Realizable differential matrices for depolarizing media

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The evolution of a Stokes vector through depolarizing media is considered. A general form for the differential matrix is found that is appropriate in the presence of depolarization and it is parameterized in a manner that ensures that it yields, upon integration, a valid Mueller matrix for any choice of parameters. The form expands the more limited form for a nondepolarizing matrix given by Azzam [J. Opt. Soc. Am. **68**, 1756 (1978)] and which was extended recently by others to include depolarization. A Mueller matrix decomposition is proposed that is based upon the new parameterization.

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The Stokes–Mueller formalism for describing the optical and polarization properties of media has proven to be extremely valuable [1]. The four-element Stokes vector defines the intensity and polarization of a stream of light. A 4×4 Mueller matrix describes the linear transformation between an incoming Stokes vector and an outgoing Stokes vector. In some applications, such as the analysis of liquid crystals [2], optical fibers [3], and plasmas [4], one is interested in a differential formalism [5], where the Stokes vector $\mathbf{S}(z)$ evolves as it propagates through a medium in the z direction, such as with the equation

$$\frac{\mathrm{d}\mathbf{S}(z)}{\mathrm{d}z} = \mathbf{m}\mathbf{S}(z), \tag{1}$$

where $\mathbf{m}$ is the differential matrix. Given an initial condition $\mathbf{S}(0)$, Eq. (1) has the well-known solution [6]

$$\mathbf{S}(z) = \mathbf{M}\mathbf{S}(0), \tag{2}$$

where the Mueller matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \mathbf{W}\mathrm{diag}(e^{\sigma_0 z}, e^{\sigma_1 z}, e^{\sigma_2 z}, e^{\sigma_3 z})\mathbf{W}^{-1}, \tag{3}$$

σ_0, σ_1, σ_2, and σ_3 are the eigenvalues of $\mathbf{m}$, and the columns of the orthogonal matrix $\mathbf{W}$ are the respective eigenvectors of $\mathbf{m}$.

Azzam [5] showed that the general differential matrix for nondepolarizing media is of the form

$$\mathbf{m} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \alpha & \mu & \nu \\ \gamma & -\mu & \alpha & \eta \\ \delta & -\nu & -\eta & \alpha \end{pmatrix}, \tag{4}$$

where α is the absorption coefficient; β, γ, and δ are diattenuation coefficients for light polarized in each of the three axes of the Poincaré sphere, respectively; and η, ν, and μ are birefringence coefficients for the three pairs of axes of the Poincaré sphere, respectively. For lossy media, the absorption coefficient α is negative. The other parameters in Eq. (4) are unconstrained.

In his work, Azzam did not describe the form the differential matrix would take if there were depolarization. Coupled rate equations occur when solving diffuse scattering problems, where scattering couples radiation traveling in different directions. In such systems, depolarization plays an important role, and having a formalism for treating depolarization would be beneficial. The purpose of this Letter is to describe a more general form for the differential matrix that includes the effects of depolarization and has no constraints.

A four-element Stokes vector must satisfy the condition that the first element must be positive and its square must be greater than or equal to the sum of squares of the other elements. A Mueller matrix, in order to be valid, must transform the space of valid Stokes vectors into a subspace of valid Stokes vectors. Givens and Kostinski [7] showed that a valid Mueller matrix $\mathbf{M}$ must therefore satisfy a relatively simple condition: if $\mathbf{G} = \mathrm{diag}(1, -1, -1, -1)$ is the Lorentz metric, then all of the eigenvalues of $\mathbf{G}\mathbf{M}^T\mathbf{G}\mathbf{M}$ must be real, and the eigenvector associated with the largest eigenvalue must be a valid Stokes vector. Any combination of real values for the variables in the matrix Eq. (4) will yield a valid, realizable Mueller matrix when Eq. (3) is applied.

