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Modeling the transmission of curved neutron guides with non-perfect reflectivity

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ABSTRACT

Analytic expressions for the transmission of a curved neutron guide including reflection losses are contrasted. The expressions are derived by considering the distribution of the number of reflections as a function of grazing angle at the outer surface. The results of different analytic expressions are compared with simulation results to find the model that best fits the simulations.

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1. Introduction

Thermal and cold neutrons may be transported from their source to different instruments using long neutron guides applying the principle of the total reflection from the inside surface of the guide. Curved guides composed of short straight sections as a polygonal approximation to uniform curvature can remove the direct streaming of fast neutrons and gammas from the beam. Maier-Leibnitz and Springer [1] have given the exact expression for the transmission through a curved guide assuming continuous curvature and perfect reflectivity. It is also seen that various transmission properties of the curved guide may be derived from the acceptance diagram [2] that describes in (z,ψ) space the available positions and directions of successfully transmitted trajectories for a given wavelength, where z and ψ are the spatial and angular coordinates of the neutron trajectory relative to the guide axis at any plane normal to the axis, including the entrance. This method assumes that the guide is uniformly and completely illuminated, so that all possible successful trajectories are accounted, and that the curved guide is at least as long as the line-of-sight length. However there can be significant reductions in transmitted intensity when the reflectivity R of the guide surface is less than unity.

Although detailed design of neutron guides require computer simulation, initial estimates of the performance and parameter dependence are useful. By considering the number of reflections as a function of the grazing angle and the distribution of grazing angles, Mildner and Hammouda [3] have shown that the transmission of a neutron bender or curved guide with a given value of R can be determined exactly using exponential integral functions. They have derived various properties of the transmission as a function of wavelength λ. These expressions reduce correctly to those for the straight guide in the limit of large wavelengths. However, such special functions are less easy to apply than elementary functions. Consequently a suitable correction to the transmission for the perfect reflecting curved guide is sought [4,5] that can reasonably approximate that for non-perfect reflection.

The length of direct sight for the curved guide is given by $L_0 = 4W/\psi_c$, where W is the width of the guide, $\psi_c = \sqrt{(2W/\rho)}$ is the characteristic angle of the curved guide, and ρ is the radius of curvature of the guide. This defines a characteristic wavelength $\lambda_c = \psi_c/\gamma_c$, for the guide, where γ_c depends on the particular reflecting surface and ψ_c on the geometry of the curved guide. (For instance, $\gamma_c = 1.73$ mrad Å^{-1} for a nickel reflecting surface.) Assuming perfect reflectivity, the transmission in the plane of curvature (usually horizontal) relative to the straight guide is given [2,6]

$$T_0(x) = \begin{cases} (2/3)x^2 & x \le 1, \\ (2/3)x^2[1-(1-x^{-2})^{3/2}] & x > 1, \end{cases} \tag{1}$$

where $x = \lambda/\lambda_c$. In practice, the transmission is reduced by the reflectivity and the reduction is best determined by computer simulation assuming a model for this reflection coefficient R as a function of grazing angle χ and wavelength λ. However, it is also

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useful to find an approximate analytic expression for this correction.

The transmission for a given successful trajectory must be modified by a factor R^n, where n is the number of reflections that the trajectory undergoes throughout the length of the guide. We require an adequate estimate for the correction to account for the inability to determine an analytic expression for the computer simulation average $\langle R^n \rangle$, where $\langle \ldots \rangle$ indicates an average in (z,ψ) space over all *successful* trajectories at a given wavelength. The value of n depends not only on the guide geometric characteristics, but also on the coordinates (z,ψ) of the trajectories at the guide entrance. It is convenient to express the angular dependence in terms of χ at the surface of the outer guide wall. A larger grazing angle results not only in a greater number of reflections for zig-zag trajectories, but also in a lower value of the reflectivity, producing greater losses. However, in the following we assume that the reflectivity may be modeled by a constant value of R for χ less that the critical angle for the given wavelength, and zero above.

