On parameter differentiation for integral representations of associated Legendre functions

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(Received: ?? January 2011)

For integral representations of associated Legendre functions in terms of modified Bessel functions, we establish justification for differentiation under the integral sign with respect to parameters. With this justification, derivatives for associated Legendre functions of the first and second kind with respect to the degree are evaluated at odd-half-integer degrees, for general complex-orders, and derivatives with respect to the order are evaluated at integer-orders, for general complex-degrees. We also discuss the properties of the complex function $f : \mathbf{C}\setminus\{-1,1\} \to \mathbf{C}$ given by $f(z) = z/\sqrt{(z+1)(z-1)}$.

Keywords: Legendre functions; Modified Bessel functions; Derivatives

AMS Subject Classification: 31B05, 31B10, 33B10, 33B15, 33C05, 33C10

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1 Introduction

This paper is a continuation of work which is presented in Cohl (2010) [6]. In [6], formulae were presented for derivatives of associated Legendre functions of the first kind $P_\nu^\mu(z)$ and the second kind $Q_\nu^\mu(z)$, for $|z|>1$, with respect to their parameters, namely the degree ν and the order μ. The strategy applied in [6] was to differentiate integral representations of associated Legendre functions, which were given in terms of modified Bessel functions of the first and second kind, with respect to their parameters. The derivatives of the integrands, for the integral representations of associated Legendre functions given in [6], which include the derivatives with respect to order evaluated at integer-orders for modified Bessel functions of the first and second kind, are well known (see for instance §3.2.3 in Magnus, Oberhettinger & Soni (1966) [13]).

Unfortunately, in [6], no justification for differentiation under the integral sign of the chosen integral representations of associated Legendre functions is given. In this paper, we give justification for differentiation under the integral sign for the integral representations of associated Legendre functions given in [6] and hence complete our proof for the validity of the parameter differentiation formulae given therein. The parameter differentiation formulae given in [6] are derivatives for associated Legendre functions of the first and second kind with respect to the degree, evaluated at odd-half-integer degrees, for general complex-orders, and for derivatives with respect to the order evaluated at integer-orders, for general complex-degrees. See [6] for a discussion of other known formulae for derivatives with respect to parameters for associated Legendre functions.

This paper is organized as follows. In §2 we investigate properties of the complex function $z\mapsto \dfrac{z}{\sqrt{z^2-1}}$. In §3 we present a description of a map on a subset of the complex plane which leads to the Whipple formulae for associated Legendre functions. In §4 we give justification for differentiation under the integral sign for the integral representations of associated Legendre functions given in [6].

Throughout this paper we use the following conventions. First $\displaystyle\sum_{n=i}^{j} a_n=0$ for all $a_1,a_2,\ldots\in\mathbf{C}$, and $i,j\in\mathbf{Z}$ with $j<i$. Secondly, for any expression of the form $(z^2-1)^\alpha$, read this as

$$(z^2-1)^\alpha := (z+1)^\alpha(z-1)^\alpha,$$

for any fixed $\alpha\in\mathbf{C}$ and $z\in\mathbf{C}\setminus\{-1,1\}$.

2 Properties of the function $z\mapsto z/\sqrt{z^2-1}$

Proposition 2.1. *Define the function $f:\mathbf{C}\setminus\{-1,1\}\to\mathbf{C}$ by*

$$f(z)=\frac{z}{\sqrt{z^2-1}}:=\frac{z}{\sqrt{z+1}\sqrt{z-1}}.$$

This function f has the following properties.

1. $f\big|_{\mathbf{C}\setminus[-1,1]}$ *is even and* $f\big|_{(-1,1)}$ *is odd.*
2. *The sets $(0,1)$ and $(-1,0)$ are mapped onto $i(-\infty,0)$ and $i(0,\infty)$ respectively.*
3. *The sets $i(-\infty,0)$ and $i(0,\infty)$ are both mapped to $(0,1)$.*
4. $f(0)=0$.

5. *If* $z \in \mathbf{C} \setminus [-1,1]$ *then* $\operatorname{Re} f(z) > 0$.

Proof. When $z \neq 0$ and the exponent w is any complex number, then z^w is defined by the equation

$$z^w := \exp(w \log z),$$

where the exponential function can be defined over the entire complex plane using the power series definition

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

and the logarithmic function is defined for points $z = re^{i \arg z}$, with $r > 0$, as

$$\log z := \log r + i \arg z.$$

Recall that if $z \in \mathbf{C} \setminus \{0\}$, then $\arg z$ (often referred to as the argument, amplitude or phase) is given by the angle measured from the positive real axis to the vector representing z. The angle is positive if measured anticlockwise and we choose the $\arg z \in (-\pi, \pi]$. Note that

$$\arg(\sqrt{w}) = \frac{1}{2} \arg w.$$

If $z \in \mathbf{C}$ and $\operatorname{Im} z > 0$ then

$$\arg(-(z \pm 1)) = -\pi + \arg(z \pm 1),$$

so

$$\arg\left(\sqrt{-(z \pm 1)}\right) = -\frac{\pi}{2} + \arg\left(\sqrt{z \pm 1}\right),$$

and we have

$$\sqrt{-(z \pm 1)} = -i\sqrt{z \pm 1}.$$

Hence

$$f(-z) = \frac{-z}{i^2 \sqrt{z+1}\sqrt{z-1}} = f(z).$$

Similarly if $\operatorname{Im} z < 0$ then

$$\sqrt{-(z \pm 1)} = i\sqrt{z \pm 1},$$

and we have the same result.

Let $x > 1$. Then

$$\arg \sqrt{-(x \pm 1)} = \frac{\pi}{2},$$

so

$$f(-x) = \frac{-x}{\sqrt{-(x+1)}\sqrt{-(x+1)}} = \frac{x}{\sqrt{x+1}\sqrt{x-1}} = f(x).$$

Therefore $f\big|_{\mathbf{C}\setminus[-1,1]}$ is even.

If $x \in (0,1)$ then

$$f(x) = \frac{-ix}{\sqrt{1+x}\sqrt{1-x}},$$

and

$$f(-x) = \frac{ix}{\sqrt{1+x}\sqrt{1-x}} = -f(x).$$

Moreover, $f(0) = 0$. Therefore $f\big|_{(-1,1)}$ maps to the imaginary axis and is odd.

If $x \in (0, \infty)$ then

$$f(ix) = \frac{ix}{\sqrt{ix+1}\sqrt{ix-1}} = \frac{x}{\sqrt{1+x^2}},$$

and

$$f(-ix) = \frac{-ix}{\sqrt{-ix+1}\sqrt{-ix-1}} = \frac{x}{\sqrt{1+x^2}},$$

so f maps both the positive and negative imaginary axes to the real interval $(0, 1)$. Clearly $f(0) = 0$. This completes the proof of 1, 2, 3 and 4.

