2011 IEEE International Conference on Control Applications (CCA)
Part of 2011 IEEE Multi-Conference on Systems and Control
Denver, CO, USA. September 28-30, 2011

A Mathematical Model of Joint Congestion Control and Routing in Multisource Networks

Fern Y. Hunt
Applied Mathematics and Computation Division
National Institute of Standards and Technology
Gaithersburg, Maryland, USA 20899
fern.hunt@nist.gov

Vladimir Marbukh
Applied Mathematics and Computation Division
National Institute of Standards and Technology
Gaithersburg, Maryland, USA 20899
vladimir.marbukh@nist.gov

Yi Wang
Department of Mathematics
University of Minnesota
Minneapolis, Minnesota, USA

Abstract— **In this paper we study a model of joint congestion control and routing in a ring network of sources with a single destination at the center (Figure 2). A utility maximization problem subject to routing constraints is posed and equations for its solution are presented. The distribution of traffic on routes available to a source is subject to an entropy constraint that controls the path diversity or degree of robustness of the allocation. Thus the utility/stability issue can be addressed directly and quantitatively in a way that differs from previous work on multiroute NUM problems.**

The dynamics of the model equations will be analyzed in the case of a constant route allocation defined by the allocation distribution entropy for a source. Motivated by earlier work on a two link network, the dynamics of the mean route costs for each source in the ring network are studied by deriving a continuous time approximation of the equations they satisfy. The equilibrium solutions of this approximation are used to greatly simplify the analysis of the model equations and the solution of the original optimization problem. We conclude with a discussion of the tradeoff between utility and path diversity (robustness) for two contrasting assignment of link capacities. Given a homogeneous assignment of capacities the network behaves like a two link model (Fig 1), while a heterogeneous assignment produces utilities displaying different tradeoffs for different sources.

I. INTRODUCTION

In recent years, network protocols have been interpreted as algorithms that solve a convex optimization problem (see e.g. [1], [2]). In this set-up overall congestion control is formulated as a network utility maximization problem (NUM) that is solved in a distributed fashion by the various network layers. The network topology and the capacity of its links introduce constraints on the optimal solution. Then algorithms solve the problem by computing and modifying the primal and dual variables based on efficient communication between users, link and router layers of the network. Since the work of Kelly et al [2] congestion control protocols have been seen as regulating user network transmission rates so that the objective function that is, the aggregate utility, is maximized subject to capacity constraints ([3], [4]). The utility function incorporates efficient utilization and fair allocation of resources among users. Significantly such functions have been identified for existing protocols such as TCP and BGP (Border Gateway Protocols) through a process of "reverse engineering", thus opening up opportunities for analysis and improvement of existing protocols as well as the development of new ones.

The paradigm just described has been extended to the problem of characterizing protocols that jointly control congestion and routing ([1], and see references in [5]. Aside from the obvious improvement in the utilization of network resources that could be gained by such an approach, there are benefits in this time of cyber security concerns, to building adequate network robustness against route disruptions. However as noted in [1] there is a tradeoff between robustness through path diversity on the one hand and network performance or utility on the other. Single path routing based on e.g. the OSPF (Open Shortest Path First) protocol can lead to route flapping instability. However splitting traffic equally across all paths regardless of cost would also imply reduced utility. The best tradeoff if it exists would have to navigate between these extremes.

In previous work ([5],[6]), we analyzed an algorithm for joint congestion control and routing where this tradeoff could be assessed. A generalized NUM (network utility maximization) problem was developed for an arbitrary network and following [1], equations for the solving the dual optimization problem were developed and analyzed for simple cases. As in these references, the use of a dual formulation introduces a decentralized algorithm that results in a lower utility than could be achieved theoretically in the unconstrained problem on the one hand but on the other hand is a more stable and robust implementation. In our model, each source s, was assigned a route according to an allocation defined by a route probability distribution. The distribution entropy h_s (see [6]) measured the path diversity or alternatively, the degree of flexibility in the choice of routes available available to source s. Equilibrium solutions of the model equations were used to find the maximal network or aggregate utility over all bandwidth rates and route allocations whose distribution entropy is greater or equal to h_s.

In this paper we want to consider the aggregate utility/path diversity trade-off in the context of larger networks. In principle this implies systems with more source-destination pairs and more complex topology. Given the nature of the optimization problem solved by the algorithm we have chosen the relatively simple but still more realistic ring network-with an arbitrary number of sources and a single destination. The ring; a circle of pairwise connected sources with a destination at the center, is a natural extension of a single two link network and yet it allows us to investigate conditions for dual optimality and the effect of link capacities on the aggregate utility as route allocation distribution(s) vary. In [5] and [6], the entropy constraints created an implicit equation for each source to be solved at each iteration step of the algorithm. For a network with numerous sources however, the amount of numerical computation is prohibitive. Therefore we take two steps to simplify the problem. The analysis of the two link case revealed that the optimal route allocation distribution is piecewise constant over convex regions that depend on the link costs. Thus it is reasonable to assume that an investigation of the variation of the aggregate utility as a function h_s for this simplified problem can provide some information about the original and therefore give us some insight about the tradeoff between aggregate utility and diversity in the cases we discuss (see below). We approximate the model equations by its continuous relaxation, a multiroute version of the dual equation in [2],[3], to obtain an equation for the mean route cost of each source over time. In particular if the sources are assumed to have the same route allocation distribution $\hat{\beta} = \{\beta_{s1}, \beta_{s2}\}_s$ and thus the same route entropy, the costs converge to a unique equilibrium.

