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We present arbitrary moments of the univariate and bivariate truncated complex Gaussian distribution. Using these moment expressions, we investigate the convergence of a particular infinite series of moments encountered in recent statistical analyses of scattering parameters measured in reverberation chambers. We find that the infinite series converges for particular parameterizations of the truncated distribution and may be expressed in closed form for the univariate case.

Key words: bivariate, circular random variables, complex Gaussian, complex normal, truncated complex Gaussian, truncated complex normal, univariate.

1. Introduction

The complex Gaussian distribution has proven to be a useful statistical model for describing a wide range of physical phenomena including thermal noise [1, 2], signal fluctuations in wireless links [3], and the complex electromagnetic fields within reverberation chambers [4]. For physical systems, it is important to recognize that the infinite tails of the complex Gaussian distribution allow for realizations that may violate energy conservation principles. As an example, in the presence of a continuous wave-transmitter, the maximum power received by an antenna is constrained by the power transmitted by the source. However, modeling the antenna's received signal as a complex Gaussian random variable allows for the possibility that the received power may exceed the transmitted power!

In particular, this work is motivated by ongoing statistical studies of the scattering parameters measured in reverberation chambers. Typically, these scattering parameters are modeled as complex Gaussian random variables [5, 6]. However, this model neglects the fact that a reverberation chamber is a passive system such that the magnitude of the scattering parameters cannot exceed unity [7]. In this light, it is expected that the scattering-parameter measurements made in reverberation chambers may be more accurately modeled as realizations of a *truncated* complex Gaussian random variable, wherein the complex Gaussian distribution's probability density function is forced to zero outside of the unit circle and re-normalized within the unit circle such that the probability density function integrates to unity.

Whereas both the truncated *real* Gaussian distribution [8–11] and the non-truncated *complex* Gaussian distribution [12–16] have been studied extensively, there have been very few analyses of the truncated complex Gaussian distribution [17–19]. In all of these instances, the truncated and underlying non-truncated Gaussian distributions were zero-mean, and the probability density function was truncated along circles of constant magnitude in the complex plane. In [17], the variance and spectral kurtosis were evaluated for a one-sided truncated complex Gaussian distribution, wherein the probability density function was nonzero within a disk centered at the origin. In [18], the one-sided truncated complex Gaussian distribution's entropy was evaluated. In [19], the variance and fourth moment were evaluated for a two-sided truncated complex Gaussian distribution, wherein the probability density function was nonzero within an annulus centered about the origin.

Here, we present expressions for arbitrary moments of the truncated complex Gaussian distribution for both the univariate (i.e., single complex random variable) and bivariate (i.e., a pair of complex random variables) cases. Additionally, we examine an infinite series of moments that we have encountered in recent statistical analyses of reverberation chambers and determine the convergence of this infinite series when the random variables are drawn from both non-truncated and truncated complex Gaussian distributions. We show that for the non-truncated case, the series diverges regardless of the distribution's parameterization. In contrast, for the truncated case, the series converges for certain distribution parameterizations and may even be expressed in closed form if the distribution is univariate.

We restrict our analysis to complex random variables that are "circular" (see [15, 16, 20, 21]), wherein the (joint) probability density function of the random variable(s) is rotationally invariant in the complex plane. Due to this rotational symmetry, circular random variables are inherently complex and zero mean, and have the unique property that expectations of products containing different numbers of conjugated and non-conjugated random variables are always zero [16]. The latter property is particularly convenient, because it reduces the number of non-zero moments that need to be considered.

We begin by reviewing the derivations for the moments of the univariate non-truncated circular Gaussian distribution in Section 2.1.. Then, following a similar analysis, we derive the moments of the univariate truncated circular Gaussian distribution in Section 2.2.. In Section 3., we consider the bivariate distributions. We first review the derivation for the moments of the bivariate non-truncated distribution in Section 3.1., and then apply this analysis to the bivariate truncated distribution in Section 3.2.. In Section 4., we examine the convergence of an infinite series of univariate and bivariate moments drawn from non-truncated and truncated circular Gaussian distributions. Section 5. summarizes the main contributions of this report.

2. Univariate Distribution

2.1. Non-Truncated Gaussian

Consider a realization z of univariate circular Gaussian random variable Z characterized by a mean of zero and a variance of σ^2. For many analyses, z is commonly decomposed into its real and imaginary components:

$$z = x + jy, \tag{1}$$

where x and y are realizations of two independent and identically distributed zero-mean Gaussian random variables, each with a variance of $\sigma^2/2$. Here, we will favor an alternative phasor decomposition where we express z in terms of its magnitude and phase [13]:

$$z = re^{j\phi}. \tag{2}$$

In (2), the magnitude $r \in [0,\infty)$ and phase $\phi \in [0,2\pi)$ are realizations of a Rayleigh and uniform distribution, respectively. Due to the circularity of Z, the magnitude and phase distributions are independent. Importantly, (1) and (2) are equivalent representations for a circular Gaussian random variable. Using (2), we may express the probability density function of Z as

$$f_Z(r,\phi) = \frac{r}{\pi\sigma^2} e^{-\frac{r^2}{\sigma^2}}. \tag{3}$$

By use of (3), the most general expression for the moments of Z is given by [16]

$$\mathcal{E}\{Z^m Z^{*n}\} = \int_0^\infty dr \int_0^{2\pi} d\phi\, z^m z^{*n} f_Z(r,\phi), \tag{4}$$

where $\mathcal{E}\{\cdot\}$ denotes the expectation operator, $\cdot^*$ denotes the complex conjugate, and $Z^{*n} = (Z^n)^* = (Z^*)^n$. However, because we have assumed that Z is a *circular* random variable, we need consider only moments for which $m = n$. For $m \neq n$, it may be shown that [16]

$$\mathcal{E}\{Z^m Z^{*n}\} = 0 \text{ if } m \neq n. \tag{5}$$

For the case of $m = n$, (4) becomes

$$\mathcal{E}\{[ZZ^*]^n\} = \int_0^\infty dr \int_0^{2\pi} d\phi\, [zz^*]^n \frac{r}{\pi\sigma^2} e^{-\frac{r^2}{\sigma^2}}. \tag{6}$$