2. Analytic corrections

We seek a factor that modifies T_0, the transmission assuming perfect reflectivity, which accounts for the loss in transmission of the curve guide with non-unity reflection coefficients. A simple multiplicative factor $R^{\langle n \rangle}$, where $\langle n \rangle$ indicates an average in (z,ψ) space over *all* (rather than all *successful*) trajectories at a given λ, is only good for small losses in reflectivity and for wavelengths close to λ_c. Schirmer and Mildner [4] have shown that the average number of reflections for successful trajectories, $\langle n \rangle_R \approx \langle nR^n \rangle / \langle R^n \rangle$, can be approximated in terms of the variance of the number of reflections for the curved guide for perfect reflectivity, when the reflectivity R is unity. That is,

$$\langle n \rangle_R \approx \langle n \rangle - (1-R)(\langle n^2 \rangle - \langle n \rangle^2). \tag{2}$$

We can determine analytic values of $\langle n \rangle$ and $\langle n^2 \rangle$ in the horizontal plane for a curved guide of length L_0 equal to the line-of-sight. Note that $\langle n \rangle_R$ is less than $\langle n \rangle$, and $R^{\langle n \rangle_R}$ is a better approximation than $R^{\langle n \rangle}$ to the simulation result. This result is used to estimate the true transmission factor, assuming that the curved guide is continuous, but reasonable agreement is found with simulation results only over a restricted range of R and x. However, expanding R about unity to second order, we obtain an expression

$$T_{SM} = T_0[R^{\langle n \rangle} + (\langle n^2 \rangle - \langle n \rangle^2)(1-R)^2/2] \tag{3}$$

which is a better approximation to $\langle R^n \rangle$. This approximation has been shown [4] to be useful for describing the transmission through a neutron bender with low reflection losses (R close to unity).

A trajectory that has n reflections per length of direct sight has its transmission modified by a factor $R^{n(L/L_0)}$ for a guide of length L ($L > L_0$), assuming a constant loss at each reflection. This factor may be written as $\exp[n(L/L_0)\log R] = \exp(-\alpha n)$, where

$$\alpha = -(L/L_0)\log R = (L/L_0)|\log R|. \tag{4}$$

Mildner and Hammouda [3] have expressed their transmission results in terms of $|\log R|$. Using a different approach, Dubbers [5,7] has shown that the transmission of the curved guide with non-perfect reflectivity relative to the straight guide with perfect reflectivity is given by

$$T_D(x) = (1/x)\int_G^{\infty} (2/n^4)\, dn \exp(-\alpha n) + (1/x)\int_2^{2Z} (1/4 - 4/n^4)\, dn \exp(-\alpha n) \tag{5}$$

where for $x \le 1$ (garland reflections only) the limits of integration are $G = 1/x$ and $2Z = 2$, and for $x > 1$ (both garland and zig-zag reflections) the limits of integration are $G = 1$ and $2Z = 2[x + (x^2-1)^{1/2}]$. This exact result for the wavelength-dependent neutron transmission can be shown to be equivalent to the earlier results [3]. For a guide with perfect reflectivity, $\alpha = 0$, these reduce to Eq. (1) above. In practice for small reflectivity losses, the transmission may be estimated by expanding the $\exp(-\alpha n)$ factor to second order. For garland reflections only ($x \le 1$) and for both garland and zig-zag reflections ($x > 1$)

$$T_D(x) = \begin{cases} T_0(x) - \alpha x + \alpha^2, & x \le 1 \\ T_0(x) - [2x - x^{-1}]\alpha + [(4/3)x^2(1 + (1 - x^{-2})^{3/2}) - 1/3x]\alpha^2, & x > 1. \end{cases} \tag{6}$$

The integrals in Eq. (5) are derived by expressing the number of reflections, n as a function of the coordinates (z,ψ) at the guide entrance. The loss in the transmission caused by non-perfect reflection becomes an integration of the distribution of the number of reflections per line-of-sight length. We have found a more convenient derivation of the integrals in Eq. (5) by expressing the transmission in terms of the spatial coordinate z and the grazing angle at the outer (concave) surface (see Appendix A). The evaluation of the distribution of the number of reflections is available from the curved guide analysis [2].