II. Derivation of Model Equations

A. Utility optimization with routing constraints

Consider a planar network to be a graph with nodes representing physical nodes in the network and edges representing links. The link capacities are defined by the vector $\mathbf{c} = (c_1, c_2, \cdots, c_L)$. A user who requires bandwidth to transmit from one node to another in the network (or a single TCP session between those two nodes) is indexed by s, the index of the source-destination pair. For each s the network operator assigns a bandwidth rate x_s and uses a twice differentiable strictly concave utility function $U_s : [m_s, M_s] \to \mathbb{R}$. Here m_s and M_s are lower and upper bounds respectively on the bandwidth rate. $U_s(x_s)$ measures the degree of user satisfaction, network fairness and efficiency for s. Users in a source-destination class s are assigned a path by edge routers so that the fraction of users allocated to path or route r is β_{sr} with $0 < \beta_{sr} \leq 1$. We can regard $\beta_s = \{\beta_{sr}\}_{r \in \mathcal{R}_s}$ as a probability distribution because the fractions sum to 1. Here $\mathcal{R}_s$ is the set of all paths available to s. The distribution is constrained so that the traffic on any link does not exceed the link capacity.

The optimization problem that our protocol seeks to solve is therefore:

$$\max_{\beta \geq 0, \mathbf{x} \in X \; \mathbf{x} \geq 0} \sum_s U_s(x_s) \tag{1}$$

$$\sum_s \sum_{r \in R_s(l)} \beta_{sr} x_s \leq c_l \tag{2}$$

$$\forall s \sum_{r \in \mathcal{R}_s} \beta_{sr} = 1, \quad \beta_{sr} \geq 0 \tag{3}$$

$$-\sum_{r \in \mathcal{R}_s} \beta_{sr} \log \beta_{sr} \geq h_s \tag{4}$$

$\mathbf{x} = \{x_s : s = 1, 2, \cdots S\}$ is the vector of source rates with each $x_s \in [m_s, M_s]$ where $m_s \geq 0$ and $R_s(l)$ is the set of routes used by source s that require link l. In this paper we will take $m_s = 0$ for all s and $U_s(x_s) = w_s(1 - \alpha_s)x_s^{1-\alpha_s}$ with $\alpha_s = 2$. The matrix $\beta = \{\beta_{sr}\}_{s,r}$ is a set of probability distributions that define path allocations for each source s. The constraints in (2) state that all routes that use link l i.e. routes r in $R_s(l)$ of source s be assigned bandwidth rates $\beta_{sr} x_s$ so that the total link load does not exceed the capacity c_l. Equation (3) is the usual requirement for probability distributions and equation (4) places a lower bound on the degree of randomness for the distribution $\{\beta_{sr}\}$ for source s. Indeed recall that for any allocation $\beta_\mathbf{s}$, the entropy of the associated probability distribution is:

$$\mathsf{H}(\beta_\mathbf{s}) = -\left(\sum_{r \in \mathcal{R}_s} \beta_{sr} \log \beta_{sr} \right). \tag{5}$$

Thus equation (4) is just $\mathsf{H}(\beta_\mathbf{s}) \geq h_s$. For example if $h_s = 0$, then all allocations are permissible because $\mathsf{H}(\beta_\mathbf{s}) \geq 0$ holds for all allocations $\beta_\mathbf{s}$ including non-robust allocations that place all source traffic on just one path ($\mathsf{H}(\beta_\mathbf{s}) = 0$). If that path maximizes utility the allocation would be optimal. When $h_s > 0$, these paths are excluded and the more robust paths are retained. The degree of robustness of the set of feasible allocations is dictated by the size of h_s.

Problem (1-4) is not convex in $(\mathbf{x}, \beta)$ but it can be made to be convex by performing the invertible change of variables $(x_s, \beta_{sr}) \to (x_s, y_{sr}) : y_{sr} = \beta_{sr} x_s \;\; r \in \mathcal{R}_s \;, s = 1, \cdots, S$. The transformed system is convex in $(\mathbf{x}, \mathbf{y})$ [1].

B. Description of Model

As is customary in a constrained optimization problem, dual variables- in our case the link costs, play an integral role in the solution of equations (1)-(4). To solve the dual problem, a projection gradient procedure is employed and as in previous works, (see references [3], [1]), the dynamics of the link costs $\{p_l, \; l = 1, 2, \cdots, L\}$ and route allocations at each iteration step is a model of the behavior of the protocol at each time step. Under appropriate conditions on U_s and the initial link costs (see reference [6]) the iterations converge to an equilibrium cost vector p^*, the solution of the dual optimization problem. The solutions of the original optimization $\mathbf{x}^*$ and β^* of equations (1)-(4) are functions of p^* determined by the optimization conditions (see (7) and (9)).