Performing the integration with respect to ϕ yields

$$\mathcal{E}\{[ZZ^*]^n\} = \frac{2}{\sigma^2} \int_0^\infty dr\, r^{2n+1} e^{-\frac{r^2}{\sigma^2}}. \tag{7}$$

By way of a change of variables and the introduction of the gamma function $\Gamma(x)$, defined as

$$\Gamma(x) = \int_0^\infty dt\, t^{x-1} e^{-t}, \tag{8}$$

we may further simplify (7), yielding

$$\mathcal{E}\{[ZZ^*]^n\} = \sigma^{2n}\Gamma(n+1). \tag{9}$$

Noting that for integer n,

$$\Gamma(n+1) = n!, \tag{10}$$

where $\cdot!$ denotes the factorial, we get

$$\mathcal{E}\{[ZZ^*]^n\} = n!\sigma^{2n}. \tag{11}$$

Equation (11) is well known [12]. Furthermore, inspection of (11) confirms that

$$\mathrm{Var}(Z) = \mathcal{E}\{ZZ^*\} = \sigma^2, \tag{12}$$

where $\mathrm{Var}(Z)$ denotes the variance of Z.

2.2. Truncated Gaussian

Consider now a realization $\tilde{z} = \tilde{r}e^{j\tilde{\phi}}$ of a univariate *truncated* circular Gaussian random variable $\tilde{Z}$. Here, $\tilde{r} \in [0, a]$ and $\tilde{\phi} \in [0, 2\pi)$ are realizations of a truncated Rayleigh distribution and uniform distribution, respectively. Again due to the circularity of $\tilde{Z}$, its magnitude and phase distributions are independent. In contrast, the real and imaginary components of $\tilde{Z}$ are dependent random variables.

We define the probability density function for the truncated distribution as

$$f_{\tilde{Z}}(\tilde{r},\tilde{\phi}) = \begin{cases} C_1 f_Z(\tilde{r},\tilde{\phi}) & \text{for} \quad \tilde{r} \le a \\ 0 & \text{otherwise} \end{cases}, \tag{13}$$

where $f_Z(\tilde{r},\tilde{\phi})$ is the probability density function corresponding to the univariate non-truncated circular Gaussian distribution defined in (3), and C_1 is a normalization constant defined such that

$$1 = \int_0^{\infty} d\tilde{r} \int_0^{2\pi} d\tilde{\phi}\, f_{\tilde{Z}}(\tilde{r},\tilde{\phi}). \tag{14}$$

Inserting (13) into (14) and solving for C_1 yields

$$C_1 = \frac{1}{1 - e^{-\frac{a^2}{\sigma^2}}}. \tag{15}$$

It is important to emphasize that σ^2 is a shape parameter for the truncated distribution – it is *not* the variance of $\tilde{Z}$, as was the case for the non-truncated circular Gaussian distribution. The relationship between σ^2 and the variance of $\tilde{Z}$ will be more fully explored toward the end of this section.

Analogous to the non-truncated univariate case, we express the moments of the truncated random variable $\tilde{Z}$ as

$$\mathcal{E}\{\tilde{Z}^m\tilde{Z}^{*n}\} = \int_0^a d\tilde{r}\int_0^{2\pi} d\tilde{\phi}\,\tilde{z}^m\tilde{z}^{*n} f_{\tilde{Z}}(\tilde{r},\tilde{\phi}). \tag{16}$$

Again, due to the circularity of $\tilde{Z}$, we have

$$\mathcal{E}\{\tilde{Z}^m\tilde{Z}^{*n}\} = 0 \text{ if } m \neq n, \tag{17}$$

whereby we need to consider only moments of the form

$$\mathcal{E}\{[\tilde{Z}\tilde{Z}^*]^n\} = \int_0^a d\tilde{r}\int_0^{2\pi} d\tilde{\phi}\,[\tilde{z}\tilde{z}^*]^n f_{\tilde{Z}}(\tilde{r},\tilde{\phi}). \tag{18}$$

Substituting (13) into (18) and performing the $\tilde{\phi}$ integration yields

$$\mathcal{E}\{[\tilde{Z}\tilde{Z}^*]^n\} = C_1\frac{2}{\sigma^2}\int_0^a d\tilde{r}\,\tilde{r}^{2n+1}e^{-\frac{\tilde{r}^2}{\sigma^2}}. \tag{19}$$

By introducing the incomplete gamma function $\gamma(n,x)$, defined as [22]

$$\gamma(n,x) = \int_0^x dt\,t^{n-1}e^{-t}, \tag{20}$$

we can evaluate the $\tilde{r}$ integral, whereby (19) becomes

$$\mathcal{E}\{[\tilde{Z}\tilde{Z}^*]^n\} = C_1\gamma\left(n+1,\frac{a^2}{\sigma^2}\right)\sigma^{2n}. \tag{21}$$

Substituting (15) for C_1 in (21) leads to our final result for the moments of $\tilde{Z}$:

$$\mathcal{E}\{[\tilde{Z}\tilde{Z}^*]^n\} = \frac{\sigma^{2n}}{1-e^{-\frac{a^2}{\sigma^2}}}\gamma\left(n+1,\frac{a^2}{\sigma^2}\right). \tag{22}$$

Inspection of (22) shows that the variance of $\tilde{Z}$ is given by

$$\mathrm{Var}(\tilde{Z}) = \mathcal{E}\{\tilde{Z}\tilde{Z}^*\} = \frac{\sigma^2}{1-e^{-\frac{a^2}{\sigma^2}}}\gamma\left(2,\frac{a^2}{\sigma^2}\right). \tag{23}$$

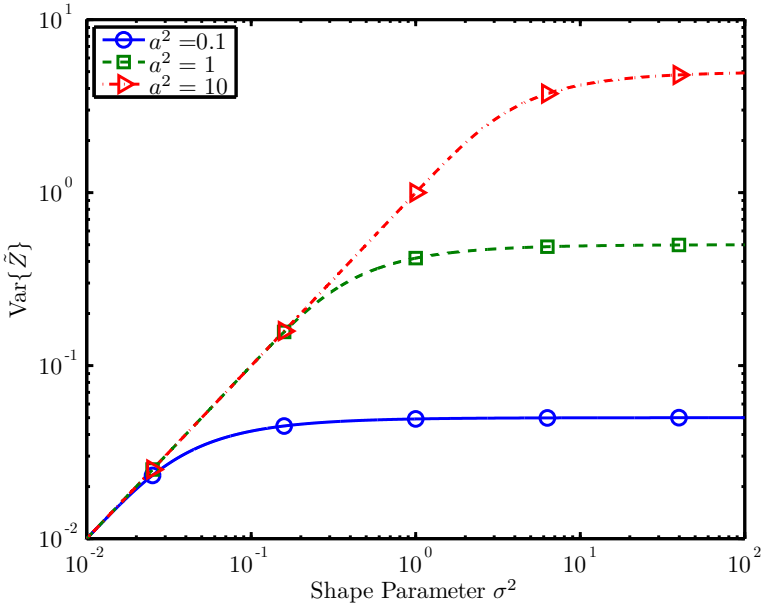

Figure 1: Comparison of the truncated circular Gaussian distribution's shape parameter σ^2 and the distribution's corresponding variance $\mathrm{Var}\{\tilde{Z}\}$ for different values of a^2.