3. Comparison of results

Fig. 1 shows a comparison of the different approximations together with the results from computer simulations as a function of the reduced wavelength for various values of reflectivity. This indicates the region of (x,R) space where the agreement for each approximation breaks down and deviates from the simulation result. Neither model is exact, and disagreement becomes more marked as $x = \lambda/\lambda_c$ increases, and the reflectivity R decreases. In fact, both models produce transmissions that eventually diverge upwards relative to the true transmission factor defined by the simulation results. In each case an expansion to third order would

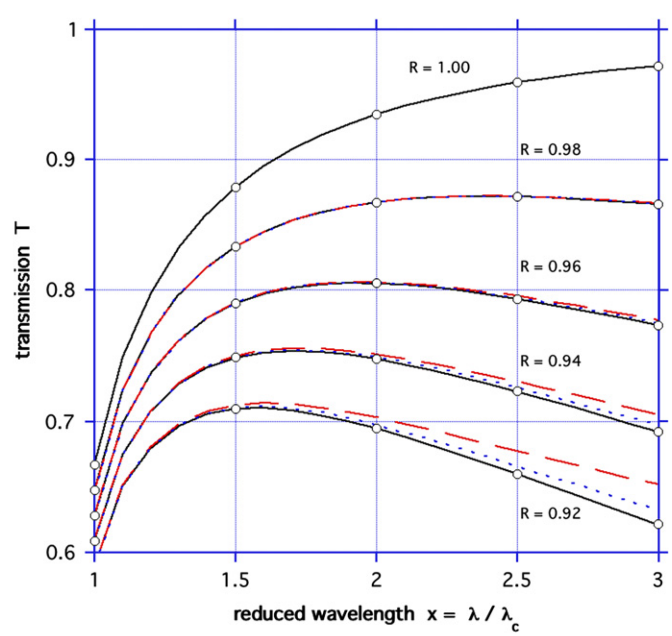

Fig. 1. The transmission of a fully illuminated curved guide of line-of-sight length for different values of the reflectivity R as a function of the reduced wavelength x. The open circles are the computer simulation results. The dotted curves are T_{SM}, the dashed curves are T_D, and the full curves are T_M.

Table 1
Transmission of a fully illuminated curved guide of line-of-sight length for different values of the reflectivity R, as a function of the reduced wavelength $x=\lambda/\lambda_c$, where λ_c is the characteristic wavelength of the guide. The table gives exact values for the transmission for $R=1$, and the values obtained by simulation have 1σ precision. The other values are those obtained through various expressions: (1) T_D [Eq. (6)], (2) T_{SM} [Eq. (3)], (3) $T_0R^{\langle n\rangle}$, (4) $T_0R^{\langle n\rangle_R}$, and (5) $T_M=T_0/2[R^{\langle n\rangle}+R^{\langle n\rangle_R}]$.