If $p_l^{(k)}$ is the link cost at time k and c_l is the capacity of the lth link then the equations of the model are

$$p_l^{(k+1)} = \left[p_l^{(k)} - h \left\{ c_l - \sum_s x_s(k) \sum_{r \in R_s(l)} \beta_{sr}^{(k)} \right\} \right]^+$$
$$l = 1, \cdots L \tag{6}$$

$$\beta_{sr}^{(k)} = \exp(-\gamma_s^{(k)} d_r(k))/Z_s(k) \tag{7}$$

where $d_r(k) = \sum_{l \in r} p_l^{(k)}$ is the cost of route r at time k, h is a step size , $[a]^+ = a$ if $a > 0$ and is 0 otherwise.
$Z_s(k) = \sum_{r \in R(s)} \exp(-\gamma_s^{(k)} d_r(k))$ is the normalization factor for the route distribution and, the variable $\gamma_s^{(k)}$ is the solution of the implicit equation,

$$\gamma_s^{(k)} D_s(k) + \log(Z_s(k)) = h_s, \quad D_s = \sum_{r \in R(s)} \beta_{sr}^{(k)} d_r(k) \tag{8}$$

The model equations are completed by a relation between the bandwidth rate $x_s(k)$ and $D_s(k)$, the mean route cost at time k for positive constants w_s and M.

$$x_s(k) = \min \left(\left(\frac{w_s}{D_s(k)} \right)^{1/2}, \ M \right) \tag{9}$$

If $x_s < M$, then (9) is a necessary optimality condition that it must satisfy.

Equations (7) and (8) force the route distribution $\beta_{sr}^{(k)}$ to be the unique distribution of entropy h_s with the smallest mean route cost at each time step k. Thus the condition on the route distributions is $\mathsf{H}(\beta_{\mathbf{s}}) = h_s$. This constant entropy requirement will constrain the set of values $\{p_l \mid l = 1 \cdots L\}$ for which a bounded $\gamma_s^{(k)}$ exists. The precise set depends on the network topology and capacity of the links. In [6] we determined the regions of p space for the TwoLinks and Diamond networks.

C. TwoLinks Network

A network consisting of a single source (the left node), destination (on the right) connected by two links is depicted in Figure 1. The equations (6) become:

$$p_1^{(k+1)} = \left[p_1^{(k)} - h\{c_1 - \beta_1^k x_s(k)\} \right]^+,$$
$$p_2^{(k+1)} = \left[p_2^{(k)} - h\{c_2 - \beta_2^k x_s(k)\} \right]^+ \tag{10}$$

Equations (7)-(8) are added to these. Here each link is a path or route so $d_i = p_i \quad i = 1, \ 2$.
The dynamics of the TwoLinks systems is conveniently described in terms of a critical entropy $h_T(c) < \log 2$ given by,

$$h_T(c) = - \left[\frac{c_1}{c_1 + c_2} \log \frac{c_1}{c_1 + c_2} + \frac{c_2}{c_1 + c_2} \log \frac{c_2}{c_1 + c_2} \right] \tag{11}$$

with a corresponding critical route distribution,

$$\beta_1^* = \frac{c_1}{c_1 + c_2} \ , \ \beta_2^* = \frac{c_2}{c_1 + c_2} \tag{12}$$

In [6], we showed that the optimal route distribution given

Fig. 1. A two links network (*TwoLinks*)

by (7) , (8) , converges in a single step to one of two distributions corresponding to the value of h_s when $h_s < \log(2)$. The specific distribution depends on the initial link cost vector and link capacities. An important implication of this fact is that once these variables are defined, *we can dispense with the implicit equation* and greatly simplify the computations and analysis of the link equations. Calculations based on the use of (10) and (8), and then on (10) and the constant distribution were compared and the results were identical.

Let us now describe the dynamics of the link equations for $h_s = h_T(c)$. Within the quadrant sector $\mathsf{P} = \{\hat{p} = (p_1, p_2,) : p_1 < p_2 \}$, the route probabilities satisfy $\beta_1^{(k)} > \beta_2^{(k)}$ as long as $(p_1^{(k)}, p_2^{(k)}) \in \mathsf{P}$. If $h_s = h_T(c)$, and $c_1 > c_2$, then (β_1^*, β_2^*) is the unique optimal probability distribution and $\beta_i^{(k)} = \beta_i^* \ i = 1, 2$ in P. At the value, $h_T(c)$, if $(p_1^{(0)}, p_2^{(0)}) \in \mathsf{P}$, either $(p_1^{(k)}, p_2^{(k)}) \in \mathsf{P}$ for $k \geq 1$ or $(p_1^{(k)}, p_2^{(k)})$ reaches the line $L = \{\hat{p} = (p_1, p_2) : p_1 = p_2\}$, where the solution of (8) is undefined. Therefore away from L, the route distribution is constant and equal to the values in (12) for $k \geq 1$. There is an open subset $\mathcal{H}$ of P where it can be shown that all subsequent iterates of (10) remain away from L and converge to a point on the line of points with constant mean route value, $L^* = \{(p_1, p_2) : \beta_1^* p_1 + \beta_2^* p_2 = \bar{x}\}$ where $\bar{x} = w(\frac{\beta_2^*}{c_2})^2$. The position of the equilibrium point on L^* depends on the initial value $(p_1^{(0)}, p_2^{(0)})$, so it cannot be unique. However, $\bar{x}$ is unique and from this one obtains a unique optimal bandwidth rate $x^* = \left(\frac{w}{\bar{x}}\right)^{1/2}$. Orbits beginning outside of $\mathcal{H}$ limit to a point on the p_2 axis.