Before we prove 5 we first show that f maps quadrant I into quadrant IV. This is non-trivial. Let $r \in (0, \infty)$ and $\theta \in (0, \pi/2)$. Then

$$f(re^{i\theta}) = \frac{r \exp\left[i(\theta - \frac{1}{2}\phi - \frac{1}{2}\psi)\right]}{(r^4 - 2r^2\cos(2\theta) + 1)^{1/4}},$$

where

$$\phi := \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta + 1}\right),$$

and

$$\psi := \begin{cases} \pi + \tan^{-1}\left(\dfrac{r\sin\theta}{r\cos\theta - 1}\right) & \text{if } r\cos\theta < 1, \\ \dfrac{\pi}{2} & \text{if } r\cos\theta = 1, \\ \tan^{-1}\left(\dfrac{r\sin\theta}{r\cos\theta - 1}\right) & \text{if } r\cos\theta > 1. \end{cases}$$

Firstly we would like to prove that $\theta - \phi/2 - \psi/2 < 0$, or equivalently

$$\phi + \psi > 2\theta. \tag{1}$$

We will break the problem into nine main cases with

$$\left[\begin{array}{ll} \text{I.} & \theta \in \left(0, \frac{\pi}{4}\right) \\ \text{II.} & \theta = \frac{\pi}{4} \\ \text{III.} & \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \end{array}\right.$$

and

$$\left[\begin{array}{ll} \text{A.} & r\cos\theta < 1 \\ \text{B.} & r\cos\theta = 1 \\ \text{C.} & r\cos\theta > 1. \end{array}\right.$$

Case IA. We need to show that

$$\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta + 1}\right) + \pi + \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta - 1}\right) > 2\theta,$$

for all $r < 1/\cos\theta$ and $\theta \in (0, \pi/4)$. First note that

$$\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta + 1}\right) \in \left(0, \frac{\pi}{2}\right),$$

and

$$\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta-1}\right)\in\left(-\frac{\pi}{2},0\right),$$

so

$$\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta+1}\right)+\pi+\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta-1}\right)\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right).$$

Since $2\theta\in(0,\pi/2)$, we have the desired result.

Case IB. We need to show that

$$\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta+1}\right)+\frac{\pi}{2}>2\theta,$$

for $r=1/\cos\theta$ and $\theta\in(0,\pi/4)$. Since $r=1/\cos\theta$ this reduces to

$$\tan^{-1}\left(\frac{1}{2}\tan\theta\right)+\frac{\pi}{2}>2\theta.$$

This is true since $2\theta\in(0,\pi/2)$ and $\tan^{-1}\left(\frac{1}{2}\tan\theta\right)>0$.

Case IC. Define $g:\{(\theta,r):\theta\in(0,\pi/4),r\in(1/\cos\theta,\infty)\}\to\mathbf{R}$ by

$$g(\theta,r):=\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta+1}\right)+\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta-1}\right)-2\theta.$$

We need to show that $g(\theta,r)>0$ for all $\theta\in(0,\pi/4)$ and $r>1/\cos\theta$. Fix $\theta\in(0,\pi/4)$. Then

$$\frac{\partial g}{\partial r}(\theta,r)=-\frac{4r\cos\theta\sin\theta}{r^4-2r^2\cos(2\theta)+1}<0.$$

Therefore $r\mapsto g(\theta,r)$ is strictly decreasing. Moreover

$$\lim_{r\to\infty}g(\theta,r)=\tan^{-1}(\tan\theta)+\tan^{-1}(\tan\theta)-2\theta=0.$$

Hence $g(\theta,r)>0$ for all $r\in(1/\cos\theta,\infty)$.

Case IIA. This follows as in Case IA.

Case IIB. Trivial.

Case IIC. In this case $\theta=\pi/4$ and $r>\sqrt{2}$. Consider the function $g:(\sqrt{2},\infty)\to\mathbf{R}$ defined by

$$g(r):=\tan^{-1}\left(\frac{1}{1+\frac{\sqrt{2}}{r}}\right)+\tan^{-1}\left(\frac{1}{1-\frac{\sqrt{2}}{r}}\right).$$

We need to show that $g>\pi/2$. The derivative of g is given by

$$\frac{dg(r)}{dr}=-\frac{2r}{1+r^4}<0.$$

This implies that g is a strictly decreasing function. Taking the limit

$$\lim_{r\to\infty}g(r)=\tan^{-1}(1)+\tan^{-1}(1)=\frac{\pi}{2}.$$

Since g is a strictly decreasing function of r, we have the desired result.

Case IIIA. Define $g:\{(\theta,r):\theta\in(\pi/4,\pi/2),r\in(0,1/\cos\theta)\}\to\mathbf{R}$ by

$$g(\theta,r):=\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta+1}\right)+\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta-1}\right)-2\theta.$$

We need to show that $g(\theta, r) > -\pi$ for all $\theta \in (\pi/4, \pi/2)$ and $r < 1/\cos\theta$. Fix $r \in (0, \infty)$. If $\theta > \cos^{-1}(1/r)$ and $\theta \in (\pi/4, \pi/2)$, then

$$\frac{\partial g}{\partial r}(\theta, r) = \frac{2(r^2\cos(2\theta) - 1)}{r^4 - 2r^2\cos(2\theta) + 1} < 0,$$

since $\cos(2\theta) \in (-1, 0)$. Therefore $\theta \mapsto g(\theta, r)$ is strictly decreasing. Since

$$\lim_{\theta \to \frac{\pi}{2}-} g(\theta, r) = \tan^{-1}(r) + \tan^{-1}(-r) - \pi = -\pi,$$

the required inequality follows.

Case IIIB. For $\theta \in (\pi/4, \pi/2)$, would like to prove the inequality

$$\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta + 1}\right) + \frac{\pi}{2} > 2\theta,$$

with $r = 1/\cos\theta$, or equivalently,

$$\tan^{-1}\left(\frac{1}{2}\tan\theta\right) + \frac{\pi}{2} > 2\theta.$$

Consider $g : (\pi/4, \pi/2) \to \mathbf{R}$ defined by

$$g(\theta) := \tan^{-1}\left(\frac{1}{2}\tan\theta\right) - 2\theta + \frac{\pi}{2}.$$

We need to show that $g > 0$. Then

$$\frac{\partial g}{\partial \theta}(\theta) = -\frac{6}{4 + \tan^2\theta} < 0$$

and

$$\lim_{\theta \to \frac{\pi}{2}-} g(\theta) = \lim_{\theta \to \frac{\pi}{2}-} \tan^{-1}\left(\frac{1}{2}\tan\theta\right) - \pi + \frac{\pi}{2} = 0.$$

The required estimate follows.

