad (x,t) \in Q$$

$$p(x,T) = -\gamma_F (y(x,T) - Y_F(x)), \quad x \in \mathbf{\Omega},$$

$$p(x,t) = 0, \quad (x,t) \in \Gamma \times (0,T),$$

$$\int_\Omega (y(x,T) - Y_L(x))^2 dx \le \beta,$$

with a_L and a_F fixed.

At this stage there are several possible approaches. One approach would be to incorporate the control variables $a(t)$ directly into the problem (so in effect (IP2) becomes (OP)) and solve the resulting problem. However, this approach severely restricts the numerical methods that we can apply since the state variable occurs in a nonlinear inequality constraint. For example, a reduced variable approach could not be employed. Another approach would be to obtain the optimality conditions for this problem (as was done for (IP1)) and then use these conditions in the formulation of the outer problem. If we take this approach then we are forced to include complementary slackness conditions as part of the necessary conditions which is an added nonlinear difficulty. Both of these methods also suffer from the fact that an *a priori* choice of β is required.

As a result of these complications, we have chosen, following to include the leader target goal as a penalty term in the objective function. That is, we reformulate (IP2) as

$$\min_{v_L,y,p} \int_0^T (v_L^2(t) + p(a_F(t),t)^2)dt$$

$$+ \frac{\gamma_L}{2} \int_\Omega (y(x,T) - Y_L(x))^2 dx$$

subject to:

$$y_t - \mathbf{\Delta} y = f(x,t) + v_L(t)\delta(x - a_L(t))$$

$$+ \; p(x,t))\delta(x - a_F(t)), \quad (x,t) \in Q$$

$$y(x,0) = b_0(x), \quad x \in \mathbf{\Omega},$$

$$y(x,t) = b_1(x,t), \quad (x,t) \in \Gamma \times (0,T),$$

$$p_t + \Delta p = 0, \quad (x,t) \in Q$$

$$p(x,T) = -\gamma_F (y(x,T) - Y_F(x)), \quad x \in \mathbf{\Omega},$$

$$p(x,t) = 0, \quad (x,t) \in \Gamma \times (0,T),$$

where is a specified constant. By choosing γ_L sufficiently large we can, in theory, force the deviation from the leader target to be less than β although such a solution will not, in general, be the solution to the original problem (IP2).

We now derive the optimality conditions for this reformulated problem.

Proposition 2.

Let a_L and a_F be fixed. If v_L, y and p are optimal for the problem (IP3) given above, then there exist functions $P(x,t)$ and $Y(x,t)$ in $L^2(0,T;\Omega)$ satisfying

$$Y_t + \mathbf{\Delta} Y = 0, \quad (x,t) \in Q \qquad (15)$$

$$Y(x,T) = -\gamma_F P(x,T) - \gamma_L (y(x,T) - Y_L(x)), \; x \in \mathbf{\Omega}, \qquad (16)$$

$$Y(x,t) = 0\,, (x,t) \in \Gamma \times (0,T) \qquad (17)$$

$$P_t - \mathbf{\Delta} P = -\delta(x - a_F)(p(x,t) - Y(x,t)), \quad (x,t) \in Q \qquad (18)$$

$$P(x,0) = 0\,, \quad x \in \mathbf{\Omega}, \qquad (19)$$

$$P(x,t) = 0\,, (x,t) \in \Gamma \times (0,T), \qquad (20)$$

and v_L is given by

$$v_L(t) = Y(a_L(t),t), \; t \in (0,T). \qquad (21)$$

Proof: If v_L, y and p are optimal for the problem (IP3) then the variation equation

$$\int_0^T (v_L(t)\widehat{v_L}(t) + p(a_F(t),t)\widehat{p(a_F(t),t)})dt \; +$$

$$\frac{\gamma_L}{2}\int_\Omega (y(x,T) - Y_L(x))\hat{z}(x,T)dx = 0 \qquad (22)$$

must be satisfied for every admissible $(\widehat{v_L}, \hat{z}, \hat{p})$, i.e., for every $(\widehat{v_L}, \hat{z}, \hat{p})$ satisfying

$$\hat{z}_t - \Delta\hat{z} = \widehat{v_F}(t)\delta(x - a_F(t)) + \hat{p}(x,t)\delta(x - a_F(t)),$$