For $h_s > h_T(c)$, $\hat{p} \to (0, p^*), \quad p^* > \bar{x}$. For $h_s < h_T(c)$, there is no convergence to an equilibrium. In cases where $\hat{p}$ converges, we can use the results of [3] to conclude that

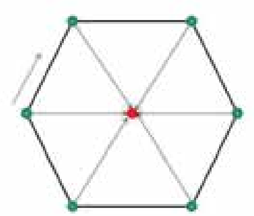

Fig. 2. RING MULTISOURCE NETWORK

the limiting value is an optimal solution of the dual problem (see [6]).

III. RING NETWORK

Our goal in this section is to solve the optimization problem for a more complex but still tractable multisource network. In Section II-C, the optimal route allocation distribution was constant after the first time step. By choosing region P, a convex subset of the non-negative quadrant of $\mathbb{R}^2_+$, the solutions of the dual problem could be determined by substituting this route distribution into the equations and determining the equilibrium behavior. Given a set of route entropies,for the multisource network we will assume a constant allocation distribution for each source. Some rationale for this assumption comes from the fact that each source in our multisource network has just two links.

We will sketch an approach to the analysis of the link equations and determining its equilibria, based on the dynamics of the mean route cost for each source. A system of ordinary differential equations for the mean route costs is formally derived and is approximately valid for small step sizes h for some interval of time whose length it is assumed, does not shrink to zero with step size. In section III-A we identify sufficient conditions for the existence of a positive equilibrium for the differential equations. The analysis of (13-15) can be simplified by setting the mean route cost equal to the positive equilibrium of 18 when it exists. Therefore the source bandwidth rate is determined by(9). Any equilibrium values of the link equations are solutions of the dual problem for the network and the bandwidth rates are optimal. Setting the mean route costs in advance can be justified in cases where the mean route costs converge before the individual link costs. A numerical example of this is shown. More numerical calculations are found in section III-C.

In the multisource network we consider, all the nodes (sources of the network) are arranged in a circular pattern around a single destination at the circle's center. Each node is connected by a (direct) link to the destination and a second link from the node to its nearest neighbor in a clockwise direction. This means that each source has two routes to the destination. The first through a direct link and the second, a path consisting of a link to the clockwise nearest neighbor followed by a direct link from the neighbor source to the destination. Given S sources, index the sources using $i = 1\ 2, \cdots S$. There are $L = 2 \cdot S$, links. The equations for the direct links, using (6) are:

$$p_1^{(k+1)} = \left[p_1^{(k)} - h\left\{c_1 - \beta_{1,1}x_1(k) - \beta_{S,2}x_S(k)\right\}\right]^+ \quad (13)$$

$$p_i^{(k+1)} = \left[p_i^{(k)} - h\left\{c_i - \beta_{i,1}x_i(k) - \beta_{i-1,2}x_{i-1}(k)\right\}\right]^+ \qquad i = 2\ ,\cdots,\ S$$

$$p_S^{(k+1)} = \left[p_S^{(k)} - h\left\{c_S - \beta_{S,1}x_S(k) - \beta_{S-1,2}x_{S-1}(k)\right\}\right]^+$$

For the indirect (or exterior) links joining the sources we have:

$$p_{i+S}^{(k+1)} = \left[p_{i+S}^{(k)} - h\left\{c_{i+S} - \beta_{i,2}x_i(k)\right\}\right]^+ \qquad i = 1\ \cdots, S-1 \quad (14)$$

$$p_L^{(k+1)} = \left[p_L^{(k)} - h\left\{c_L - \beta_{S,2}x_S(k)\right\}\right]^+ \quad (15)$$

Here $\beta_{i,1}$ and $\beta_{i,2}$ are the probabilities of choosing the direct or indirect paths respectively and are constants in the problem using the assumptions discussed at the beginning of this section. Fixed points, i.e. equilibrium values of the system (13)-(15) are solutions of the dual problem and can be used to generate the solution of (1)-(4). Sufficient conditions for an internal equilibrium where all link prices p_i are positive are:

$$\beta_{1,1} = \frac{c_1 - c_L}{x_1} \quad \beta_{i,1} = \frac{c_i - c_{i-1+S}}{x_i} \qquad i = 1\ ,2\ \cdots S \quad (16)$$

$$x_1 = c_1 + c_{S+1} - c_L$$
$$x_i = c_i + c_{S+i} - c_{S+i-1}$$
$$i = 1\ ,2\ ,\cdots S$$

Other equilibria occur when one or more of the $p_i = 0$. Rather than investigate this system in full generality we will consider a simplified version of (13)-(15), based on a differential equation approximation. This technique of approximating the equations for the dual problem by a continuous or relaxed system (for a different utility function and single path routing) was used by Kelly et al in [2].