Using the recurrence relationships for the incomplete gamma function, [22]

$$\gamma(n+1,x) = n\gamma(n,x) - x^n e^{-x} \text{ for } n > 0 \tag{24}$$

and

$$\gamma(1,x) = 1 - e^{-x}, \tag{25}$$

we may alternatively express the variance of $\tilde{Z}$ as

$$\mathrm{Var}(\tilde{Z}) = \sigma^2 \frac{1 - \left(1 + \frac{a^2}{\sigma^2}\right) e^{-\frac{a^2}{\sigma^2}}}{1 - e^{-\frac{a^2}{\sigma^2}}}. \tag{26}$$

Equation (26) agrees with that reported in [17]. Taking the limit of (26) as $a^2/\sigma^2 \to 0$ and $a^2/\sigma^2 \to \infty$, we find that for $a^2 \gg \sigma^2$, $\mathrm{Var}(\tilde{z}) \sim \sigma^2$ and for $a^2 \ll \sigma^2$, $\mathrm{Var}(\tilde{z}) \sim a^2/2$. That is, when the truncation radius a is large compared to the shape parameter σ, the variance of the truncated circular Gaussian distribution approaches that of the non-truncated circular Gaussian distribution, as defined in (12). Conversely, when a is small compared to σ, the probability density function of the truncated circular Gaussian distribution is approximately constant within a circle of radius a in the complex plane. Thus, for $a^2 \ll \sigma^2$, the variance of $\tilde{Z}$ approaches that of a "circular uniform" distribution characterized by a constant probability density function within a circle of radius a in the complex plane. This behavior is demonstrated in Figure 1 for different values of a^2/σ^2. As expected, we observe that the variance approaches $a^2/2$ when $\sigma^2 \gg a^2$ and σ^2 when $\sigma^2 \ll a^2$.

3. Bivariate Distribution

We next consider moments of a bivariate distribution. As in Section 2., we first review the derivation for the non-truncated circular Gaussian distribution and then proceed to derive the moments for the truncated circular Gaussian distribution.

3.1. Non-Truncated Gaussian

Consider the moments of a pair of identically distributed and potentially correlated non-truncated circular Gaussian random variables, Z_1 and z_2. As in Section 2.1., we assume the variables are zero mean with a variance of σ^2. Additionally, we denote the complex correlation coefficient between Z_1 and Z_2 as ρ. Denoting $z_1 = r_1 e^{j\phi_1}$ and $z_2 = r_2 e^{j\phi_2}$ as realizations of Z_1 and Z_2, the probability density function for the associated *bivariate* circular Gaussian distribution is given by [13]

$$f_{Z_1,Z_2}(r_1,\phi_1,r_2,\phi_2) = \frac{r_1 r_2}{\pi^2\sigma^4(1-|\rho|^2)} e^{-\frac{r_1^2+r_2^2+2r_1r_2|\rho|\cos(\phi_2-\phi_1+\pi+\angle\rho)}{\sigma^2(1-|\rho|^2)}} \tag{27}$$

for $|\rho| \in [0,1)$. In (27), $\angle\rho$ denotes the phase of the complex phasor ρ such that $\rho = |\rho|e^{j\angle\rho}$, and we consider $|\rho| = 1$ in the one-sided limit that $|\rho| \to 1^-$.

Again, allowing for moments of conjugated and non-conjugated terms, we find that the most general expression for the moments of the bivariate distribution is

$$\mathcal{E}\left\{Z_1^m Z_2^n Z_1^{*p} Z_2^{*q}\right\} = \int_0^\infty dr_1 \int_0^{2\pi} d\phi_1 \int_0^\infty dr_2 \int_0^{2\pi} d\phi_2\, z_1^m z_2^n z_1^{*p} z_2^{*q} f_{Z_1,Z_2}(r_1,\phi_1,r_2,\phi_2). \tag{28}$$

Due to the circularity of Z_1 and Z_2, we get

$$\mathcal{E}\left\{Z_1^m Z_2^n Z_1^{*p} Z_2^{*q}\right\} = 0 \text{ if } m+n \neq p+q, \tag{29}$$

whereby we need consider only moments of the form

$$\mathcal{E}\left\{Z_1^m Z_2^{N-m} Z_1^{*p} Z_2^{*(N-p)}\right\} = \int_0^\infty dr_1 \int_0^{2\pi} d\phi_1 \int_0^\infty dr_2 \int_0^{2\pi} d\phi_2\, z_1^m z_2^{N-m} z_1^{*p} z_2^{*(N-p)} f_{Z_1,Z_2}(r_1,\phi_1,r_2,\phi_2) \tag{30}$$

for $N \leq m, p \leq 0$. Substituting (27) and the phasor forms of z_1 and z_2 into (30) yields

$$\begin{aligned}\mathcal{E}\left\{Z_1^m Z_2^{N-m} Z_1^{*p} Z_2^{*(N-p)}\right\} &= \frac{1}{\pi^2\sigma^4(1-|\rho|^2)} \\ &\times \int_0^\infty dr_1 \int_0^{2\pi} d\phi_1 \int_0^\infty dr_2 \int_0^{2\pi} d\phi_2\, r_1^{m+p+1} r_2^{2N-m-p+1} e^{j(m-p)[\phi_1-\phi_2]} e^{-\frac{r_1^2+r_2^2+2r_1r_2|\rho|\cos(\phi_2-\phi_1+\pi+\angle\rho)}{\sigma^2(1-|\rho|^2)}}.\end{aligned} \tag{31}$$