		$x=1$	$x=1.5$	$x=2.0$	$x=2.5$	$x=3.0$
$R=1$	Exact	0.6667	0.8788	0.9346	0.9589	0.9717
	Simulation	0.66667(2)	0.87887(2)	0.93463(1)	0.95887(1)	0.97168(1)
$R=0.98$	Simulation	0.64692(2)	0.83335(1)	0.86735(1)	0.87166(1)	0.86569(1)
	T_D	0.6469	0.8334	0.8674	0.8719	0.8662
	T_{SM}	0.6469	0.8333	0.8674	0.8717	0.8659
	$T_0R^{\langle n\rangle}$	0.6468	0.8330	0.8665	0.8703	0.8637
	$T_0R^{\langle n\rangle_R}$	0.6470	0.8337	0.8681	0.8729	0.8676
	T_M	0.6469	0.8333	0.8673	0.8716	0.8656
$R=0.96$	Simulation	0.62747(2)	0.79005(1)	0.80518(1)	0.79331(1)	0.77303(1)
	T_D	0.6275	0.7903	0.8061	0.7954	0.7769
	T_{SM}	0.6275	0.7901	0.8055	0.7941	0.7745
	$T_0R^{\langle n\rangle}$	0.6271	0.7886	0.8021	0.7883	0.7658
	$T_0R^{\langle n\rangle_R}$	0.6278	0.7913	0.8080	0.7980	0.7799
	T_M	0.6275	0.7900	0.8051	0.7932	0.7729
$R=0.94$	Simulation	0.60847(1)	0.74887(1)	0.74770(1)	0.72297(1)	0.69205(1)
	T_D	0.6086	0.7499	0.7511	0.7302	0.7051
	T_{SM}	0.6085	0.7491	0.7488	0.7255	0.6969
	$T_0R^{\langle n\rangle}$	0.6076	0.7457	0.7413	0.7126	0.6774
	$T_0R^{\langle n\rangle_R}$	0.6093	0.7516	0.7537	0.7326	0.7060
	T_M	0.6084	0.7487	0.7475	0.7226	0.6917
$R=0.92$	Simulation	0.58985(2)	0.70969(1)	0.69475(1)	0.65975(1)	0.62112(1)
	T_D	0.5902	0.7123	0.7028	0.6769	0.6518
	T_{SM}	0.5899	0.7101	0.6972	0.6657	0.6322
	$T_0R^{\langle n\rangle}$	0.5883	0.7043	0.6840	0.6427	0.5975
	$T_0R^{\langle n\rangle_R}$	0.5912	0.7143	0.7045	0.6756	0.6437
	T_M	0.5898	0.7093	0.6942	0.6592	0.6206

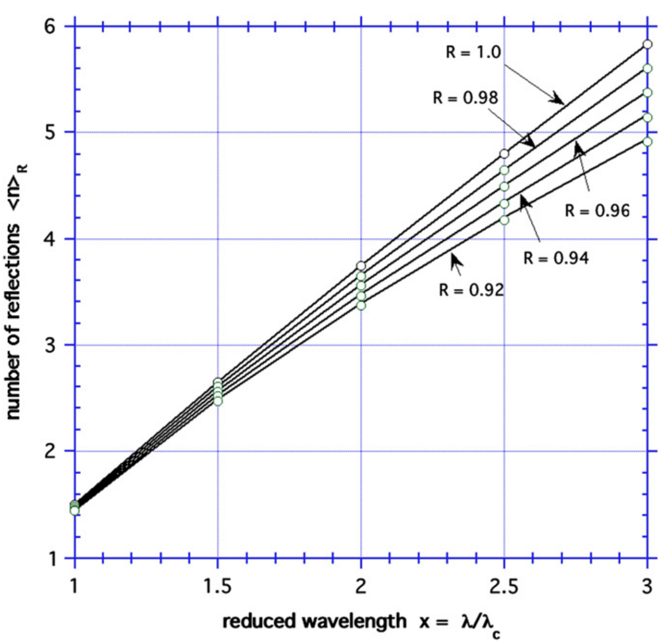

Fig. 2. The average number of reflections $\langle n\rangle_R$ for successfully transmitted trajectories through a fully illuminated curved guide of line-of-sight length for different values of the reflectivity R as a function of the reduced wavelength x. The curves are obtained from Eq. (2) and (A.12). The open circles are obtained from simulation.

enable agreement to higher values of x and lower values of reflectivity.

Table 1 gives a comparison of the transmission at specific values of x. All model transmissions for $x=1$ are comparable within less than 0.2% for $R>0.92$. Both T_D (Eq. (6)) and T_{SM} (Eq. (3)) are excellent approximations for $R\geq 0.96$. For lower values of reflectivity, T_{SM} appears to be the better.

Fig. 2 shows that Eq. (2) is an excellent approximation to the average number of reflections for successfully transmitted trajectories for $R>0.92$ compared to the simulation results that are within 10^{-4} of the exact value. For $x=1$ ($\lambda=\lambda_c$), the approximation $\langle n\rangle_R$ is within 10^{-3} of the simulation results for $R\geq 0.94$. For $x>1$ and $R<1$, $\langle n\rangle_R$ is always greater than the simulation values. The difference increases as x becomes larger and also as R becomes smaller. This difference is less than $\langle n\rangle_R(1-R)/16$ for $R>0.92$.