The matrix $\mathbf{m}$ in Eq. (4) has a clear symmetry. Azzam pointed out that any breaking of this symmetry will result in depolarization. It is tempting, therefore, to make the generalization [8,9]

$$\mathbf{m}' = \begin{pmatrix} \alpha & \beta-\beta' & \gamma-\gamma' & \delta-\delta' \\ \beta+\beta' & \alpha-\alpha' & \mu'+\mu & \nu'+\nu \\ \gamma+\gamma' & \mu'-\mu & \alpha-\alpha'' & \eta'+\eta \\ \delta+\delta' & \nu'-\nu & \eta'-\eta & \alpha-\alpha''' \end{pmatrix}, \tag{5}$$

where α', α'', α''', β', γ', δ', μ', ν', and η' are nine depolarization coefficients that fill out the degrees of freedom of the differential matrix, and each of which breaks a specific symmetry found in Eq. (4). However, with the exception of α', α'', and α''', setting any one of these depolarization coefficients to a nonzero value will result in an invalid Mueller matrix.

We thus seek a differential matrix parameterization that guarantees the resulting Mueller matrix will be valid. Consider the generating depolarizing matrix

$$\mathbf{m}_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

The differential matrix $\mathbf{m}_1$ has no effect on the Stokes vector $(1,1,0,0)^T$ and depolarizes the Stokes vector $(1,-1,0,0)^T$ to the greatest extent.

We construct an orthogonal rotation matrix designed to rotate the polarization state $(1,1,0,0)^T$ to $(1,q/I,u/I,v/I)^T$, where q, u, and v can be any set of three real numbers and $I=(q^2+u^2+v^2)^{1/2}$:

$$\mathbf{R}(q,u,v)=\frac{1}{I}\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & q & \frac{Iu}{\sqrt{q^2+u^2}} & -\frac{qv}{\sqrt{q^2+u^2}} \\ 0 & u & -\frac{Iq}{\sqrt{q^2+u^2}} & -\frac{uv}{\sqrt{q^2+u^2}} \\ 0 & v & 0 & \sqrt{q^2+u^2}\end{pmatrix}. \quad (7)$$

We can use this matrix to orient the axis of the depolarization matrix $\mathbf{m}_1$, so that it is oriented along a vector $\mathbf{a}=(a_q,a_u,a_v)^T$, and scale it so that it has a depolarization coefficient $A=(a_q^2+a_u^2+a_v^2)^{1/2}$:

$$\begin{aligned}\mathbf{m}_a &= \mathbf{R}(a_q,a_u,a_v)\mathbf{m}_1\mathbf{R}^T(a_q,a_u,a_v)A \\ &= \frac{1}{A}\begin{pmatrix} A^2 & -a_qA & -a_uA & a_vA \\ a_qA & -a_q^2 & -a_qa_u & -a_qa_v \\ a_uA & -a_ua_q & -a_u^2 & -a_ua_v \\ a_vA & -a_va_q & -a_va_u & -a_v^2\end{pmatrix}. \quad (8)\end{aligned}$$

The matrix $\mathbf{m}_a$ has no effect on the Stokes vector $(A,a_q,a_u,a_v)^T$ and depolarizes to a maximum degree the Stokes vector $(A,-a_q,-a_u,-a_v)^T$. Each of the parameters a_q, a_u, and a_v can take any real value. Notice that there are apparent retardance elements in what one may have thought to be a diattenuative depolarization differential matrix.

Next, we consider the generator

$$\mathbf{m}_2=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{pmatrix}. \quad (9)$$

The matrix $\mathbf{m}_2$ depolarizes to a maximum degree the Stokes vectors $(1,1,0,0)^T$ and $(1,-1,0,0)^T$ and has no effect on the Stokes vector $(1,0,\cos\phi,\sin\phi)^T$ for any ϕ. The matrix $\mathbf{m}_2$ is independent of $\mathbf{m}_a$, in that no vector $\mathbf{a}$ can be chosen so that $\mathbf{m}_a=\mathbf{m}_2$. As we did above, we can orient the matrix $\mathbf{m}_2$ so that its axis is aligned along a vector $\mathbf{b}=(b_q,b_u,b_v)^T$ and scale it so that it has a depolarization coefficient $B=(b_q^2+b_u^2+b_v^2)^{1/2}$:

$$\begin{aligned}\mathbf{m}_b &= \mathbf{R}(b_q,b_u,b_v)\mathbf{m}_2\mathbf{R}^T(b_q,b_u,b_v)B \\ &= \frac{1}{B}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -b_q^2 & -b_qb_u & -b_qb_v \\ 0 & -b_ub_q & -b_u^2 & -b_ub_v \\ 0 & -b_vb_q & -b_vb_u & -b_v^2\end{pmatrix}. \quad (10)\end{aligned}$$