$$(x,t) \in Q \qquad (23)$$

$$\hat{z}(x,0) = 0,\; x \in \Omega, \qquad (24)$$

$$\hat{z}(x,t) = 0,\; (x,t) \in \Gamma\times(0,T), \qquad (25)$$

$$\hat{p}_t - \Delta\hat{p} = 0, \quad (x,t) \in Q \qquad (26)$$

$$\hat{p}(x,T) = -\gamma_F\,\hat{z}(x,T),\; x \in \Omega, \qquad (27)$$

$$\hat{p}(x,t) = 0,\; (x,t) \in \Gamma\times(0,T), \qquad (28)$$

Multiplying (23) by $Y(x,t)$ and (26) by $P(x,t)$, integrating over Q, and applying Green's theorem we obtain

$$\int_\Omega (Y_t + \Delta Y)\hat{z}(x,t)dxdt$$

$$+ \int_\Omega (Y(x,T)\hat{z}(x,T) - Y(x,0)\hat{z}(x,0))dx$$

$$+ \int_{\Gamma\times(0,T)} (Y(x,t)\frac{d\hat{z}}{d\eta}(x,t) - \frac{dY}{d\eta}(x,t)\hat{z}(x,t))dxdt$$

$$= \int_Q \widehat{v_L}(x,t)\delta(x - a_L(t))$$

$$+ \hat{p}(a_F(t),t)\delta(x - a_F(t))Y(x,t)dx,dt \qquad (29)$$

and

$$\int_Q (P_t + \Delta P)\hat{p}(t)dxdt$$

$$+ \int_\Omega (P(x,T)\hat{p}(x,T) - P(x,0)\hat{p}(x,0))dx$$

$$+ \int_{\Gamma\times(0,T)} (P(x,t)\frac{d\hat{p}}{d\eta}(x,t) - \frac{dP}{d\eta}(x,t)\hat{p}(x,t))dxdt = 0$$

$$(30)$$

Using the various PDE's and boundary conditions for the functions in (29) and (30) we arrive at

$$-\gamma_F \int_\Omega P(x,T)\,\hat{z}(x,T)dx$$

$$-\gamma_L \int_\Omega (y(x,T) - Y_L(x))\hat{z}(x,T)dx$$

$$= \int_0^T (\widehat{v_L}(t)Y(a_L(t),t) + \hat{p}(a_F(t),t)Y(a_F(t),t))dt. \quad (31)$$

and

$$-\int_0^T (p(a_F(t),t) - Y(a_F(t),t))\hat{p}(a_F(t),t)dt +$$

$$\int_\Omega P(x,T)\hat{p}(x,T)dx = 0 \qquad (32)$$

Using (27) and rearranging terms in (32) yields

$$\gamma_F \int_\Omega P(x,T)\,\hat{z}(x,T)dx$$

$$= \int_0^T \hat{p}(a_F(t),t)(p(a_F(t),t) - Y(a_F(t),t))dt\,. \quad (33)$$

Substituting this last equation into (31) yields (22).

With this derivation the formulated optimization problem (OP) becomes

$$\min_{a_L,a_F} \frac{1}{2}\int_0^T (p(a_F(t),t)^2 + Y(a_L(t),t)^2)dt$$

$$+ \frac{\gamma_L}{2}\int_\Omega (y(x,T) - Y_L(x))^2\,dx$$

subject to: equations (2)-(7) and (15)-(21)