Case IIIC. Define $g : \{(\theta, r) : \theta \in (0, \pi/2), r > 1/\cos\theta\} \to \mathbf{R}$ by

$$g(\theta, r) := \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta + 1}\right) + \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta - 1}\right).$$

We would like to prove the inequality $g(\theta, r) > 2\theta$ for all $\theta \in (\pi/4, \pi/2)$ and $r > 1/\cos\theta$. We first show that $g(\theta, r) > \pi/2$ for all $\theta \in (\pi/4, \pi/2)$ and $r > 1/\cos\theta$. Then

$$\frac{\partial g}{\partial \theta}(\theta, r) = \frac{2r^2(r^2 - \cos(2\theta))}{(r^2 + 2r\cos\theta + 1)(r^2 - 2r\cos\theta + 1)} > 0,$$

for all $\theta \in (\pi/4, \cos^{-1}(1/r))$ since $\cos(2\theta) < 0$ and all factors are positive. Hence $g(\theta, r) > g(\pi/4, r)$ for all $\theta \in (\pi/4, \pi/2)$ and $r > 1/\cos\theta$. Next

$$\frac{dg}{dr}\left(\frac{\pi}{4}, r\right) = \frac{-2r}{(r^2 + \sqrt{2}r + 1)(r^2 - \sqrt{2}r + 1)} < 0,$$

for all $r \in (\sqrt{2}, \infty)$ and

$$\lim_{r \to \infty} g\left(\frac{\pi}{4}, r\right) = \frac{\pi}{2}.$$

Therefore $g(\pi/4, r) > \pi/2$ for all $r \in (\sqrt{2}, \infty)$ and hence $g(\theta, r) > \pi/2$ for all $\theta \in (\pi/4, \pi/2)$ and $r > 1/\cos\theta$. We have shown that

$$\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta + 1}\right) + \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta - 1}\right) > \frac{\pi}{2},$$

for all $\theta \in (\pi/4, \pi/2)$ and $r > 1/\cos\theta$. Since also $2\theta > \pi/2$, the inequality

$$\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta+1}\right) + \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta-1}\right) > 2\theta,$$

is equivalent to the inequality

$$\tan(2\theta) < \tan\left[\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta+1}\right) + \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta-1}\right)\right]. \tag{2}$$

Using the addition formula for the tangent function

$$\tan(z_1+z_2) = \frac{\tan z_1 + \tan z_2}{1-\tan z_1 \tan z_2},$$

the inequality (2) reduces to

$$\tan(2\theta) < \frac{r^2\sin(2\theta)}{r^2\cos(2\theta)-1},$$

which is trivially verified. Thus (1) is valid for all $\theta \in (0, \pi/2)$.

Secondly we would like to show that $\theta - \phi/2 - \psi/2 > -\pi/2$ or equivalently

$$\phi + \psi - 2\theta < \pi. \tag{3}$$

This inequality if clear for cases B, C, IIA and IIIA. All that remains is to prove (3) for case IA. Define $g : \{(\theta, r) : \theta \in (0, \pi/4), r \in (0, 1/\cos\theta)\} \to \mathbf{R}$ by

$$g(\theta, r) := \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta+1}\right) + \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta-1}\right) - 2\theta.$$

We need to show that $g < 0$. Fix $\theta \in (0, \pi/4)$. Then

$$\frac{\partial g}{\partial r}(\theta, r) = -\frac{4r\cos\theta\sin\theta}{r^4 - 2r^2\cos(2\theta) + 1} < 0.$$

Therefore $r \mapsto g(\theta, r)$ is strictly decreasing. Moreover

$$\lim_{r \to \frac{1}{\cos\theta}-} g(\theta, r) = \tan^{-1}\left(\frac{1}{2}\tan\theta\right) - \frac{\pi}{2} - 2\theta.$$

However $2\theta + \frac{\pi}{2} \in (\pi/2, \pi)$. It follows that $g(\theta, r) < 0$ for all $\theta \in (0, \pi/4)$ and $r \in (0, 1/\cos\theta)$.

Thus f maps quadrant I into quadrant IV.

Due to the evenness of the f, quadrants I & III are mapped to quadrants IV, and quadrants II & IV are mapped to quadrant I. Therefore if $z \in \mathbf{C} \setminus [-1, 1]$ then $\mathrm{Re}\,\frac{z}{\sqrt{z^2-1}} > 0$. This completes the proof of 5.

The range of f is $\{z \in \mathbf{C} : \mathrm{Re}\,z \geq 0 \text{ and } z \neq 1\}$. Every complex number in the range of the function is taken twice except for elements in $(0, 1)$ and on the imaginary axis. These complex numbers are taken only once.

3 The Whipple formulae for associated Legendre functions

There is a transformation over an open subset of the complex plane which is particularly useful in studying associated Legendre functions (see Abramowitz & Stegun (1972) [1] and Hobson (1955) [11]). This transformation, which is valid on a certain domain of the complex numbers, accomplishes the following

$$
\left.\begin{array}{ccc}
\cosh z & \leftrightarrow & \coth w \\
\coth z & \leftrightarrow & \cosh w \\
\sinh z & \leftrightarrow & (\sinh w)^{-1}
\end{array}\right\}. \tag{4}
$$

This transformation is accomplished using the map $w : \mathfrak{D} \to \mathbf{C}$, with

$$
\mathfrak{D} := \mathbf{C} \setminus \big\{ z \in \mathbf{C} : \operatorname{Re} z \le 0 \text{ and } \operatorname{Im} z = 2\pi n,\ n \in \mathbf{Z} \big\},
$$

and w defined by

$$
w(z) := \log \coth \frac{z}{2}. \tag{5}
$$

The map w is periodic with period $2\pi i$ and is locally injective. The map w restricted to $\mathfrak{D} \cap \{z \in \mathbf{C} : -\pi < \operatorname{Im} z < \pi\}$ is verified to be an involution. The transformation (4) is the restriction of the mapping w to this restricted domain.

This transformation is particularly useful for certain associated Legendre functions such as toroidal harmonics (see Cohl *et al.* (2001) [7], Cohl & Tohline (1999) [8]), associated Legendre functions of the first and second kind with odd-half-integer degree and integer-order, and for other associated Legendre functions which one might encounter in potential theory. The real argument of toroidal harmonics naturally occur in $(1, \infty)$, and these are the simultaneous ranges of both the real hyperbolic cosine and cotangent functions. One application of this map occurs with the Whipple formulae for associated Legendre functions (Whipple (1917) [17], Cohl *et al.* (2000) [9]) under index (degree and order) interchange. See for instance, (8.2.7) and (8.2.8) in Abramowitz & Stegun (1972) [1], namely

$$
P_{-\mu-1/2}^{-\nu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right) = \sqrt{\frac{2}{\pi}} \frac{(z^2-1)^{1/4} \mathrm{e}^{-i\mu\pi}}{\Gamma(\nu+\mu+1)} Q_\nu^\mu(z), \tag{6}
$$

and

$$
Q_{-\mu-1/2}^{-\nu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right) = -i(\pi/2)^{1/2} \Gamma(-\nu-\mu)(z^2-1)^{1/4} \mathrm{e}^{-i\nu\pi} P_\nu^\mu(z),
$$

which are valid for $\operatorname{Re} z > 0$ and for all complex ν and μ, except where the functions are not defined.