Eq. (2) shows that $\langle n\rangle_R$ is always less than $\langle n\rangle$, and the difference becomes larger with lower reflectivity R, and longer wavelength λ (increasing n), so that $R^{\langle n\rangle_R}$ is always larger than $R^{\langle n\rangle}$. Note that the values in Table 1 show that $T_0R^{\langle n\rangle_R}$ overcorrects for reflection losses, whereas $T_0R^{\langle n\rangle}$ underestimates the correction. The simulation provides an estimate of $\langle R^n\rangle$. If we expand both $R^{\langle n\rangle_R}$ and $\langle R^n\rangle$ around unity to second order, we obtain, using Eq. (2), then

$$R^{\langle n\rangle_R}-\langle R^n\rangle=[\langle n^2\rangle-\langle n\rangle^2](1-R)^2/2 \quad (7)$$

A similar expansion of $R^{\langle n\rangle}$ gives

$$\langle R^n\rangle-R^{\langle n\rangle}=[\langle n^2\rangle-\langle n\rangle^2](1-R)^2/2 \quad (8)$$

Hence we have not only the inequality

$$R^{\langle n\rangle}<\langle R^n\rangle<\langle R^{\langle n\rangle_R} \quad (9)$$

but also the simulation result may be expressed as the mean of $R^{\langle n\rangle_R}$ and $R^{\langle n\rangle}$. That is,

$$T_M=T_0\langle R^n\rangle=T_0/2[R^{\langle n\rangle_R}+R^{\langle n\rangle}]. \quad (10)$$

This may be observed from the values in Table 1, and can also be seen in Fig. 1. Evidently this expression tracks the simulation remarkably well and is clearly best of all.

In this region of (x,R) space, the result would be no different had the geometric mean rather than the arithmetic mean been taken. It differs from Dubbers' approximation (Eq. (6)) to first order by $(T_0/2)\alpha(\langle n^2\rangle-\langle n\rangle^2)\{\alpha-(1-R)\}$.

4. Discussion

All expressions assume that the reflectivity has a constant value R for χ less than the critical angle corresponding to the given λ, and zero above. We have found a form for the correction that best fits the simulation results. However, an advantage of Dubbers' expression (Eq. (6)) is that it can also take into account a reflectivity model with a step function in which there is a second constant value above some grazing angle. This is accomplished by attributing different ranges over which the integral in Eq. (5) is evaluated, each with different values of α and therefore R.

A further advantage of Dubbers' form for the correction is that it can be used when the guide has dissimilar coatings on the outer and inner surfaces. The grazing angle at the inner surface is always smaller than that at the outer surface for trajectories that have zig-zag reflections in the curved guide. Consequently the inner surface may have a lower critical angle on the inner surface than on the outer surface. We treat the case when the reflectivity of the inner surface is greater than that of the outer surface in Appendix B. This might occur in a phase-space tailoring guide [8] to correct for the spatial asymmetry of the curved guide.

5. Conclusions

We have compared various expressions for calculating the loss in transmission of curved guides that do not have perfect reflectivity. (Note that further losses caused by the straight guide sections that are the polygonal approximation to the curved guide have been ignored.) By comparison with simulation results, we find that there is little difference for a reflectivity greater than 0.94 and for wavelengths less than a few times the characteristic wavelength of the guide. The assumption is that the reflectivity coefficient is a constant that goes abruptly to zero above the critical grazing angle, with no variation in reflectivity over the entire guide surface. We have found a practical expression that is easy to use and best describes the transmission loss over a wide range of wavelengths and reflectivity. We have also given a more simple derivation for Dubbers' result using the distribution of the number of reflections as function of grazing angle at the outer surface rather as a function of the incident spatial coordinate at the entrance of the guide.

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Appendix A

From grazing angle χ to number of reflections n.