Given a route allocation $\hat{\beta}$, the mean or average route cost to source i is:

$$\begin{aligned} D_i = \beta_{i,1} p_i + \beta_{i,2} \left(p_{i+S} + p_{i+1} \right) \ , \\ i = 1 \ , 2 \ , \cdots , S-1 \\ D_S = \beta_{S,1} p_S + \beta_{S,2} \left(p_L + p_1 \right) . \end{aligned} \quad (17)$$

To derive equations for the mean route costs we assume that:

- There exist $k_1 > k_0 \geq 0$ with $k_1 - k_0 \nrightarrow 0$ as $h \to 0$, such that $R_l(k) \geq 0 \ , \ l = 1 \cdots L$ for all $k_0 \leq k \leq k_1$.

Here $[R_l(k)]^+$ is the right hand side of the equation for $p_l^{(k+1)}$ in (13-15). Under this assumption the equations for mean route costs can be obtained by using (17) and adding the corresponding right hand sides of the link costs system. On taking the limit of small h as $h \to 0$, we formally derive the following equations:

$$\frac{dD_i}{dt} = - \left[C_i - \sum_{j=1}^{S} W_{ij} f_j(D_j) \right] \ , \ i = 1 \ \cdots \ S \quad (18)$$

with

$$f_i(D_i) = \min\{ \left(\frac{w_i}{D_i} \right)^{\frac{1}{2}} , M \} \ ,$$

where $C_i = \beta_{i,1} c_i + \beta_{i,2} \left(c_{i+S} + c_{i+1} \right)$ for $i = 1 \ 2 \ \cdots S-1$, and $C_S = \beta_{S,1} c_S + \beta_{S,2} \left(c_L + c_1 \right)$. The elements W_{ij} for $i < S$ are,

$$W_{ij} = \begin{cases} 0, & \text{if } |i-j| > 1 \ ; \\ \beta_{i,1}\beta_{i-1,2} & \text{if } j = i-1 \ ; \\ \beta_{i,2}\beta_{i+1,1} & \text{if } j = i+1 \ ; \\ \beta_{i,1}^2 + 2\beta_{i,2}^2 & \text{if } i = j; \end{cases} \quad (19)$$

and $W_{S,S-1} = \beta_{S-1,2}\beta_{S,1}$, $W_{S,1} = \beta_{S,2}\beta_{1,1}$ with other elements $W_{S,j} = 0$ if $j \neq 1, S$.

We can expect the differential equation to approximate the dynamic behavior of the mean route costs while the $\{R_l(k) \ l = 1 \cdots L\}$ are non-negative. Under the assumption this remains true for all sufficiently small h.

A. Ring Network Dynamics in Continuous Model

The dynamics of (18) simplify if we introduce the function

$$V(D) = - \sum_{i=1}^{S} C_i f_i(D_i) + \frac{1}{2} \sum_{i,j=1}^{S} W_{ij} f_i(D_i) f_j(D_j) \quad (20)$$

We are interested in the situation where $f_i(D_i) = \left(\frac{w_i}{D_i} \right)^{\frac{1}{2}}$ for $i = 1 \cdots S$ so the $\{f_i\}$ are differentiable. We discuss when this is appropriate later in the discussion. It follows from computing the gradient of V and using (18) that $\frac{dV}{dt} = \sum_{i=1}^{S} f_i'(D_i) \left(\frac{dD_i}{dt} \right)^2 \leq 0$, with equality holding only at the equilibrium points. Orbits of (18) move in the direction of decreasing values of V, perpendicular to $V = constant$ level curves. The system (18) can be rewritten as ([8])

$$\frac{dD_i}{dt} = \frac{1}{f_i'(D_i)} \frac{\partial V}{\partial D_i} \ , \ i = 1 \cdots S \quad (21)$$

Thus, any equilibrium point of (18) is a critical point of V and if the matrix $W = (W_{ij})$ is positive definite, an equilibrium corresponding to the solution of the equation $Wx^* = C$ exists, and it is unique. If in addition $x_i^* > 0$ we can then get the equilibrium bandwidth rates from the relations $x_i^* = f_i(D_i^*) > 0$. Sufficient conditions for the existence of a positive x^* involve verifying a version of Farkas' lemma with strict inequalities involving the route probabilities and link capacities [9]. In this case we can choose M large enough so that the corresponding value of D is far from the equilibrium point of (18). We can then situate the initial mean route cost vector D^0 very close to the equilibrium point. Since it is asymptotically stable, $D(t)$ will remain close to it for $t \geq 0$. Then the $\{f_i(D_i(t))\}$ defined in (18) are differentiable.

Specific use of the equilibria of (18) in the link equations has only been carried out so far for the very restricted homogeneous capacity case discussed in section III-C. Extension to more general cases is left for future work. However, we can present choices of capacities where convergence of the mean route costs occurs before the link costs converge, as illustrated in Figure 3. For such cases the large time behavior of (13-15) can be inferred by substituting the equilibrium mean route cost into these equations. The rationale comes from the observation (proved for example networks in [6], [5]) that if the cost vector $\hat{p}$ is in or sufficiently close to the region defined by the equilibrium mean route cost D^*, $\{\hat{D} \in \mathbb{R}^S : D_i = D_i^*\}$, it remains there for all subsequent time.