4 Justification for differentiation under the integral sign

In this section, we present and derive formulae for parameter derivatives of associated Legendre functions of the first kind P_ν^μ and the second kind Q_ν^μ, with respect to their parameters, namely the degree ν and the order μ. We cover parameter derivatives of associated Legendre functions for argument $z \in \mathbf{C} \setminus (-\infty, 1]$.

We incorporate derivatives with respect to order evaluated at integer-orders for modified Bessel functions (see Abramowitz & Stegun (1972) [1], Brychkov & Geddes (2005) [5], Magnus, Oberhettinger & Soni (1966) [13]) to compute derivatives with respect to the degree and the order of associated Legendre functions. Below we apply these results through certain integral representations of associated Legendre functions in terms of modified Bessel functions. Modified Bessel functions of the first and second kind respectively can be defined for all $\nu \in \mathbf{C}$ (see for instance §3.7 in Watson (1944) [16]) by

$$
I_\nu(z) := \sum_{m=0}^{\infty} \frac{(z/2)^{\nu+2m}}{m!\Gamma(\nu+m+1)},
$$

and

$$
K_\nu(z) := \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \pi\nu}.
$$

For $\nu = n \in \mathbf{N}_0$, the first definition yields

$$I_n(z) = I_{-n}(z).$$

It may be verified that

$$K_n(z) = \lim_{\nu \to n} K_\nu(z)$$

is well defined. The modified Bessel function of the second kind is commonly referred to as a Macdonald function.

The strategy applied in this section is to use integral representations of associated Legendre functions, expressed in terms of modified Bessel functions, and justify differentiation under the integral sign with respect to the relevant parameters.

4.1 Parameter derivative formulas from $K_\nu(t)$

It follows from Gradshteyn & Ryzhik (2007) (6.628.7) [10] (see also Prudnikov *et al.* (1988) (2.16.6.3) [15]) that

$$\begin{aligned}\int_0^\infty e^{-zt} K_\nu(t) t^{\mu-1/2} dt &= \sqrt{\frac{\pi}{2}} \Gamma\left(\mu - \nu + \frac{1}{2}\right) \Gamma\left(\mu + \nu + \frac{1}{2}\right) \left(z^2 - 1\right)^{-\mu/2} P^{-\mu}_{\nu-1/2}(z) \\ &= \Gamma\left(\mu - \nu + \frac{1}{2}\right) \left(z^2 - 1\right)^{-\mu/2-1/4} e^{-i\pi\nu} Q^\nu_{\mu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right), \end{aligned} \tag{7}$$

where we used the Whipple formulae (6), for $\operatorname{Re} z > -1$ and $\operatorname{Re} \mu > |\operatorname{Re} \nu| - 1/2$. We would like to generate an analytical expression for the derivative of the associated Legendre function of the second kind with respect to its order, evaluated at integer-orders. In order to do this our strategy is to solve the above integral expression for the associated Legendre function of the second kind, differentiate with respect to the order, evaluate at integer-orders, and take advantage of the corresponding formula for differentiation with respect to order for modified Bessel functions of the second kind (see Abramowitz & Stegun (1972) [1], Brychkov (2010) [4], Brychkov & Geddes (2005) [5], Magnus, Oberhettinger & Soni (1966) [13]). Using the expression for the associated Legendre function of the second kind in (7), we solve for $Q^\mu_{\nu-1/2}(z)$ and re-express using the map in (5). This gives us the expression

$$Q^\mu_{\nu-1/2}(z) = \frac{\left(z^2 - 1\right)^{-\nu/2-1/4} e^{i\pi\mu}}{\Gamma\left(\nu - \mu + \frac{1}{2}\right)} \int_0^\infty \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right) K_\mu(t) t^{\nu-1/2} dt. \tag{8}$$

In order to justify differentiation under the integral sign we use the following well-known corollary of the bounded convergence theorem (cf. §8.2 in Lang (1993) [12]).

Proposition 4.1. *Let (X, μ) be a measure space, $U \subset \mathbf{R}$ open and $f : X \times U \to \mathbf{R}$ a function. Suppose*

1. *for all $y \in U$ the function $x \mapsto f(x, y)$ is measurable,*
2. *$\frac{\partial f}{\partial y}(x, y)$ exists for all $(x, y) \in X \times U$,*
3. *there exists $g \in \mathcal{L}^1(X)$ such that $\left|\frac{\partial f}{\partial y}(x, y)\right| \le g(x)$ for all $(x, y) \in X \times U$.*

Then the function $y \mapsto \int_X f(x, y) d\mu(x)$ is differentiable on U and

$$\frac{d}{dy}\left(\int_X f(x, y) d\mu(x)\right) = \int_X \frac{\partial f}{\partial y}(x, y) d\mu(x).$$

We call g a $\mathcal{L}^1$-majorant.

We wish to differentiate (8) with respect to the order μ and evaluate at $\mu_0 = \pm m$, where $m \in \mathbf{N}_0$. The derivative of the modified Bessel function of the second kind with respect to its order (see Abramowitz & Stegun (1972) [1], Brychkov (2010) [4], Brychkov & Geddes (2005) [5], Magnus, Oberhettinger & Soni (1966) [13]) is given by

$$\left[\frac{\partial}{\partial\mu}K_\mu(t)\right]_{\mu=\pm m} = \pm m! \sum_{k=0}^{m-1} \frac{2^{m-1-k}}{k!(m-k)} t^{k-m} K_k(t) \tag{9}$$

(see for instance (1.14.2.2) in Brychkov (2008) [3]). For a fixed t, $K_\mu(t)$ is an even function of $\mu \in \mathbf{R}$ (see (9.6.6) in Abramowitz & Stegun (1972) [1]), i.e.

$$K_{-\mu}(t) = K_\mu(t),$$

and for $\mu \in [0,\infty)$, $K_\mu(t)$ is a strictly increasing function of μ. Also, for a fixed t, $\partial K_\mu(t)/\partial\mu$ is an odd function of $\mu \in \mathbf{R}$ and for $\mu \in [0,\infty)$, $\partial K_\mu(t)/\partial\mu$ is also a strictly increasing function of μ. Using (9) we can make the following estimate

$$\left|\frac{\partial}{\partial\mu}K_\mu(t)\right| < \left.\frac{\partial K_\tau}{\partial\tau}\right|_{\tau=\pm(m+1)}, \tag{10}$$

for all $\mu \in (\mu_0 - 1, \mu_0 + 1)$.