$$g(a) \le 0\,.$$

The relative sizes of the constants γ_L and γ_F affect how accurately the different targets can be approximated. In order to approximate the leader target as accurately as possible, γ_L must be made large. However, the effect of increasing its size is influenced by the size of γ_F. Thus, as in the first formulation of this section, (SD), with a single objective function incorporating both targets, the magnitudes of γ_F and γ_L required to achieve the desired target deviations must be determined by experimentation. Our preliminary numerical studies have suggested that if both targets are in the objective function and both constants are large, then there can be difficulties in achieving convergence to the optimal solution. It should be emphasized that in order to provide useful results the optimal control generated by the model must be implementable, e.g., wildly oscillating optimal controls are undesirable. Our studies to date have indicated that the controls achieved in the hierarchical formulation are better-behaved than those from (SD) for large values of the parameters. Both of these conjectures need further testing and, if possible, theoretical grounding. It is clear that this formulation of the problem is fundamentally different from other models. As is well-documented in the finite-dimensional cases of bilevel programming [7], an optimal solution to a bilevel optimization problem need not be a *Pareto optimal* solution in the sense of multiobjective optimization and there is no reason to assume that this is not the case here. Also, the inclusion of the follower control sites a_F as part of the outer optimization, rather than the inner optimization problem, may seem inconsistent. In formulating the problem in this manner, we were again motivated by an effort to make the problem tractable; complicated (and possibly nonconvex) inequality constraints in the control locations would seriously degrade the ability to express concisely the necessary conditions for the inner problem. All of these points speak to the difficulty in formulating state equations and in solving large scale multicriteria optimization problems. The results presented here represent an initial effort in this direction.

We observe that hierarchical control might profitably be used to formulate a multitude of important scientific applications. For example, in the area of oil reservoir simulation [10] one can formulate optimal well placement problems as hierarchical control problems where desired well productions might form mandatory (or leader) targets while revenue or efficiency based goals are a secondary (follower) targets. Cryobiology applications employ complicated cell freezing models that can be controlled in a similar hierarchical way where temperatures form leader targets and desired concentrations form follower targets. Remote manipulator systems, like those employed by space-craft, are required to solve optimal control problems rapidly. In some instances, these systems must accomplish a goal while maintaining prescribed distances from other pieces of machinery. One could formulate a class of hierarchical control in which leader targets include primary objectives and follower targets maintain minimal separation from sensitive machinery whenever possible.

III. Prepare Your Paper Before Styling

The problem addressed is that of the preceding section with the domain Ω taken to be the unit square with the boundary conditions chosen to be zero. Moreover, we have assumed that the control sites are not functions of t but constant. These simplifications don't prohibit us from making preliminary assessments about the prospects for this type of formulation.

We had several goals for these preliminary numerical experiments. First we wanted to determine the possibility of efficiently solving the problem in its hierarchical formulation. Second, we wanted to determine how sensitive the solutions were to different choices of the constants γ_L and γ_F and to compare these results with those obtained by solving the problem with a single objective function containing a weighted sum of the target discrepancies. Finally, we wanted to ascertain if we could solve a problem with nonconvex constraints on the control locations. We begin by describing the time discretization. Let N_T be the number of time steps and $N_T\Delta t = T$. We denote the estimate of y at the n th time step by y^n where $n = 1 \ldots N_T$. If N_X denotes the number of spatial steps in the x_1 and in the x_2 directions, the spatial step is denoted by h with $N_X h = 1$. The discrete approximation to y is

$$y(n\Delta t, ih, jh) \approx y^n_{i,j}.$$

We follow the *two-step implicit scheme* for parabolic problems

$$\frac{\partial y}{\partial t}((n+1)\Delta t) \approx \frac{1}{2\Delta t}\left(3y^{n+1}_{i,j} - 4y^n_{i,j} + y^{n-1}_{i,j}\right).$$

Experience with this time discretization has led us to use it on stiff problems when we need to integrate to large values of T (see [6]). In such cases, the fact that it assures unconditional stability and produces an accuracy to second order in time amply justifies the storage costs.

At each time step we must solve an elliptic problem to obtain $y^{n+1}_{i,j}$. The domain is so simple that we use the very common finite-element triangulation of Ω consisting of bisected squares. The space of polynomials of degree ≤ 1 is used to form a finite dimensional approximation to $L^2(\Omega)$ and $H^1(\Omega)$. More sophisticated schemes are certainly available for both linear and nonlinear parabolic equations.