The analysis of the transmission of neutrons through the curved guide is performed in terms of the grazing angle χ of the outer surface and the characteristic angle ψ_c of the guide [2]. This enables an alternative and easier derivation of Eq. (5) and (6) originally given by Dubbers [5]. This analysis is based on two expressions for the curved guide, namely the number of reflections per length of direct sight given [2,6] by

$$n(\chi)=\begin{cases}\psi_c/\chi & \chi\le\psi_c \quad \text{(garland)}\\ 2\psi_c[\chi-(\chi^2-\psi_c^2)^{1/2}]^{-1}=2[\chi+(\chi^2-\psi_c^2)^{1/2}]/\psi_c, & \chi>\psi_c \quad \text{(zig-zag)}\end{cases} \tag{A.1}$$

and the distribution $m(\chi)$ of grazing angles χ given by

$$m(\chi)=\begin{cases}4W\chi^2/\psi_c^2 & \chi\le\psi_c \quad \text{(garland)}\\ 4W[\chi^2-\chi(\chi^2-\psi_c^2)^{1/2}]/\psi_c^2, & \chi>\psi_c \quad \text{(zig-zag)}.\end{cases} \tag{A.2}$$

The transmission for garland reflections with grazing angles up to χ assuming perfect reflectivity is given by

$$T_G=(2/3)W\chi^3/\psi_c^2=(2/3)W\psi_c x^3 \tag{A.3}$$

where $x=\chi/\psi_c=\lambda/\lambda_c$, so that $\partial T_G/\partial x=2x^2W\psi_c$. The number of garland reflections per line of sight is given by $n_G(\chi)=\psi_c/\chi=1/x$, so that $dx/dn=-n^{-2}$. Then the transmission for trajectories with only garland reflections up to a limiting glancing angle χ, including transmission loss, is

$$T_G=W\psi_c\int_0^{\chi/\psi_c}2x^2dx\exp(-\alpha n)=W\psi_c\int_\infty^{1/x}(-2/n^4)dn\exp(-\alpha n). \tag{A.4}$$

Relative to the straight guide with perfect reflectivity, $T_S=W\chi$, the transmission factor is given by

$$T_D(x)=T_G/T_S=(1/x)\int_{1/x}^{\infty}(2/n^4)dn\exp(-\alpha n), \qquad x\le 1 \tag{A.5}$$

as given by Dubbers.

The transmission for zig-zag reflections with grazing angles up to a limiting value of χ assuming perfect reflectivity is given by

$$T_Z=(2/3)[x^3-(x^2-1)^{3/2}]W\psi_c \tag{A.6}$$

so that $\partial T_Z/\partial x=2[x^2-x(x^2-1)^{1/2}]W\psi_c$. The number of zig-zag reflections per line-of-sight length is given by

$$n_Z(\chi)=2\psi_c[\chi-(\chi^2-\psi_c^2)^{1/2}]^{-1}=2[x-(x^2-1)^{1/2}]^{-1}=2[x+(x^2-1)^{1/2}], \tag{A.7}$$

so that $dx/dn=1/4-1/n^2$. Then the transmission for trajectories with only zig-zag reflections is the integral of $n_Z(\chi)$ for grazing angles greater than ψ_c,

$$T_Z=W\psi_c\int_1^{\chi/\psi_c}2[(x^2-x(x^2-1)^{1/2}]dx\exp(-\alpha n)$$
$$=W\psi_c\int_2^{2Z}(1/4-4/n^4)dn\exp(-\alpha n) \tag{A.8}$$

where $Z=x+(x^2-1)^{1/2}$. We now need to add this result to that for garland reflections to obtain the transmission factor of a curved guide with non-perfect reflectivity relative to the straight guide given by Eq. (5).