To justify differentiation under the integral sign in (8), with respect to μ, evaluated at μ_0, we use Proposition 4.1. If we fix z and ν, the integrand of (8) can be given by the function $f : \mathbf{R} \times (0,\infty) \to \mathbf{C}$ defined by

$$f(\mu,t) := \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right) t^{\nu-1/2} K_\mu(t).$$

Since $\partial K_\mu(t)/\partial\mu$ is a strictly increasing function of $\mu \in [0,\infty)$, we have for all $\mu \in (\mu_0 - 1, \mu_0 + 1)$

$$\begin{aligned}
\left|\frac{\partial f}{\partial\mu}(\mu,t)\right| &= \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right) t^{\nu-1/2} \left|\frac{\partial}{\partial\mu}K_\mu(t)\right| \\
&< \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right) t^{\nu-1/2} \left|\left[\frac{\partial}{\partial\tau}K_\tau(t)\right]_{\tau=\pm(m+1)}\right| \\
&= \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right) t^{\nu-1/2} \left|\left[\frac{\partial}{\partial\tau}K_\tau(t)\right]_{\tau=m+1}\right|, \\
&\leq \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right) t^{\nu-1/2} (m+1)! \sum_{k=0}^{m} \frac{2^{m-k}}{k!(m+1-k)} t^{k-m-1} K_k(t), \\
&\leq \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right) t^{\nu-1/2} (m+1)! 2^m t^{-1} K_m(t) =: g(t),
\end{aligned}$$

where we used (9) and the fact that $K_k(t) \leq K_m(t)$ for all $k \in \{0,\ldots,m-1\}$. Then g is a $\mathcal{L}^1$-majorant for the derivative of the integrand, since the integral (8) converges for $\operatorname{Re}\left(z/\sqrt{z^2-1}\right) > -1$ and $\operatorname{Re}\nu > m - 1/2$.

The conditions for differentiating under the integral sign have been satisfied and we can re-write (8) as

$$\left[\frac{\partial}{\partial\mu}Q^{\mu}_{\nu-1/2}(z)\right]_{\mu=\pm m} = \left(z^2-1\right)^{-\nu/2-1/4}\left[\frac{\partial}{\partial\mu}\frac{e^{i\pi\mu}}{\Gamma\left(\nu-\mu+\frac{1}{2}\right)}\right]_{\mu=\pm m} \times\int_0^\infty \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right)K_{\pm m}(t)t^{\nu-1/2}dt + \frac{\left(z^2-1\right)^{-\nu/2-1/4}(-1)^m}{\Gamma\left(\nu\mp m+\frac{1}{2}\right)} \times\int_0^\infty \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right)t^{\nu-1/2}\left[\frac{\partial}{\partial\mu}K_\mu(t)\right]_{\mu=\pm m}dt. \tag{11}$$

The derivative from the first term is given as

$$\left[\frac{\partial}{\partial\mu}\frac{e^{i\pi\mu}}{\Gamma\left(\nu-\mu+\frac{1}{2}\right)}\right]_{\mu=\pm m} = \frac{(-1)^m}{\Gamma\left(\nu\mp m+\frac{1}{2}\right)}\left[i\pi+\psi\left(\nu\mp m+\frac{1}{2}\right)\right],$$

where the ψ is the digamma function defined in terms of the derivative of the gamma function,

$$\frac{d}{dz}\Gamma(z) := \psi(z)\Gamma(z),$$

for $z\in\mathbf{C}\setminus(-\mathbf{N}_0)$.

Substituting these expressions for the derivatives into the two integrals and using the map in (5) to re-evaluate these integrals in terms of associated Legendre functions gives the following general expression for the derivative of the associated Legendre function of the second kind with respect to its order evaluated at integer-orders as

$$\frac{\Gamma(\nu\mp m+\frac{1}{2})}{\Gamma(\nu-m+\frac{1}{2})}\left[\frac{\partial}{\partial\mu}Q^{\mu}_{\nu-1/2}(z)\right]_{\mu=\pm m} = \left[i\pi+\psi\left(\nu\mp m+\frac{1}{2}\right)\right]Q^m_{\nu-1/2}(z) \pm m!\sum_{k=0}^{m-1}\frac{(-1)^{k-m}\left(z^2-1\right)^{(k-m)/2}}{k!(m-k)2^{k-m+1}}Q^k_{\nu+k-m-1/2}(z).$$

If we start with the expression for the associated Legendre function of the first kind in (7) and solve for $P^{-\mu}_{\nu-1/2}(z)$ we have

$$P^{-\mu}_{\nu-1/2}(z) = \sqrt{\frac{2}{\pi}}\frac{\left(z^2-1\right)^{\mu/2}}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)\Gamma\left(\mu+\nu+\frac{1}{2}\right)}\int_0^\infty e^{-zt}K_\nu(t)t^{\mu-1/2}dt. \tag{12}$$

To justify differentiation under the integral sign in (12), with respect to ν, evaluated at $\nu=\pm n$, where $n\in\mathbf{N}_0$, we use as similar argument as in (8) only with modification $\mu\mapsto\nu$ and $m\mapsto n$. The same modified $\mathcal{L}^1$-majorant will work for the derivative of this integrand, since the integral (12) converges for $\operatorname{Re} z>-1$ and $\operatorname{Re}\nu > |\operatorname{Re}\mu|-1/2$.

The conditions for differentiating under the integral sign have been satisfied and we can re-write (12) as

$$\left[\frac{\partial}{\partial\nu}P_{\nu-1/2}^{-\mu}(p)\right]_{\nu=\pm n}=\sqrt{\frac{2}{\pi}}\left(z^2-1\right)^{\mu/2}\left[\frac{\partial}{\partial\nu}\frac{1}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)\Gamma\left(\mu+\nu+\frac{1}{2}\right)}\right]_{\nu=\pm n} \tag{13}$$

$$\times\int_0^\infty e^{-zt}K_{\pm n}(t)t^{\mu-1/2}dt$$

$$+\sqrt{\frac{2}{\pi}}\frac{\left(z^2-1\right)^{\mu/2}}{\Gamma\left(\mu\mp n+\frac{1}{2}\right)\Gamma\left(\mu\pm n+\frac{1}{2}\right)}$$

$$\times\int_0^\infty e^{-zt}t^{\mu-1/2}\left[\frac{\partial}{\partial\nu}K_\nu(t)\right]_{\nu=\pm n}dt.$$

The derivative from the first term in (13) is given as

$$\left[\frac{\partial}{\partial\nu}\frac{1}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)\Gamma\left(\mu+\nu+\frac{1}{2}\right)}\right]_{\nu=\pm n}=\frac{\psi\left(\mu\mp n+\frac{1}{2}\right)-\psi\left(\mu\pm n+\frac{1}{2}\right)}{\Gamma\left(\mu\pm n+\frac{1}{2}\right)\Gamma\left(\mu\mp n+\frac{1}{2}\right)}.$$