B. Symmetric Ring Network

Suppose that all the sources use the same route allocation distribution so that $\beta_{i,1} = a$ and $\beta_{i,2} = b = 1 - a$. Let us (without loss of generality) also assume that $w_i = w$ in (18). Then the orbits of (18) approach a unique equilibrium point given by the solution of the equation $Wx^* = C$ when W is non-singular and therefore positive definite. If x^* is positive,then for each i, $x_i^* = f(D_i^*) = \left(\frac{w}{D_i^*}\right)^{\frac{1}{2}}$. The rest of this subsection sketches a proof that W is positive definite and in the next subsection we discuss several numerical examples illustrating the dynamics of (13,15) and the behavior of the aggregate utility as a function of h_s.

Under the single route allocation W takes the form,

$$\begin{pmatrix} a^2+2b^2 & ab & 0 & \cdots & \cdots & ab \\ ab & a^2+2b^2 & ab & 0\cdots\cdots & \cdots\cdots 0 & 0 \\ 0 & ab & a^2+2b^2 & ab & 0\cdots\cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ ab & 0\cdots\cdots & 0\cdots\cdots & \cdots\cdots & ab & a^2+2b^2 \end{pmatrix} \quad (22)$$

To show that W is positive definite we first observe that W is symmetric for any choice of allocation $\hat{\beta}$. Positive definiteness holds if and only if the determinants of the principal minors of W are positive. This can be proved by induction for minors up to order $S-1$. For order S, note that (22) implies that W is a circulant matrix. Therefore the determinant can be written explicitly as

$$det(W) = \prod_{k=1}^{S} q(\omega_k)$$

where $\omega_k = \exp\left(\frac{-2\pi ik}{S}\right)$ and $q(z) = a^2 + 2b^2 + abz + abz^{S-1}$. Therefore $det(W) \geq \prod_{k=1}^{S}\left((a-b)^2 + b^2\right) > 0$.

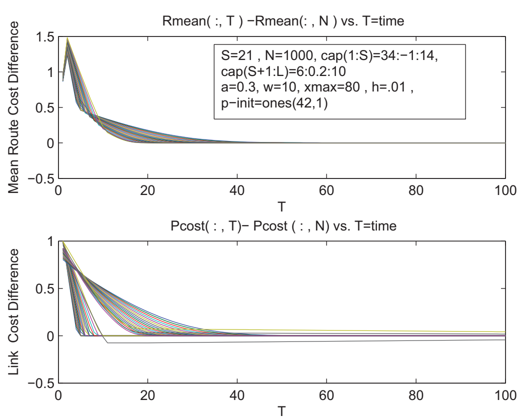

Fig. 3. Plots showing the approach of link costs (Pcosts) and route mean costs (Rmean) to their limiting values. The mean route costs converge more quickly (after 60 time steps) than the link costs (more than 100 time steps)

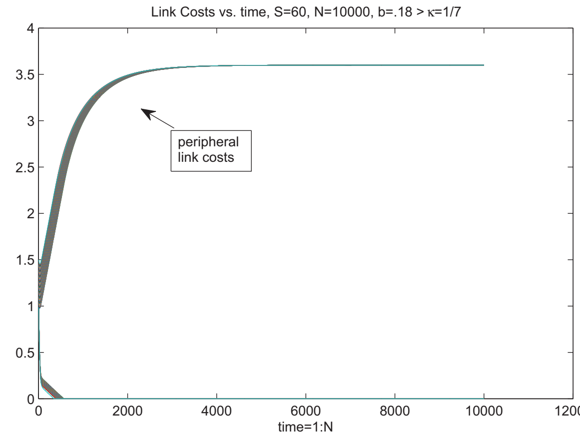

Fig. 4. Link Cost versus time for $b = .18 > \kappa = 1/7$. Other parameters were h=.01, xmax=80, w=20. See the text for the initial values used.

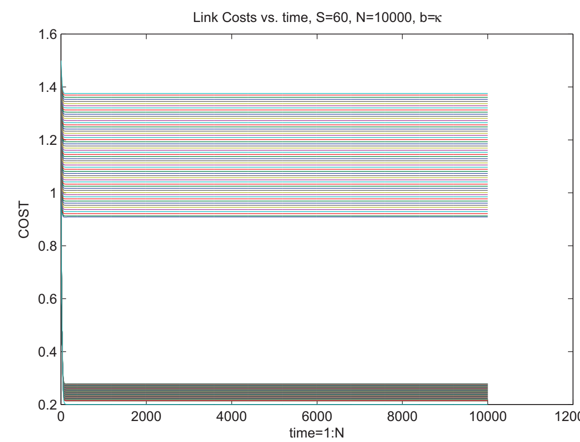

Fig. 5. Link Cost versus time for $b = \kappa$. Upper lines show peripheral link costs, lower lines are direct link costs. Other parameter values are the same as in previous figure.