Writing the average transmission $\langle R^{n(L/L_0)}\rangle$, where α is given by Eq. (4), and expanding the exponent to second order, then the transmission relative to the straight guide is given by

$$T_D=T_0\int\{1-\alpha n(\chi)+\alpha^2n^2(\chi)/2\}m(\chi)d\chi\left[\int m(\chi)d\chi\right]^{-1}$$
$$=T_0[1-\alpha\langle n\rangle+\alpha^2\langle n^2\rangle/2]. \tag{A.9}$$

The values of $\langle n\rangle$, the average number of reflections, and $\langle n^2\rangle$ in the horizontal plane for a curved guide of length L_0 equal to the line-of-sight can be determined analytically. For longer guides of length L, the number of reflections must be multiplied by (L/L_0). For garland reflections ($\lambda\le\lambda_c$, or $x\le 1$) only

$$\langle n\rangle=(3/2)x^{-1}$$
$$\text{and } \langle n^2\rangle=3x^{-2} \tag{A.10}$$

With $T_0=(2/3)x^2$, we obtain

$$T_D(x)=T_0-\alpha x+\alpha^2, \qquad x\le 1 \tag{A.11}$$

For garland and zig-zag reflections ($\lambda > \lambda_c$ or $x > 1$) we have

$$\langle n \rangle = (3/2)(2x^2-1)[x^3-(x^2-1)^{3/2}]^{-1}$$
$$\text{and } \langle n^2 \rangle = [4x^3+4(x^2-1)^{3/2}-1][x^3-(x^2-1)^{3/2}]^{-1} \quad \text{(A.12)}$$

With $T_0(x) = (2/3)x^2[1-(1-x^{-2})^{3/2}]$, we obtain

$$T_D(x) = T_0-[2x-x^{-1}]\alpha+[(4/3)x^2(1+(1-x^{-2})^{3/2})-(1/3)x^{-1}]\alpha,^2 \quad x > 1. \quad \text{(A.13)}$$

We can also determine the 'vertical' transmission for the curved guide, which can be treated as a straight guide of width H and length L. The average number n_S of reflections for a trajectory at angle χ to the guide axis, and therefore with a grazing angle χ, is given by $n_S=(L/H)\chi$. The transmission is given by

$$T_S = (H\theta_c)^{-1} \int_0^H dz \int_0^{\theta_c} d\chi \exp(-\alpha n_S)$$
$$= (H/L\alpha\theta_c)[1-\exp(-\alpha(L/H)\theta_c] \quad \text{(A.14)}$$

where $\alpha=|\log R|$, and θ_c is the critical angle at the wavelength λ. This reduces the unity for perfect reflectivity.

Appendix B

Dissimilar wall coatings

For a trajectory that makes zig-zag reflections in the curved guide, the value of χ at the outer surface is greater than the grazing angle $\sqrt{(\chi^2-\psi_c^2)}$ at the inner surface. Hence it is possible to have a coating with a lower critical angle on the inner surface than on the outer surface, such that the mean reflectivity of the inner wall R' is greater than that R of the outer wall. Let $\alpha' = (L/L_0)|\log R'|$. Then the transmission for $x=\lambda/\lambda_c > 1$ is given by

$$T_D(x) = \begin{array}{l} T_0-\alpha'[x-x^{-1}]-\alpha x+\alpha'^2[(2/3)x^2\{(1+(1-x^{-2})^{3/2}\}-(2/3)x^{-1}], \\ +\alpha^2[(2/3)x^2\{(1+(1-x^{-2})^{3/2}\}+(1/3)x^{-1}] \end{array} \quad x > 1. \quad \text{(B.1)}$$

If $R'=R$, so that $\alpha'=\alpha$, and $T_R(x)$ reduces to Eq. (6).

Alternatively, note that half the zig-zag reflections have a reflectivity R and the other half R'. Hence on average zig-zag trajectories have a reflectivity $\sqrt{(RR')}$, the geometric mean, with a transmission factor of $(RR')^{(n/2)(L/L_0)}$, whereas the reflectivity for garland reflections remains R. With $\alpha''=(L/L_0)$ $|\log \sqrt{(RR')}|$, the transmission becomes

$$T_D(x) = T_0-\alpha/x+\alpha^2/x-2\alpha''(x-x^{-1})+\alpha''^2(4/3)[x^2\{1+(1-x^{-2})^{3/2}-x^{-1}\}] \quad \text{(B.2)}$$

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