Substituting this expression for the derivative and that given in (9) yields the following general expression for the derivative of the associated Legendre function of the first kind with respect to its degree evaluated at odd-half-integer degrees as

$$\pm\left[\frac{\partial}{\partial\nu}P_{\nu-1/2}^{-\mu}(z)\right]_{\nu=\pm n}=\left[\psi\left(\mu-n+\frac{1}{2}\right)-\psi\left(\mu+n+\frac{1}{2}\right)\right]P_{n-1/2}^{-\mu}(z)$$

$$+\frac{n!}{\Gamma\left(\mu+n+\frac{1}{2}\right)}\sum_{k=0}^{n-1}\frac{\Gamma\left(\mu-n+2k+\frac{1}{2}\right)\left(z^2-1\right)^{(n-k)/2}}{k!(n-k)2^{k-n+1}}P_{k-1/2}^{-\mu+n-k}(z).$$

If one makes a global replacement $-\mu\mapsto\mu$, using the properties of gamma and digamma functions, this result reduces to

$$\pm\left[\frac{\partial}{\partial\nu}P_{\nu-1/2}^{\mu}(z)\right]_{\nu=\pm n}=\left[\psi\left(\mu+n+\frac{1}{2}\right)-\psi\left(\mu-n+\frac{1}{2}\right)\right]P_{n-1/2}^{\mu}(z)$$

$$+n!\,\Gamma\left(\mu-n+\frac{1}{2}\right)\sum_{k=0}^{n-1}\frac{\left(z^2-1\right)^{(n-k)/2}}{\Gamma\left(\mu+n-2k+\frac{1}{2}\right)k!(n-k)2^{k-n+1}}P_{k-1/2}^{\mu+n-k}(z).$$

4.2 Parameter derivative formulas from $I_\nu(t)$

Starting this time with Gradshteyn & Ryzhik (2007) (6.624.5) [10] (see also Prudnikov *et al.* (1988) (2.15.3.2) [15]), we have for $\mathrm{Re}\,z>1$ and $\mathrm{Re}\,\mu>-\mathrm{Re}\,\nu-1/2$,

$$\int_0^\infty e^{-zt}I_\nu(t)t^{\mu-1/2}dt=\sqrt{\frac{2}{\pi}}e^{-i\pi\mu}\left(z^2-1\right)^{-\mu/2}Q_{\nu-1/2}^{\mu}(z)$$

$$=\Gamma\left(\mu+\nu+\frac{1}{2}\right)\left(z^2-1\right)^{-\mu/2-1/4}P_{\mu-1/2}^{-\nu}\left(\frac{z}{\sqrt{z^2-1}}\right), \tag{14}$$

where we used again the Whipple formulae (6).

We will use this particular integral representation of associated Legendre functions to compute certain derivatives of the associated Legendre functions with respect to the degree and order. We start with the integral representation of the associated Legendre function of the second kind (14), namely

$$Q^{\mu}_{\nu-1/2}(z) = \sqrt{\frac{\pi}{2}} e^{i\pi\mu} \left(z^2-1\right)^{\mu/2} \int_0^\infty e^{-zt} t^{\mu-1/2} I_\nu(t) dt. \tag{15}$$

To justify differentiation under the integral sign in (15), with respect to ν, evaluated at $\nu_0 = \pm n$, where $n \in \mathbf{N}$, we use again Proposition 4.1. If we fix z and μ, the integrand of (15) can be given by the function $f : \mathbf{R} \times (0,\infty) \to \mathbf{C}$ defined by

$$f(\nu, t) := e^{-zt} t^{\mu-1/2} I_\nu(t).$$

We use the following integral representation for the derivative with respect to order of the modified Bessel function of the first kind (see (75) in Apelblat & Kravitsky (1985) [2])

$$\frac{\partial I_\nu(t)}{\partial \nu} = -\nu \int_0^t K_0(t-x) I_\nu(x) x^{-1} dx. \tag{16}$$

Let $\delta \in (0,1)$ and $M > 2$. Consider $g : (0,\infty) \to [0,\infty)$ defined by

$$g(t) := M e^{-t \operatorname{Re} z} t^{\operatorname{Re}\mu - 1/2} \int_0^t K_0(t-x) I_\delta(x) x^{-1} dx.$$

Using (16) we have for all $\nu \in (\delta, M)$

$$\begin{aligned}
\left|\frac{\partial f(\nu,t)}{\partial \nu}\right| &= e^{-t\operatorname{Re} z} t^{\operatorname{Re}\mu-1/2} \left|\frac{\partial I_\nu(t)}{\partial \nu}\right| \\
&= \nu e^{-t\operatorname{Re} z} t^{\operatorname{Re}\mu-1/2} \int_0^t K_0(t-x) I_\nu(x) x^{-1} dx \\
&\le M e^{-t\operatorname{Re} z} t^{\operatorname{Re}\mu-1/2} \int_0^t K_0(t-x) I_\delta(x) x^{-1} dx \\
&= g(t),
\end{aligned}$$

since for fixed t, $\nu \mapsto I_\nu(t)$ is strictly decreasing. Now we show that $g \in \mathcal{L}^1$. The integral of g over its domain is

$$\int_0^\infty g(t) dt = M \int_0^\infty e^{-t\operatorname{Re} z} t^{\operatorname{Re}\mu-1/2} \int_0^t K_0(t-x) I_\delta(x) x^{-1} dx dt.$$

Making a change of variables in the integral, $(x,t) \mapsto (x,y)$ with $y = t - x$, yields

$$\int_0^\infty g(t) dt = M \int_0^\infty e^{-y\operatorname{Re} z} K_0(y) \int_0^\infty e^{-x\operatorname{Re} z} (x+y)^{\operatorname{Re}\mu-1/2} x^{-1} I_\delta(x) dx\, dy.$$

First we show that g is integrable in a neighbourhood of zero. Suppose $\operatorname{Re}\mu - 1/2 < 0$, $x, y \in (0,1]$ and $a \in (0,1)$. Then

$$(x+y)^{\operatorname{Re}\mu-1/2} = (x+y)^{-a} (x+y)^{\operatorname{Re}\mu-1/2+a} \le y^{-a} \max\left(2^{\operatorname{Re}\mu-1/2+a}, x^{\operatorname{Re}\mu-1/2+a}\right).$$

Since $K_0(y) \sim -\log(y)$ ((9.6.8) in Abramowitz & Stegun (1972) [1]) it follows that

$$\int_0^1 K_0(y) y^{-a} dy < \infty.$$

Furthermore since $I_\delta(x) \sim (x/2)^\delta / \Gamma(\delta+1)$ ((9.6.7) in Abramowitz & Stegun (1972) [1]) it follows that

$$\int_0^1 I_\delta(x) x^{-1} dx < \infty.$$

Now we show that
$$\int_0^1 I_\delta(x)x^{\mathrm{Re}\,\mu-1/2+a-1}dx, \tag{17}$$
is convergent if $\mathrm{Re}\,\mu - 1/2 + a + \delta > 0$. If we define
$$\epsilon := \frac{\mathrm{Re}\,\mu + \nu_0 + \frac{1}{2}}{3} > 0,$$
then $\mathrm{Re}\,\mu = -\nu_0 - 1/2 + 3\epsilon$. Therefore if we take $a := 1 - \epsilon$ and $\delta := \nu_0 - \epsilon < \nu_0$ then
$$\mathrm{Re}\mu - \frac{1}{2} + a + \delta = \epsilon > 0,$$
and hence (17) is convergent and thus g is integrable near the origin. If $\mathrm{Re}\,\mu - 1/2 \geq 0$ then similarly g is integrable near the origin.