The matrix $\mathbf{m}_b$ depolarizes the Stokes vectors $(B,b_q,b_u,b_v)^T$ and $(B,-b_q,-b_u,-b_v)^T$ to the greatest extent and has no effect on those Stokes vectors orthogonal to them. Each of the b_q, b_u, and b_v can take on any real value. Notice that the matrix $\mathbf{m}_b$, like $\mathbf{m}_a$, has elements one might attribute to retardance depolarization.

Finally, we consider the generator

$$\mathbf{m}_3=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1\end{pmatrix}. \quad (11)$$

The matrix $\mathbf{m}_3$ has no effect on the Stokes vectors $(1,1,0,0)^T$ and $(1,-1,0,0)^T$ and depolarizes to a maximum degree the Stokes vector $(1,0,\cos\phi,\sin\phi)^T$ for any ϕ. The matrix $\mathbf{m}_3$ is independent of $\mathbf{m}_a$ and $\mathbf{m}_b$, in that no vectors $\mathbf{a}$ and $\mathbf{b}$ can be chosen so that $\mathbf{m}_a+\mathbf{m}_b=\mathbf{m}_3$. As we did above, we can orient the matrix $\mathbf{m}_3$ so that its axis is aligned along a vector $\mathbf{c}=(c_q,c_u,c_v)^T$ and scale it so that it has a depolarization coefficient $C=(c_q^2+c_u^2+c_v^2)^{1/2}$:

$$\begin{aligned}\mathbf{m}_c &= \mathbf{R}(c_q,c_u,c_v)\mathbf{m}_3\mathbf{R}^T(c_q,c_u,c_v)C \\ &= \frac{1}{C}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -c_u^2-c_v^2 & c_qc_u & c_qc_v \\ 0 & c_uc_q & -c_q^2-c_v^2 & c_uc_v \\ 0 & c_vc_q & c_vc_u & -c_q^2-c_u^2\end{pmatrix}. \quad (12)\end{aligned}$$

The matrix $\mathbf{m}_c$ has no effect on the polarization states $(C,c_q,c_u,c_v)^T$ and $(C,-c_q,-c_u,-c_v)^T$ and depolarizes mostly strongly those states orthogonal to these. Each of the c_q, c_u, and c_v can take on any real value. Once again, the matrix $\mathbf{m}_c$, like $\mathbf{m}_a$ and $\mathbf{m}_b$, has elements one might attribute to retardance depolarization.

It is worth mentioning that the generating matrices for the nondepolarizing differential matrices are

$$\mathbf{m}_4=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{pmatrix}, \quad (13)$$

$$\mathbf{m}_5=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0\end{pmatrix}, \quad (14)$$

$$\mathbf{m}_6=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{pmatrix}. \quad (15)$$

The matrix $\mathbf{m}$ given in Eq. (4) is the sum of $\mathbf{m}_4$, rotated by $\mathbf{R}(\beta,\gamma,\delta)$, $\mathbf{m}_5$, rotated by $\mathbf{R}(\eta,-\nu,\mu)$, and $\mathbf{m}_6$, scaled by α. The complete depolarizing differential matrix can therefore be parameterized by the sum $\mathbf{m}'=\mathbf{m}+\mathbf{m}_a+\mathbf{m}_b+\mathbf{m}_c$, with seven nondepolarizing coefficients and nine depolarizing coefficients, all of which can take on any real values.