The ϕ_1 and ϕ_2 integrals may be evaluated by way of a change of variables and the following integral valid for integer n [23]:

$$I_{|n|}(x) = \frac{1}{\pi}\int_0^{\pi} d\theta\, e^{x\cos\theta \pm jn\theta}, \tag{32}$$

where $I_n(x)$ is the modified Bessel function of the first kind of order n. Thereby, (31) becomes

$$\begin{aligned}\mathcal{E}\{Z_1^m Z_2^{N-m} Z_1^{*p} Z_2^{*(N-p)}\} &= \frac{4e^{j(m-p)\angle\rho}}{\sigma^4(1-|\rho|^2)} \\ &\times \int_0^{\infty} dr_1 \int_0^{\infty} dr_2 r_1^{m+p+1} r_2^{2N-m-p+1} e^{-\frac{r_1^2+r_2^2}{\sigma^2(1-|\rho|^2)}} I_{|m-p|}\left(\frac{2r_1r_2|\rho|}{\sigma^2(1-|\rho|^2)}\right).\end{aligned} \tag{33}$$

To carry out the r_1 and r_2 integrations, we first substitute for $I_n(x)$ its power series [24]

$$I_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k}}{k!(n+k)!}, \tag{34}$$

whereby (33) becomes

$$\begin{aligned}\mathcal{E}\{Z_1^m Z_2^{N-m} Z_1^{*p} Z_2^{*(N-p)}\} &= \frac{4|\rho|^{|m-p|}e^{j(m-p)\angle\rho}}{\sigma^2} \\ &\times \sum_{k=0}^{\infty} \frac{1}{k!(k+|m-p|)!} \frac{|\rho|^{2k}}{[\sigma^2(1-|\rho|^2)]^{2k+|m-p|+1}} \\ &\times \int_0^{\infty} dr_1 r_1^{2k+m+p+|m-p|+1} e^{-\frac{r_1^2}{\sigma^2(1-|\rho|^2}} \int_0^{\infty} dr_2 r_2^{2k+2N-m-p+|m-p|+1} e^{-\frac{r_2^2}{\sigma^2(1-|\rho|^2}}.\end{aligned} \tag{35}$$

After evaluating the integrals with the aid of the gamma function defined in (8), (35) becomes

$$\begin{aligned}\mathcal{E}\{Z_1^m Z_2^{N-m} Z_1^{*p} Z_2^{*(N-p)}\} &= |\rho|^{|m-p|}e^{j(m-p)\angle\rho}\sigma^{2N}(1-|\rho|^2)^{N+1} \\ &\times \sum_{k=0}^{\infty} \frac{\left(k+\frac{|m-p|+m+p}{2}\right)!\left(k+N+\frac{|m-p|-m-p}{2}\right)!}{k!(k+|m-p|)!}|\rho|^{2k},\end{aligned} \tag{36}$$

where we have used $\Gamma(n+1) = n!$.

For arbitrary N, m, p, the summation in (36) cannot be evaluated in closed form. However, if either $m = 0$, $p = 0$, or $N = m + p$, we may apply the binomial series [23]

$$\frac{1}{(1-x)^{n+1}} = \sum_{k=0}^{\infty} \frac{(n+k)!}{n!k!} x^k \tag{37}$$

valid for $|x| < 1$. Thereby, for $m = 0$, (36) becomes

$$\mathcal{E}\{Z_2^N Z_1^{*p} Z_2^{*(N-p)}\} = N!|\rho|^p e^{-jp\angle\rho}\sigma^{2N}; \tag{38}$$

for $p = 0$,

$$\mathcal{E}\{Z_1^m Z_2^{N-m} Z_2^{*N}\} = N!|\rho|^m e^{jm\angle\rho}\sigma^{2N}; \tag{39}$$

and for $N = m + p$,

$$\mathcal{E}\{Z_1^m Z_2^p Z_1^{*p} Z_2^{*m}\} = (m+p)!|\rho|^{|m-p|} e^{j(m-p)\angle\rho}\sigma^{2(m+p)}. \tag{40}$$

In terms of expectations, the correlation between two random variables X_1 and X_2 is defined as

$$\text{Corr}\{X_1, X_2\} = \frac{\text{Cov}\{X_1, X_2\}}{\sqrt{\text{Var}\{X_1\}\text{Var}\{X_2\}}}, \tag{41}$$

where $\text{Cov}\{X_1, X_2\}$ denotes the covariance between X_1 and X_2 as given by

$$\text{Cov}\{X_1, X_2\} = \mathcal{E}\{X_1 X_2^*\}. \tag{42}$$

By use of (40) with $m = 1$ and $p = 0$, the covariance between Z_1 and Z_2 is thus

$$\text{Cov}\{Z_1, Z_2\} = \rho\sigma^2. \tag{43}$$

Substituting (43) and (12) into (41), we find that the correlation between Z_1 and Z_2 is

$$\text{Corr}\{Z_1, Z_2\} = \rho, \tag{44}$$

which is the expected result for a bivariate non-truncated circular Gaussian distribution.

3.2. Truncated Gaussian

We now consider the moments of a pair of identically distributed and potentially correlated truncated circular Gaussian random variables, $\tilde{Z}_1$ and $\tilde{Z}_2$. As in Section 2.2., σ^2 is a shape parameter for the truncated bivariate distribution that is related to the variance of $\tilde{Z}_1$ and $\tilde{Z}_2$ by (26). Similarly, $|\rho| \in [0, 1)$ is a complex shape parameter for the bivariate truncated distribution that is related to the complex correlation coefficient between $\tilde{Z}_1$ and $\tilde{Z}_2$. The exact relationship between ρ and the complex correlation coefficient will be explored toward the end of this section.