Now we show that g is integrable. Suppose $\mathrm{Re}\,\mu - 1/2 > 0$. Then
$$(x+y)^{\mathrm{Re}\,\mu-1/2} \leq [2\max(x,y))]^{\mathrm{Re}\,\mu-1/2} = 2^{\mathrm{Re}\,\mu-1/2}\max(x^{\mathrm{Re}\,\mu-1/2}, y^{\mathrm{Re}\,\mu-1/2})$$
for all $x, y \geq 0$. For $y \to \infty$ one has $K_\nu(y) \sim \sqrt{\pi/(2y)}e^{-y}$ ((8.0.4) in Olver (1997) [14]). Hence it follows that
$$\int_1^\infty K_0(y)e^{-y\mathrm{Re}\,z}y^{\mathrm{Re}\,\mu-1/2}dy < \infty,$$
and
$$\int_1^\infty K_0(y)e^{-y\mathrm{Re}\,z}dy < \infty.$$
Furthermore since for $x \to \infty$, $I_\delta(x) \sim e^x/\sqrt{2\pi x}$ (p. 83 in Olver (1997) [14]) it follows that
$$\int_1^\infty e^{-x\mathrm{Re}\,z}I_\delta(x)x^{\mathrm{Re}\,\mu-3/2}dx < \infty,$$
and
$$\int_1^\infty e^{-x\mathrm{Re}\,z}I_\delta(x)x^{-1}dx < \infty.$$
If $\mathrm{Re}\,\mu - 1/2 \leq 0$ then similarly g is integrable.

Therefore g is a $\mathcal{L}^1$-majorant for the derivative with respect to ν of the integrand in (15). It is unclear whether differentiation under the integral sign is also possible for $\nu_0 = 0$. However, we show below that our derived results for derivatives with respect to the degree for associated Legendre functions match up with the to be derived results for degree $\nu = 0$. It is true that relatively little is known about the properties of Bessel functions in relation to operations (differentiation and integration) with respect to their order (cf. Apelblat & Kravitsky (1985) [2]).

Differentiating with respect to the degree ν and evaluating at $\nu = \pm n$, where $n \in \mathbf{N}$, one obtains
$$\left[\frac{\partial}{\partial\nu}Q^\mu_{\nu-1/2}(z)\right]_{\nu=\pm n} = \sqrt{\frac{\pi}{2}}e^{i\pi\mu}\left(z^2-1\right)^{\mu/2}\int_0^\infty e^{-zt}t^{\mu-1/2}\left[\frac{\partial}{\partial\nu}I_\nu(t)\right]_{\nu=\pm n}dt. \tag{18}$$
The derivative of the modified Bessel function of the first kind (18) (see Abramowitz & Stegun (1972) [1], Brychkov (2010) [4], Brychkov & Geddes (2005) [5], Magnus, Oberhettinger & Soni (1966) [13]) is given by
$$\left[\frac{\partial}{\partial\nu}I_\nu(t)\right]_{\nu=\pm n} = (-1)^{n+1}K_n(t) \pm n!\sum_{k=0}^{n-1}\frac{(-1)^{k-n}}{k!(n-k)}\frac{t^{k-n}}{2^{k-n+1}}I_k(t) \tag{19}$$
(see for instance (1.13.2.1) in Brychkov (2008) [3]).

Inserting (19) into (18) and using (7) and (14), we obtain the following general expression for the derivative of the associated Legendre function of the second kind with respect to its degree evaluated at odd-half-integer degrees as

$$\left[\frac{\partial}{\partial\nu}Q^{\mu}_{\nu-1/2}(z)\right]_{\nu=\pm n} = -\sqrt{\frac{\pi}{2}}e^{i\pi\mu}\Gamma\left(\mu-n+\frac{1}{2}\right)\left(z^2-1\right)^{-1/4}Q^{n}_{\mu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right)$$
$$\pm n!\sum_{k=0}^{n-1}\frac{\left(z^2-1\right)^{(n-k)/2}}{2^{k-n+1}k!(n-k)}Q^{\mu+k-n}_{k-1/2}(z). \tag{20}$$

Note that

$$\left[\frac{\partial}{\partial\nu}Q^{\mu}_{\nu-1/2}(z)\right]_{\nu=0} = -\sqrt{\frac{\pi}{2}}e^{i\pi\mu}\Gamma\left(\mu+\frac{1}{2}\right)\left(z^2-1\right)^{-1/4}Q_{\mu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right),$$

by Magnus, Oberhettinger & Soni (1966) [13]. Therefore (20) is also valid if $\nu=0$.

Finally, we obtain a formula for the derivative with respect to the order for the associated Legendre function of the first kind evaluated at integer-orders. In order to do this we use the integral expression for the associated Legendre function of the first kind given by (14) and the map given in (5) to convert to the appropriate argument. Now use the negative-order condition for associated Legendre functions of the first kind (see for example (22) in Cohl *et al.* (2000) [9]) to convert to a positive order, namely

$$P^{\mu}_{\nu-1/2}(z) = \frac{2}{\pi}e^{-i\mu\pi}\sin(\mu\pi)Q^{\mu}_{\nu-1/2}(z)$$
$$+\frac{(z^2-1)^{-\nu/2-1/4}}{\Gamma(\nu-\mu+\frac{1}{2})}\int_0^\infty \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right)I_\mu(t)t^{\nu-1/2}dt. \tag{21}$$

To justify differentiation under the integral sign in (21), with respect to μ, evaluated at $\mu=\pm m$, where $m\in\mathbf{N}$, we use as similar argument as in (15) only with modification $\nu\mapsto\mu$ and $n\mapsto m$. The same modified $\mathcal{L}^1$-majorant will work for the derivative of this integrand, since the integral (21) converges for $\mathrm{Re}\,(z/\sqrt{z^2-1})>1$ and $\mathrm{Re}\,\mu>-\mathrm{Re}\,\nu-1/2$. Since we were unable to justify differentiation under the integral for $\nu=0$ before, the case for differentiation under the integral (21) with respect to μ evaluated at $\mu=0$ remains open. However, below we show that our derived results for derivatives with respect to the order for associated Legendre functions match up to previously established results in the literature for order $\mu=0$.