C. Results of Computation

In this section we will discuss the results of iterating the link cost equations (13-15) by illustrating the possible limiting behaviors of the link costs for two capacity settings. After this, the tradeoff between utility and path diversity in both cases will be analyzed in terms of a plot of the time averaged utility as a function of the entropy of the route distribution. The first case was obtained by setting the capacities of direct links of all sources equal to a common value c_1 and all the capacities of peripheral links equal to $c_L > c_1$. As before, $L = 2S$ is the number of links where S is the number of sources. The equations were iterated for N time steps. The initial link costs, were $p_i^{(0)} = 1 + \frac{\kappa i}{L}$ $i = 1 \cdots S$, where $\kappa = \frac{c_L}{c_1}$, and $p_i^{(0)} = 1 + \frac{(i-S)}{L}$ $i = S+1 \cdots L$.

In the symmetric allocation case introduced in Section B, solutions of (13-15) converged to constant values determined by κ. If $b > \kappa$, the costs of the direct links converged to zero while the corresponding indirect link prices converged to the same positive constant (see Figure 4). If $b < \kappa$, the reverse was true. At $b = \kappa$, all link costs approached non-negative limits that depended on the initial costs. This is illustrated in Figure 5. Despite the different limiting link costs, all sources had the same limiting mean route cost D^*. Thus the system behaved like a two links network with a single source and destination. For the choice of allocation probabilities and capacities under discussion, a positive equilibrium solution of (18) exists and is in fact equal to $D_i = D^*$ $i = 1 \cdots, S$. We now turn to the situation where the link capacities of different sources are different. In contrast to the previous homogeneous case, existence and convergence to a positive equilibrium is not assured for either the link equations or the mean route equations and their continous approximation (18). This complicates the analysis of the system. When there is existence and convergence,([3]) we know that the equilibrium values of the cost equations are dual optimal. Here we will discuss a choice of link capacities where convergence has been observed numerically. The initial link costs in this case including Figure 3 were $p_i = 1$ $i = 1 \cdots L$. As with homogeneous capacities, the limiting value of the direct link costs can be zero while the limits of peripheral link costs are positive. This is the case for the parameters shown in Figure 3. Other behaviors are possible however. Figure 6 shows the time course of the direct links in a network of 6 sources where links 1, 2 and 3 have zero limits while, links 4, 5, and 6 have positive limits.

We now turn to a discussion of the effect of these different capacity settings on the utility/path diversity tradeoff. The aggregate or average utility was computed as a function of the entropy of the route allocation $\{a, b = 1 - a\}$, for $a \geq 1/2$. For a fixed entropy value $h_s = -a\log(a) - b\log(b)$, the utility function U_i was computed as a function of the bandwidth $x_i(k)$ for each time step k and the average was taken over N time steps for each source i. The time average was then averaged over all the sources. The resulting plot for the link capacities and other parameters of Figure 4 is shown in Figure 7. Here the tradeoff between utility and diversity is very similar to that obtained for the two links network. There is a critical value $h^* = .4101$ of the entropy corresponding to $b = \kappa$. When $h_s < h^*$, ($b < \kappa$), traffic on the peripheral links can be increased thereby increasing path diversity without decreasing utility. This situation changes when $h_s = h^*$, ($b = \kappa$). If $h_s > h^*$, ($b > \kappa$), increasing the traffic on peripheral routes will decrease utility.

For the choice of parameters shown in Figure 6, Figure 8 shows the time averaged utility of each source as a function of the route allocation entropy. Although the utility curve for source 1 (and in fact the aggregate utility over all sources) resembles the utility curve for the previous case, the utility curves for remaining links have a maximum utility, depending on the capacity of the individual link. This contrasts with the previous case where in fact all the individual source utility curves coincide with the aggregate utility curve.

IV. Conclusion and Future Work

In this paper we studied a model of joint congestion control and routing in a ring network of sources with a single destination at the center (Figure 2) examining the effect of link capacities on the resulting optimization problem and on the tradeoff between path diversity and utility. Motivated by earlier work on the TwoLink model (Figure 1), we analyzed the dynamics of the link costs equations in the case of a constant route allocation distribution. We proposed a method for finding equation equilibria based on the analysis of the equilibria of (18), an ordinary differential equation system that approximates the dynamics of the mean route costs of the sources. The equations form a gradient system and sufficient conditions for the existence of a unique positive equilibrium can be given in terms of the link capacities and route distribution. In cases where the mean route costs converge more quickly than the link costs, their equilibrium behavior can be inferred by substituting the equilibrium mean route cost and resulting bandwidth rate (9). The resulting equilibrium link cost vector and bandwidth vector constitute a solution of the optimization problem. Numerical examples in section III-C illustrate these ideas.

The tradeoff between path diversity and utility is depicted in plots of utility (aggregate and individual source) as a function of the route distribution entropy h_s, where all sources are given the same route distribution. Two contrasting situations are discussed. The first involves assigning the same direct and peripheral link capacities to all sources. In the second case, the link capacities differ among the sources. In the first, the network behaves like the TwoLinks network and the tradeoff is expressed in terms of a critical value of h_s that depends on $\kappa = \frac{c_L}{c_1}$. The fraction of traffic b assigned to peripheral links cannot exceed κ without decreasing the utility. The utility curves in the heterogeneous case also show similar behavior but critical values vary with the source. A numerical example shows that there can be qualitative differences in the individual utilities even though the aggregate utility curve resembles utility in the homogeneous case.