Lu and Chipman [10] proposed a Mueller matrix decomposition in which a matrix $\mathbf{M}$ is expressed as the product

$$\mathbf{M} = \mathbf{M}_\Delta \mathbf{M}_\mathrm{R} \mathbf{M}_\mathrm{D}, \tag{16}$$

where $\mathbf{M}_\Delta$ represents a depolarizer, $\mathbf{M}_\mathrm{R}$ represents a retarder, and $\mathbf{M}_\mathrm{D}$ represents a diattenuator. There has been some discussion about the appropriate order of the matrices in Eq. (16), because the one given by Lu and Chipman is somewhat arbitrary [10–12]. Volume-diffusing materials, however, are not expected to consist physically of these three elements in any order, so the use of the Lu–Chipman decomposition may not provide necessary physical insight for these cases. Instead, we would expect some materials to continuously depolarize, retard, and diattenuate as light propagates through them.

Ossikovski [8] and Ortega-Quijano and Arce-Diego [13] proposed using the logarithm of the Mueller matrix as a means for characterizing it. By simply inverting Eq. (3), we have, for a given $\mathbf{M}$ and for $z = 1$, the differential matrix

$$\mathbf{m} = \mathbf{W}\mathrm{diag}(\ln\,\tau_0, \ln\,\tau_1, \ln\,\tau_2, \ln\,\tau_3)\mathbf{W}^{-1}, \tag{17}$$

where τ_0, τ_1, τ_2, and τ_3 are the eigenvalues of $\mathbf{M}$ and $\mathbf{W}$ is a matrix whose columns are the eigenvectors of $\mathbf{M}$. Parameterizing a Mueller matrix in this fashion has the advantage that there is no assumption about the ordering of the depolarizer, retarder, or diattenuator, because all three are treated as occurring simultaneously.

We propose a similar decomposition as that proposed before [8,13], albeit with the new parameterization for the differential matrix. Because of the nonlinear behavior of the new depolarization parameters, however, it is difficult to analytically obtain them from the differential matrix. Instead, Newton's method can be used to iteratively find the parameters. If the Mueller matrix or differential matrix is not valid, then there will be no real solution, because the parameterization guarantees realizable Mueller matrices for any real parameters.

Here, we present an example. Data were obtained from reflection of 633 nm light from a pressed polytetrafluoroethylene powder scattering standard, measured with an incident angle of 75° and a viewing angle of 60° [14]. In this case, the normalized Mueller matrix was found to be

$$\mathbf{M} = \begin{pmatrix} 1.000 & -0.098 & 0.005 & 0.001 \\ -0.105 & 0.442 & 0.000 & -0.002 \\ 0.002 & 0.000 & 0.323 & -0.124 \\ 0.003 & 0.004 & 0.127 & 0.276 \end{pmatrix}. \tag{18}$$

(Details of these measurements and their uncertainties are given in the original work; see [14].) The effective differential matrix is then found from Eq. (17) to be

$$\mathbf{m}' = \begin{pmatrix} -0.009 & -0.145 & 0.008 & 0.003 \\ -0.155 & -0.831 & 0.002 & -0.005 \\ 0.004 & 0.002 & -1.053 & -0.393 \\ 0.005 & 0.011 & 0.402 & -1.202 \end{pmatrix}. \tag{19}$$

The parameters that match the differential matrix are found to be

$$\begin{gathered} \alpha = -0.014, \\ (\beta, \gamma, \delta)^T = (-0.150, 0.006, 0.004)^T, \\ (\eta, \nu, \mu)^T = (-0.398, -0.008, -0.001)^T, \\ \mathbf{a} = (-0.005, -0.002, 0.001)^T, \\ \mathbf{b} = (0.682, -0.009, 0.678)^T, \\ \mathbf{c} = (0.858, -0.003, 0.584)^T. \end{gathered} \tag{20}$$

It can be seen that this decomposition finds a negligible $\mathbf{a}$, b_u, c_u, γ, δ, μ, and ν. The negligible values of γ, μ, a_u, b_u, and c_u are expected from the reflection symmetry of the sample and the measurement geometry.

The general form for differential matrices described in this Letter provides a basis for performing simulations of polarization behavior in systems where we anticipate continuous evolution of the Stokes vector as light propagates through a material. In solving radiative transfer problems, multistream generalizations of Eq. (1) are required to describe the evolution of the Stokes vector in scattering media, because the scattering process couples light propagating in different directions. Scattering media tend to cause depolarization, so the formalism described in this manuscript should assist those performing simulations for these media.

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