Differentiating both sides of the resulting expression with respect to the order μ and evaluating at $\mu=\pm m$, where $m\in\mathbf{N}$ yields

$$\left[\frac{\partial}{\partial\mu}P^{\mu}_{\nu-1/2}(z)\right]_{\mu=\pm m} = 2Q^{\pm m}_{\nu-1/2}(z)$$
$$+\left(z^2-1\right)^{-\nu/2-1/4}\left\{\frac{\partial}{\partial\mu}\left[\Gamma\left(\nu-\mu+\frac{1}{2}\right)\right]^{-1}\right\}_{\mu=\pm m}$$
$$\times\int_0^\infty \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right)I_{\pm m}(t)t^{\nu-1/2}dt$$
$$+\frac{\left(z^2-1\right)^{-\nu/2-1/4}}{\Gamma\left(\nu\mp m+\frac{1}{2}\right)}$$
$$\times\int_0^\infty \exp\left(\frac{-zt}{\sqrt{z^2-1}}\right)t^{\nu-1/2}\left[\frac{\partial}{\partial\mu}I_\mu(t)\right]_{\mu=\pm m}dt.$$

The derivative of the reciprocal of the gamma function reduces to

$$\left\{\frac{\partial}{\partial\mu}\left[\Gamma\left(\nu-\mu+\frac{1}{2}\right)\right]^{-1}\right\}_{\mu=\pm m}=\frac{\psi\left(\nu\mp m+\frac{1}{2}\right)}{\Gamma\left(\nu\mp m+\frac{1}{2}\right)}.$$

The derivative with respect to order for the modified Bessel function of the first kind is given in (19). The integrals are easily obtained by applying the map given by (5) as necessary to (7) and (14). Hence by also using standard properties of associated Legendre, gamma, and digamma functions we obtain the following compact form

$$\frac{\Gamma(\nu\mp m+\frac{1}{2})}{\Gamma(\nu-m+\frac{1}{2})}\left[\frac{\partial}{\partial\mu}P^{\mu}_{\nu-1/2}(z)\right]_{\mu=\pm m}=Q^{m}_{\nu-1/2}(z)+\psi\left(\nu\mp m+\frac{1}{2}\right)P^{m}_{\nu-1/2}(z)$$
$$\pm m!\sum_{k=0}^{m-1}\frac{(-1)^{k-m}\left(z^2-1\right)^{(k-m)/2}}{2^{k-m+1}k!(m-k)}P^{k}_{\nu+k-m-1/2}(z). \qquad (22)$$

Note that

$$\left[\frac{\partial}{\partial\mu}P^{\mu}_{\nu-1/2}(z)\right]_{\mu=0}=Q_{\nu-1/2}(z)+\psi\left(\nu+\frac{1}{2}\right)P_{\nu-1/2}(z),$$

by §4.4.3 of Magnus, Oberhettinger & Soni (1966) [13]. So (22) is also valid if $\mu=0$.

5 Acknowledgements

I would like to thank Dr A.F.M. ter Elst for valuable discussions and acknowledge funding for time to write this paper from the Dean of the Faculty of Science at the University of Auckland in the form of a three month stipend to enhance University of Auckland 2012 PBRF Performance. Part of this work was conducted while the author was a National Research Council Research Postdoctoral Associate in the Information Technology Laboratory of the National Institute of Standards and Technology.

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. U.S. Government Printing Office, Washington, D.C., 1972.

[2] A. Apelblat and N. Kravitsky. Integral representations of derivatives and integrals with respect to the order of the Bessel functions $J_\nu(t)$, $I_\nu(t)$, the Anger function $\mathbf{J}_\nu(t)$ and the integral Bessel function $\mathrm{Ji}_\nu(t)$. *IMA Journal of Applied Mathematics*, 34(2):187–210, 1985.

[3] Yu. A. Brychkov. *Handbook of Special Functions: Derivatives, Integrals, Series and Other Formulas*. Chapman & Hall/CRC Press, Boca Raton-London-New York, 2008.

[4] Yu. A. Brychkov. On the derivatives of the Legendre functions $P^\mu_\nu(z)$ and $Q^\mu_\nu(z)$ with respect to μ and ν. *Integral Transforms and Special Functions*, 21:DOI–10.1080/10652460903069660, 2010.

[5] Yu. A. Brychkov and K. O. Geddes. On the derivatives of the Bessel and Struve functions with respect to the order. *Integral Transforms and Special Functions*, 16(3):187–198, 2005.

[6] H. S. Cohl. Derivatives with respect to the degree and order of associated Legendre functions for $|z| > 1$ using modified Bessel functions. *Integral Transforms and Special Functions*, 21(8):581–588, 2010.

[7] H. S. Cohl, A. R. P. Rau, J. E. Tohline, D. A. Browne, J. E. Cazes, and E. I. Barnes. Useful alternative to the multipole expansion of $1/r$ potentials. *Physical Review A: Atomic and Molecular Physics and Dynamics*, 64(5):052509, Oct 2001.

[8] H. S. Cohl and J. E. Tohline. A Compact Cylindrical Green's Function Expansion for the Solution of Potential Problems. *The Astrophysical Journal*, 527:86–101, December 1999.

[9] H. S. Cohl, J. E. Tohline, A. R. P. Rau, and H. M. Srivastava. Developments in determining the gravitational potential using toroidal functions. *Astronomische Nachrichten*, 321(5/6):363–372, 2000.

[10] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, seventh edition, 2007. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX).

[11] E. W. Hobson. *The theory of spherical and ellipsoidal harmonics*. Chelsea Publishing Company, New York, 1955.

[12] S. Lang. *Real and functional analysis*, volume 142 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 1993.

[13] W. Magnus, F. Oberhettinger, and R. P. Soni. *Formulas and theorems for the special functions of mathematical physics*. Third enlarged edition. Die Grundlehren der mathematischen Wissenschaften, Band 52. Springer-Verlag New York, Inc., New York, 1966.

[14] F. W. J. Olver. *Asymptotics and special functions*. AKP Classics. A K Peters Ltd., Wellesley, MA, 1997. Reprint of the 1974 original [Academic Press, New York].

[15] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev. *Integrals and series. Vol. 2*. Gordon & Breach Science Publishers, New York, second edition, 1988. Special functions, Translated from the Russian by N. M. Queen.

[16] G. N. Watson. *A treatise on the theory of Bessel functions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1944.

[17] F. J. W. Whipple. A symmetrical relation between Legendre's functions with parameters $\cosh\alpha$ and $\coth\alpha$. *Proceedings of the London Mathematical Society*, 16:301–314, 1917.