Future work involves elaboration and extension of the conditions on the link capacities that permit the use of the substitution method proposed for analyzing the link costs equation and determining their equilibria. A very significant additional need is to determine ways to allow for different route distributions for different sources.

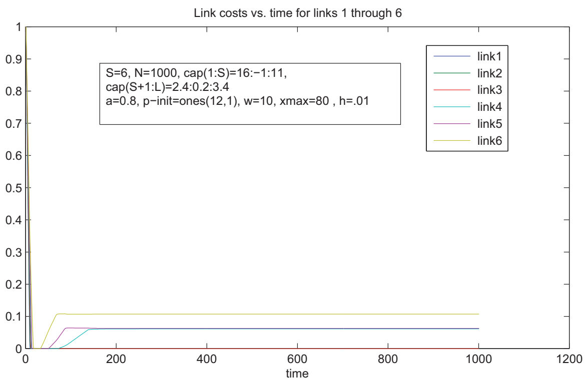

Fig. 6. Link costs versus time for heterogeneous capacities S=6. Capacities for links 1 through 6 decrease from 16 to 11 with step size 1, capacities for links 7 through 12 increase from 2.4 to 3.4 with step size 0.2

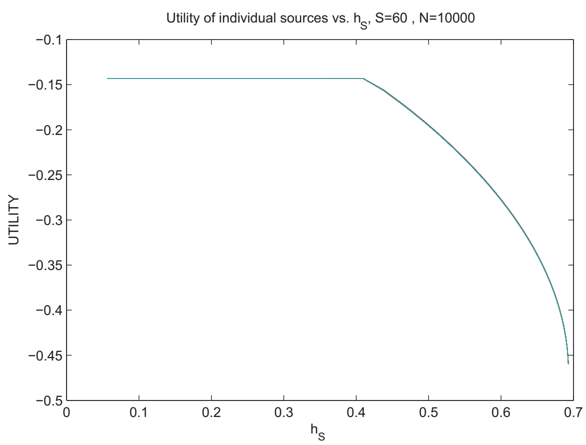

Fig. 7. Utility vs. h_s, for $a \geq 1/2$, other parameters are the same as in Figure 4.

REFERENCES

[1] J.WANG, L. LI, S.H. LOW, J.C.DOYLE "Cross-Layer Optimization in TCP/IP networks" IEEE Transactions on Networking, Vol. 13, No. 3, pp. 568-582, 2005.

[2] F.P. KELLY, A.K.MAULLOO, D.K.H. TAN "Rate control in communication networks: shadow prices, proportaion fairness and stability." Journal of the Operational Research Society, Vol. 49, pp. 237-252, 1998.

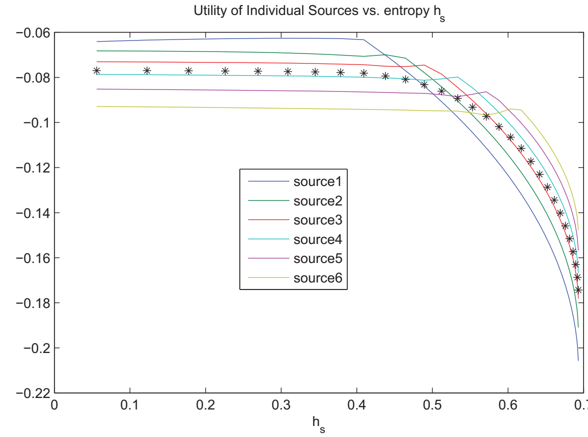

Fig. 8. Individual source utilities vs. h_s, for $a \geq 1/2$. Parameters used were same as in Figure 6. Aggregate utility is shown in black stars.

[3] S.H. LOW, D. E. LAPSLEY "Optimization Flow Control I: Basic Algorithm and Convergence" IEEE/ACM Transactions on Networking Vol. 7, No. 6, pp.861-874, 1999 available at http://netlab.caltech.edu/pub.

[4] M.CHIANG, S.H. LOW, A.R. CALDERBANK, J.C. DOYLE "Layering as optimization decomposition: A mathematical theory of network architectures", Proceedings of the IEEE, Vol. 95, no. 1, pp. 252-312, 2007.

[5] F. HUNT, V. MARBUKH "Measuring the Utility-Path Diversity Tradeoff in Multipath Protocols" Proceedings of the 4th International Conference on Performance Evaluation Methodologies and Tools, Pisa, Italy 2009

[6] F. HUNT, V. MARBUKH "A Model of Joint Congestion Control and Routing Through Random Assignment of Paths" NIST Internal Report no.7590, 2009

[7] D.P. BERTSEKAS, J.N.TSITSIKLIS "Parallel and Distributed Computation: Numerical Methods", Prentice Hall, Englewood Cliffs, N. J. 1989.

[8] R. VIVLELA MENDES "Tools for network dynamics" International Journal of Bifurcation and Chaos, 2005

[9] A.L. PERESSINI, F.E. SULLIVAN, J.J.UHL "The Mathematics of Nonlinear Programming", Spring Verlag, New York, Heidelberg, London,Tokyo, 1980