Analogous to the univariate truncated circular Gaussian case presented in Section 2.2., we denote realizations of $\tilde{Z}_1$ and $\tilde{Z}_2$ as $\tilde{z}_1 = \tilde{r}e^{j\tilde{\phi}_1}$ and $\tilde{z}_2 = \tilde{r}_2 e^{j\tilde{\phi}_2}$, respectively and define the bivariate truncated Gaussian distribution's probability density function as

$$f_{\tilde{z}_1,\tilde{z}_2}(\tilde{r}_1,\tilde{r}_2,\tilde{\phi_1},\tilde{\phi}_2) = \begin{cases} C_2 f_{z_1,z_2}(\tilde{r}_1,\tilde{r}_2,\tilde{\phi_1},\tilde{\phi}_2) & \text{for} \quad \tilde{r}_1,\tilde{r}_2 \leq a \\ 0 \quad \text{otherwise} & \end{cases}, \tag{45}$$

where $f_{Z_1,Z_2}(r_1,r_2,\phi_1,\phi_2)$ is the probability density function defined in (27) for the non-truncated bivariate distribution, and C_2 is the bivariate truncated distribution's normalization constant such that $f_{\tilde{Z}_1,\tilde{Z}_2}(\tilde{r}_1,\tilde{r}_2,\tilde{\phi_1},\tilde{\phi}_2)$ integrates to unity. This normalization constant is defined such that

$$1 = \int_0^\infty d\tilde{r}_1 \int_0^{2\pi} d\tilde{\phi}_1 \int_0^\infty d\tilde{r}_2 \int_0^{2\pi} d\tilde{\phi}_2 \, f_{\tilde{Z}_1,\tilde{Z}_2}(\tilde{r}_1,\tilde{r}_2,\tilde{\phi}_1,\tilde{\phi}_2). \tag{46}$$

To solve (46) for C_2, we use (45) and (35) with $N = m = p = 0$ and change the upper integration limits from ∞ to a. This leads to

$$1 = C_2 \frac{4}{\sigma^2} \sum_{k=0}^{\infty} \frac{|\rho|^{2k}}{[\sigma^2(1-|\rho|^2)]^{2k+1}} \int_0^a dr_1 r_1^{2k+1} e^{-\frac{r_1^2}{\sigma^2(1-|\rho|^2}} \int_0^a dr_2 r_2^{2k+1} e^{-\frac{r_2^2}{\sigma^2(1-|\rho|^2}}. \tag{47}$$

Using (20) to evaluate the integrals yields

$$1 = C_2 \frac{4}{\sigma^2} \sum_{k=0}^{\infty} \frac{1}{k!k!} \frac{|\rho|^{2k}}{[\sigma^2(1-|\rho|^2)]^{2k+1}} \left[\frac{[\sigma^2(1-|\rho|^2)]^{k+1}}{2} \gamma\left(k+1, \frac{a^2}{\sigma^2(1-|\rho|^2)}\right)\right]^2. \tag{48}$$

Introducing the regularized incomplete gamma function $P(p,x)$, defined as [22]

$$P(p,x) = \frac{\gamma(p,x)}{\Gamma(p)}, \tag{49}$$

allows us to express the reciprocal of C_2 as

$$\frac{1}{C_2} = (1-|\rho|^2) \sum_{k=0}^{\infty} \left[P\left(k+1, \frac{a^2}{\sigma^2(1-|\rho|^2)}\right)\right]^2 |\rho|^{2k}. \tag{50}$$

Recognizing that $0 \leq P(p,x) \leq 1$ for all $p,x \geq 0$, we can show with the aid of (37) that $C_2 \geq 1$, with equality approached as $a^2/\sigma^2/(1-|\rho|^2) \to \infty$.

Similar to (28), we define the moments of the bivariate truncated circular Gaussian distribution as

$$\mathcal{E}\{\tilde{Z}_1^m \tilde{Z}_2^{*n}\} = \int_0^a d\tilde{r}_1 \int_0^{2\pi} d\tilde{\phi}_1 \int_0^a d\tilde{r}_2 \int_0^{2\pi} d\tilde{\phi}_2 \, \tilde{z}_1^m \tilde{z}_2^{*n} f_{\tilde{Z}_1,\tilde{Z}_2}(\tilde{r}_1,\tilde{r}_2,\tilde{\phi}_1,\tilde{\phi}_2). \tag{51}$$

Again due to circularity, we have

$$\mathcal{E}\{\tilde{Z}_1^m \tilde{Z}_2^n \tilde{R}_1^{*p} \tilde{R}_2^{*q}\} = 0 \text{ if } m+n \neq p+q, \tag{52}$$

whereby we need consider only moments of the form

$$\mathcal{E}\{\tilde{Z}_1^m \tilde{Z}_2^{N-m} \tilde{Z}_1^{*p} \tilde{Z}_2^{*(N-p)}\} = \int_0^\infty d\tilde{r}_1 \int_0^{2\pi} d\tilde{\phi}_1 \int_0^\infty d\tilde{r}_2 \int_0^{2\pi} d\tilde{\phi}_2 \, \tilde{z}_1^m \tilde{z}_2^{N-m} \tilde{z}_1^{*p} \tilde{z}_2^{*(N-p)} f_{\tilde{Z}_1,\tilde{Z}_2}(\tilde{r}_1,\tilde{\phi}_1,\tilde{r}_2,\tilde{\phi}_2) \tag{53}$$

for $0 \leq m, p \leq N$. Following (31)-(35), (53) simplifies to

$$\begin{aligned}\mathcal{E}\{\tilde{Z}_1^m \tilde{Z}_2^{N-m} \tilde{Z}_1^{*p} \tilde{Z}_2^{*(N-p)}\} &= C_2 \frac{4|\rho|^{|m-p|} e^{j(m-p)\angle\rho}}{\sigma^2} \\ &\times \sum_{k=0}^{\infty} \frac{1}{k!(k+|m-p|)!} \frac{|\rho|^{2k}}{[\sigma^2(1-|\rho|^2)]^{2k+|m-p|+1}} \\ &\times \int_0^a d\tilde{r}_1 \tilde{r}_1^{2k+m+p+|m-p|+1} e^{-\frac{\tilde{r}_1^2}{\sigma^2(1-|\rho|^2)}} \int_0^a d\tilde{r}_2 \tilde{r}_2^{2k+2N-m-p+|m-p|+1} e^{-\frac{\tilde{r}_2^2}{\sigma^2(1-|\rho|^2)}}.\end{aligned} \tag{54}$$