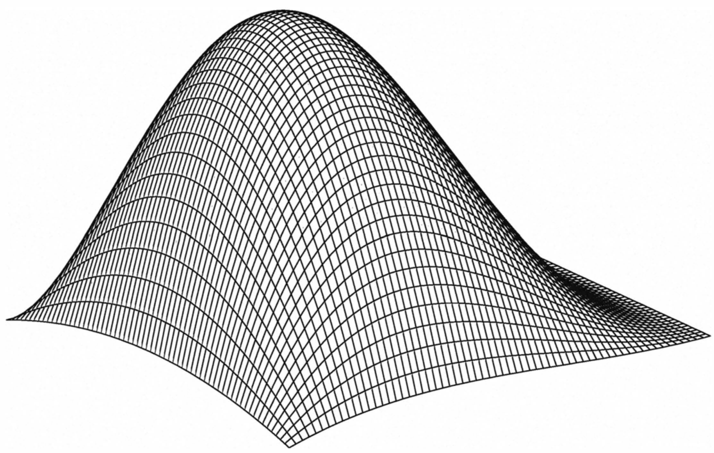

Figure 1. The leader target

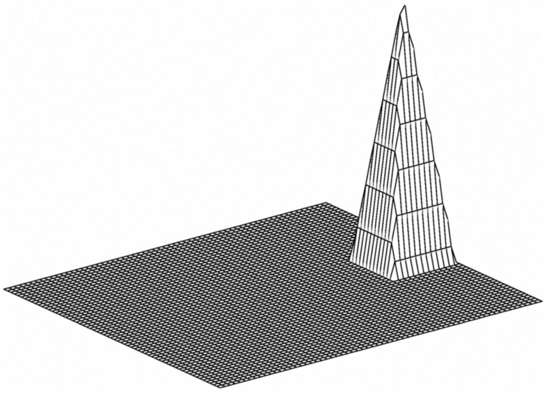

Figure 2. The follower target

For testing, this simple numerical scheme is both adequate and appropriate. Specific applications may require more specialized discretizations. Two target states, are used to test the performance of this formulation. We choose one specific pair of test shapes that illustrates behavior seen in most of our numerical experiments. The leader target shape is a smooth function and the follower is a pyramid (Figures 1 and 2). Specifically, the target functions are

$$y_L(x) = 35x_1x_2(1-x_2)(1-x_1)^2$$

$$y_F(x) = 2\min\{5x_1-4, 3-5x_2, 5-5x_1, 5x_2-2\}.$$

The optimization problem that arose from our formulation was solved using a sequential quadratic programming (SQP) algorithm. The specifics of the algorithm are contained in [1] and a theoretical analysis that can be found in [2]. Our problem formulation worked well with our numerical optimization algorithm. In numerical results not presented here we solved problems with values of γ as large as 1.e16 and

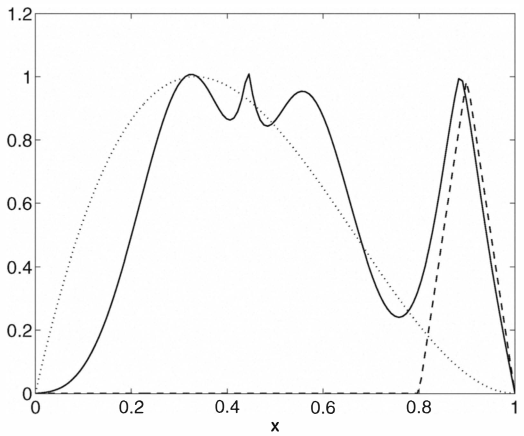

Figure 3. State variables.

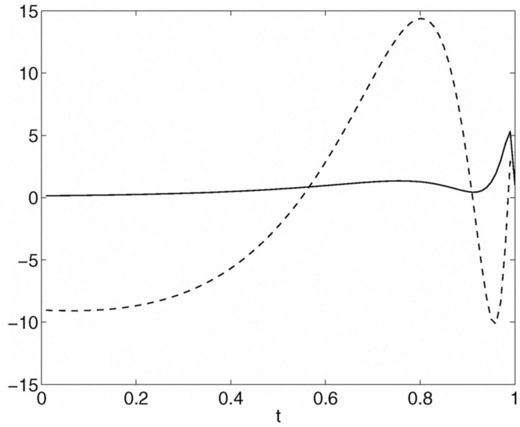

Figure 4. Control variables.

values approaching machine precision. In Figure 3 the dotted and dashed profiles respectively denote leader and follower target profiles along the line. The solid lines are the state variable y at time $T=1$ along the line $x_2 = 1/2$. Clearly both the leader and follower targets were approximately attained. In Figure 4 the leader and follower controls, $v_L(t)$ and $v_F(t)$ are shown for $t \in (0,1]$, by solid and dashed lines respectively.

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