Using (20) to evaluate the $\tilde{r}_1$ and $\tilde{r}_2$ integrals, we have

$$\begin{aligned}\mathcal{E}\{\tilde{Z}_1^m \tilde{Z}_2^{N-m} \tilde{Z}_1^{*p} \tilde{Z}_2^{*(N-p)}\} &= C_2 |\rho|^{|m-p|} e^{j(m-p)\angle\rho} \sigma^{2N} (1-|\rho|^2)^{N+1} \\ &\times \sum_{k=0}^{\infty} \frac{\gamma\left(k+\frac{|m-p|+m+p}{2}+1, \frac{a^2}{\sigma^2(1-|\rho|^2)}\right) \gamma\left(k+N+\frac{|m-p|-m-p}{2}+1, \frac{a^2}{\sigma^2(1-|\rho|^2)}\right)}{k!(k+|m-p|)!} |\rho|^{2k}.\end{aligned} \tag{55}$$

Unfortunately, the summation in (55) cannot be evaluated in closed form for *any* m, p, or N. Thus, substituting into (55) the expression for our normalization constant given in (50) and simplifying, we have

$$\begin{aligned}\mathcal{E}\{\tilde{Z}_1^m \tilde{Z}_2^{N-m} \tilde{Z}_1^{*p} \tilde{Z}_2^{*(N-p)}\} &= |\rho|^{|m-p|} e^{j(m-p)\angle\rho} \sigma^{2N} (1-|\rho|^2)^N \\ &\times \frac{\sum_{k=0}^{\infty} \frac{\gamma\left(k+\frac{|m-p|+m+p}{2}+1, \frac{a^2}{\sigma^2(1-|\rho|^2)}\right) \gamma\left(k+N+\frac{|m-p|-m-p}{2}+1, \frac{a^2}{\sigma^2(1-|\rho|^2)}\right)}{k!(k+|m-p|)!} |\rho|^{2k}}{\sum_{k=0}^{\infty} \left[P\left(k+1, \frac{a^2}{\sigma^2(1-|\rho|^2)}\right)\right]^2 |\rho|^{2k}}.\end{aligned} \tag{56}$$

By use of (56) with $m = N = 1$ and $p = 0$, the covariance between $\tilde{Z}_1$ and $\tilde{Z}_2$ may be expressed reasonably compactly as

$$\mathrm{Cov}\{\tilde{Z}_1,\tilde{Z}_2\} = \mathcal{E}\{\tilde{Z}_1\tilde{Z}_2^*\} = \rho\sigma^2(1-|\rho|^2)\frac{\sum_{k=0}^{\infty}\left[P\left(k+2,\frac{a^2}{\sigma^2(1-|\rho|^2)}\right)\right]^2|\rho|^{2k}(k+1)}{\sum_{k=0}^{\infty}\left[P\left(k+1,\frac{a^2}{\sigma^2(1-|\rho|^2)}\right)\right]^2|\rho|^{2k}}, \tag{57}$$

where we have used $\Gamma(k+2) = (k+1)\Gamma(k+1)$, along with the regularized incomplete gamma function defined in (49). Combining (57), (26), and (41), we find that the correlation between $\tilde{Z}_1$ and $\tilde{Z}_2$ is

$$\mathrm{Corr}\{\tilde{Z}_1,\tilde{Z}_2\} = \rho(1-|\rho|^2)\frac{1-e^{-\frac{a^2}{\sigma^2}}}{1-\left(1+\frac{a^2}{\sigma^2}\right)e^{-\frac{a^2}{\sigma^2}}}\frac{\sum_{k=0}^{\infty}\left[P\left(k+2,\frac{a^2}{\sigma^2(1-|\rho|^2)}\right)\right]^2|\rho|^{2k}(k+1)}{\sum_{k=0}^{\infty}\left[P\left(k+1,\frac{a^2}{\sigma^2(1-|\rho|^2)}\right)\right]^2|\rho|^{2k}}. \tag{58}$$

Figure 2 compares the actual correlation $\mathrm{Corr}\{\tilde{Z}_1\tilde{Z}_2^*\}$ to the truncated circular Gaussian distribution's shape parameter ρ for $0 < \rho < 1$ and different values of a^2/σ^2. When $a^2/\sigma^2 \gg 1$, the distribution closely resembles a non-truncated circular Gaussian distribution, and the correlation coefficient is approximately the shape parameter ρ. As a^2/σ^2 is decreased, we observe that the shape parameter becomes an increasingly poor estimate of the distribution's correlation coefficient, with $\mathrm{Corr}\{\tilde{Z}_1\tilde{Z}_2^*\} < \rho$ in general. We also find that as a^2/σ^2 decreases, the correlation coefficient's sensitivity to ρ becomes skewed, whereby moderate correlation coefficient values require ρ to be increasingly large. Although Figure 2 considers purely real and positive values of ρ, inspection of (58) indicates that similar comments apply to the magnitude of complex correlation coefficients based on the magnitude of complex values of ρ.

4. Infinite Series of Moments

In recent and ongoing statistical analyses of scattering parameters measured in reverberation chambers, we have encountered the following two infinite series:

$$\sum_{n=0}^{\infty} c^n \mathcal{E}\{[XX^*]^n\} \tag{59}$$

and

$$\sum_{n=0}^{\infty} c^n \mathcal{E}\{[X_1X_2^*]^n\}. \tag{60}$$

In (59) and (60), $c > 0$ is a real, positive, and constant coefficient, and X, X_1, and X_2 are random variables drawn from some circular distribution. For (59), the underlying distribution is univariate; for (60), the distribution is bivariate, with X_1 and X_2 being identically distributed and potentially correlated. Having derived expressions for arbitrary moments of the non-truncated and truncated

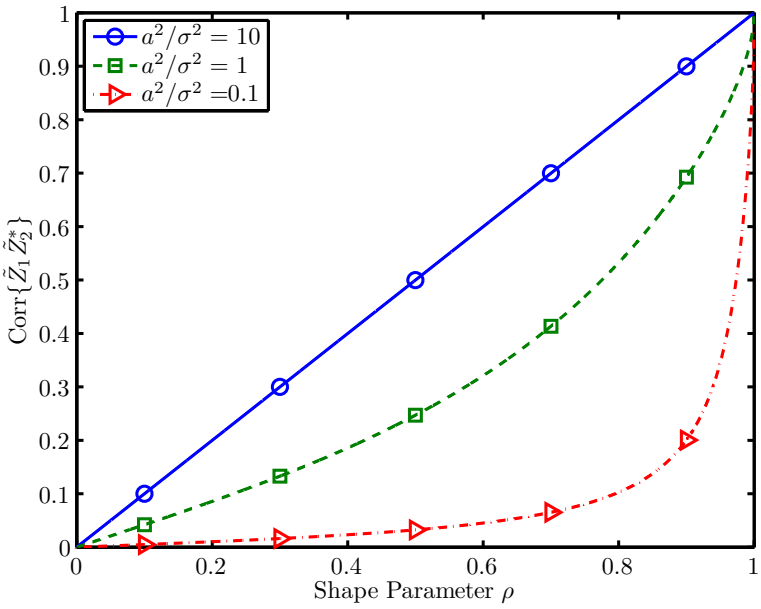

Figure 2: Comparison of the truncated circular Gaussian distribution's shape parameter ρ and the distribution's corresponding correlation coefficent $\mathrm{Corr}\{Z_1 Z_2^*\}$ for different values of a^2/σ^2.

circular Gaussian distributions in the previous sections, we are now able to ascertain the convergence of these two series for the different distributions and, where possible, evaluate the summations in closed form. As with the preceding discussion, we first consider the univariate case given by (59) and then proceed to the bivariate case given by (60).

4.1. Univariate Distribution

4.1.1. Non-Truncated Gaussian

By use of (11), the infinite series for the univariate non-truncated circular Gaussian distribution takes the form

$$\sum_{n=0}^{\infty} c^n \mathcal{E}\{[ZZ^*]^n\} = \sum_{n=0}^{\infty} c^n n! \sigma^{2n}. \tag{61}$$

The factorial $n!$ has the *lower* bound given by [25]

$$n! > \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \tag{62}$$

whereby (61) has the lower bound

$$\sum_{n=0}^{\infty} c^n \mathcal{E}\{[ZZ^*]^n\} > \sqrt{2\pi}\sum_{n=0}^{\infty}\sqrt{n}\left[c\sigma^2\frac{n}{e}\right]^n. \tag{63}$$

This lower bound diverges for all $c\sigma^2 > 0$. Thus, the infinite series given by (61) also diverges, regardless of the choice of c or the variance of the distribution.

4.1.2. Truncated Gaussian

By use of (21), the infinite series for the univariate truncated circular Gaussian distribution takes the form

$$\sum_{n=0}^{\infty} c^n \mathcal{E}\{[\tilde{Z}\tilde{Z}^*]^n\} = C_1 \sum_{n=0}^{\infty} c^n \gamma\left(n+1, \frac{a^2}{\sigma^2}\right) \sigma^{2n}. \tag{64}$$

By use of the integral representation of the incomplete gamma function given in (20), (64) becomes

$$\sum_{n=0}^{\infty} c^n \mathcal{E}\{[\tilde{Z}\tilde{Z}^*]^n\} = C_1 \int_0^{a^2/\sigma^2} dt\, e^{-t} \sum_{n=0}^{\infty} [ct\sigma^2]^n. \tag{65}$$

Convergence of the summation in (65) requires that $|ct\sigma^2| < 1$. Observing that the integration region in (65) corresponds to $0 \le t \le \frac{a^2}{\sigma^2}$, we find that convergence thus requires that $ca^2 < 1$. Under this condition, (65) simplifies to

$$\sum_{n=0}^{\infty} c^n \mathcal{E}\{[\tilde{Z}\tilde{Z}^*]^n\} = C_1 \int_0^{a^2/\sigma^2} dt\, e^{-t} \frac{1}{1-ct\sigma^2} \text{ for } ca^2 < 1. \tag{66}$$

By introducing the exponential integral function $\mathrm{Ei}(x)$ defined as [24]

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\, dt, \tag{67}$$

we may alternatively express (66) as

$$\sum_{n=0}^{\infty} c^n \mathcal{E}\{[\tilde{Z}\tilde{Z}^*]^n\} = C_1 \frac{1}{c\sigma^2} e^{-\frac{1}{c\sigma^2}} \left[\mathrm{Ei}\left(\frac{1}{c\sigma^2}\right) - \mathrm{Ei}\left(\frac{1-ca^2}{c\sigma^2}\right)\right] \text{ for } ca^2 < 1. \tag{68}$$

In the Appendix, we provide a more general proof of convergence for (59) for *any* circular random variable X whose realizations are bounded such that $|x| < 1/\sqrt{c}$ for all x.

4.2. Bivariate Distribution

4.2.1. Non-Truncated Gaussian

By use of (39) with $m = N$, the infinite series for the bivariate non-truncated circular Gaussian distribution takes the form

$$\sum_{n=0}^{\infty} c^n \mathcal{E}\{[Z_1 Z_2^*]^n\} = \sum_{n=0}^{\infty} c^n \rho^n \sigma^{2n} n!. \tag{69}$$

Re-examining the lower bound on the factorial given in (62), we find that regardless of the value of $c|\rho|$, the magnitude of the individual terms in the summation in (69) will grow toward infinity as $n \to \infty$. Thus (69) diverges.

4.2.2. Truncated Gaussian

Starting from (55) and setting $m = N = n$ and $p = 0$, we have

$$\begin{aligned} \mathcal{E}\{[\tilde{Z}_1 \tilde{Z}_2^*]^n\} &= C_2 \rho^n \sigma^{2n} (1-|\rho|^2)^{n+1} \\ &\times \sum_{k=0}^{\infty} \frac{\gamma\left(k+n+1, \frac{a^2}{\sigma^2(1-|\rho|^2)}\right) \gamma\left(k+n+1, \frac{a^2}{\sigma^2(1-|\rho|^2)}\right)}{k!(k+n)!} |\rho|^{2k}. \end{aligned} \tag{70}$$

Using (49) and (10), we re-express (70) as

$$\begin{aligned} |\mathcal{E}\{[\tilde{Z}_1 \tilde{Z}_2^*]^n\}| &= C_2 |\rho|^n \sigma^{2n} (1-|\rho|^2)^{n+1} \\ &\times \sum_{k=0}^{\infty} \frac{\gamma\left(k+n+1, \frac{a^2}{\sigma^2(1-|\rho|^2)}\right)}{\Gamma(k+1)} P\left(k+n+1, \frac{a^2}{\sigma^2(1-|\rho|^2)}\right) |\rho|^{2k}. \end{aligned} \tag{71}$$

A lower bound on the magnitude of (71) may be formulated by recognizing that $P(p+q,x) < P(p,x)$ [22], whereby

$$\begin{aligned} |\mathcal{E}\{[\tilde{Z}_1 \tilde{Z}_2^*]^n\}| &< C_2 |\rho|^n \sigma^{2n} (1-|\rho|^2)^{n+1} P\left(n+1, \frac{a^2}{\sigma^2(1-|\rho|^2)}\right) \\ &\times \sum_{k=0}^{\infty} \frac{\gamma\left(k+n+1, \frac{a^2}{\sigma^2(1-|\rho|^2)}\right)}{\Gamma(k+1)} |\rho|^{2k}. \end{aligned} \tag{72}$$

This inequality may be simplified by use of the following relation [24]

$$\frac{\gamma(n+1, t(1-x))}{(1-x)^{n+1}} = \sum_{k=0}^{\infty} x^k \frac{\gamma(k+n+1,t)}{\Gamma(k+1)} \tag{73}$$

valid for $|x| < 1$. By use of (73), (72) becomes

$$|\mathcal{E}\{[\tilde{Z}_1 \tilde{Z}_2^*]^n\}| < C_2 [|\rho|\sigma^2]^n P\left(n+1, \frac{a^2}{\sigma^2(1-|\rho|^2)}\right) \gamma\left(n+1, \frac{a^2}{\sigma^2}\right). \tag{74}$$

Substitution of (74)'s upper bound into the infinite series given in (60) yields

$$\sum_{n=0}^{\infty} c^n |\mathcal{E}\{[\tilde{Z}_1\tilde{Z}_2^*]^n\}| < C_2 \sum_{n=0}^{\infty} c^n [|\rho|\sigma^2]^n P\left(n+1, \frac{a^2}{\sigma^2(1-|\rho|^2)}\right) \gamma\left(n+1, \frac{a^2}{\sigma^2}\right). \tag{75}$$

Recognizing that $P(n+1,x) < P(1,x)$ and using (49) and (25), we find that (75)'s upper bound becomes

$$\sum_{n=0}^{\infty} c^n |\mathcal{E}\{[\tilde{Z}_1\tilde{Z}_2^*]^n\}| < C_2 \left(1 - e^{-\frac{a^2}{\sigma^2(1-|\rho|^2)}}\right) \sum_{n=0}^{\infty} [c|\rho|\sigma^2]^n \gamma\left(n+1, \frac{a^2}{\sigma^2}\right), \tag{76}$$

which, analogous to the summation in (64), may be evaluated by use of the integral representation of the incomplete gamma function in (20). The result is

$$\begin{aligned} \sum_{n=0}^{\infty} c^n |\mathcal{E}\{[\tilde{Z}_1\tilde{Z}_2^*]^n\}| < C_2 \left(1 - e^{-\frac{a^2}{\sigma^2(1-|\rho|^2)}}\right) \frac{1}{c|\rho|\sigma^2} \\ \times e^{-\frac{1}{c|\rho|\sigma^2}} \left[\mathrm{Ei}\left(\frac{1}{c|\rho|\sigma^2}\right) - \mathrm{Ei}\left(\frac{1-c\rho a^2}{c|\rho|\sigma^2}\right)\right] \text{ for } c|\rho|a^2 < 1. \end{aligned} \tag{77}$$

The normalization constant C_2 defined in (50) is the reciprocal of the area under the *unnormalized* probability density function. Thus, C_2 is guaranteed to be finite for any $a > 0$, and (77) is guaranteed to converge for $c|\rho|a^2 < 1$. This indicates that the infinite series of bivariate moments defined in (60) converges for the case of a truncated circular Gaussian distribution, provided that $c|\rho|a^2 < 1$.

5. Summary

The truncated circular Gaussian distribution is a useful alternative to the non-truncated circular Gaussian distribution, particularly when physical processes are described that have a finite range of values. Here, we have developed expressions for arbitrary moments of univariate and bivariate truncated circular Gaussian random variables. We have also examined the convergence of a particular series of infinite moments that we have encountered in recent statistical analyses of scattering parameters measured in reverberation chambers. We observed that the series always diverges for the non-truncated circular Gaussian case and converges for particular parameterizations of the truncated circular Gaussian case. Based on these findings, we suggest that statistical analyses of reverberation-chamber measurements of scattering parameters use truncated circular Gaussian random variables rather than the more conventional non-truncated circular Gaussian random variables. This is particularly important when moments of large order are being calculated.

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Appendix A: Bounds on the Infinite Series of Moments of Univariate Distributions with Finite Support

Consider an arbitrary circular random variable X with probability density function $f_X(r,\phi)$. Due to the circularity of X, the univariate probability density function $f_X(r,\phi)$ is in general given by [16]

$$f_X(r,\phi) = \frac{1}{2\pi} f_X(r), \tag{78}$$

where $f_X(r)$ is the marginal probability density function corresponding to $|X|$. The non-zero moments of X are given by

$$\mathcal{E}\left\{[XX^*]^n\right\} = \int_0^\infty dr \int_0^{2\pi} d\phi\, [xx^*]^n f_X(r,\phi). \tag{79}$$

Defining a realization of X as $x = re^{j\phi}$, and using (78), we find that (79) simplifies to

$$\mathcal{E}\{[XX^*]^n\} = \int_0^\infty dr\, r^{2n} f_X(r), \tag{80}$$

whereby the non-zero moments of X are determined by the even moments of $|X|$.

Let us suppose that $f_X(r) = 0$ for $r > a$; that is, we assume $f_X(r)$ is non-zero only for $r \in [0, a]$. Due to the monotonicity of r^{2n}, (80) will be maximized if $f_X(r) = \delta(r-a)$, where $\delta(\cdot)$ denotes the Dirac delta function. This leads to the following upper bound on the moments of X:

$$\mathcal{E}\{[XX^*]^n\} \leq a^{2n} \text{ if } f_X(r) = 0 \text{ for } r > a. \tag{81}$$

Substitution of (81) into the infinite series defined in (59) yields

$$\sum_{n=0}^{\infty} c^n \mathcal{E}\{[XX^*]^n\} \leq \sum_{n=0}^{\infty} [ca^2]^n, \tag{82}$$

which converges if $a < 1/\sqrt{c}$.

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