A Special Functions Handbook for the Digital Age

Ronald Boisvert, Charles W. Clark, Daniel Lozier, and Frank Olver

Trivia question: What is the most cited work in the mathematical literature? With an estimated 40,000 citations,[1] the *Handbook of Mathematical Functions* [1] may well be it. Edited by Milton Abramowitz and Irene Stegun (see Figure 1) and released by the National Bureau of Standards in 1964, the *Handbook* was the result of a ten-year project to compile essential information on the special functions of applied mathematics (e.g., Bessel functions, hypergeometric functions, and orthogonal polynomials) for use in applications [2]. The *Handbook* remains highly relevant today in spite of its age. In 2009, for example, the *Web of Science* records more than 2,000 citations to the *Handbook*. That is more than one published paper every five hours—quite remarkable! And the number of citations yearly has been steadily increasing since 1964 (see Figure 2).

Why so many citations? Many refer to the *Handbook* when introducing a special function in their papers; the citation relates the notation used to a precise definition. In this way, the *Handbook* has become a de facto standard for the definition and notation for the special functions. In addition, the *Handbook* succinctly displays the most important properties of the functions for those who need to use them in applications, things like series expansions, integral representations, recurrence relations, limiting forms, asymptotic expansions, and relations to other functions. The fact that the *Handbook* authors had such good taste when selecting material—one can write down an infinite number of mathematical truths for each function, of course—is demonstrated by the journals in which the *Handbook* is cited. The vast majority of citations come not from mathematics journals but from journals in physics and engineering. The *Handbook* has been undeniably *useful*!

But, by several measures, the NBS *Handbook* is also *old*. More than half of its 1,046 pages are tables of function values whose use has largely been superseded by modern numerical software. The field of special functions itself has not stood still in the intervening forty-six years. There have been enormous advances, including new methods of analysis and new classes of well-characterized functions. Finally, the revolution in modern communications has led to unique opportunities for the effective conveyance of technical information to a vast audience.

Recognizing both these problems and opportunities, the National Institute of Standards and Technology (NIST)[2] in 1997 undertook a project to update and modernize the Abramowitz and Stegun *Handbook*. The main goals of the project were to identify those mathematical functions and their properties that are most important in application, to present this information in a concise form accessible to practitioners, and to disseminate it to the widest possible audience.

Ronald Boisvert is chief of the Applied and Computational Mathematics Division at the National Institute of Standards and Technology (NIST) in Gaithersburg, Maryland. His email address is `boisvert@nist.gov`.

Charles W. Clark is a NIST Fellow at the National Institute of Standards and Technology and a fellow and codirector of the Joint Quantum Institute of NIST and the University of Maryland. His email address is `charles.clark@nist.gov`.

Daniel Lozier is leader of the Mathematical Software Group in the Applied and Computational Mathematics Division of NIST. His email address is `daniel.lozier@nist.gov`.

Frank Olver is professor emeritus at the Institute for Physical Science and Technology and Department of Mathematics at the University of Maryland at College Park. His email address is `fwjo@umd.edu`.

[1] *This estimate was produced with the help of the Thomson Reuters* Web of Science. *It is an estimate because the* Handbook *was cited in hundreds of different ways, making a complete harvest of citations quite laborious.*

[2] *NBS became NIST in 1988.*

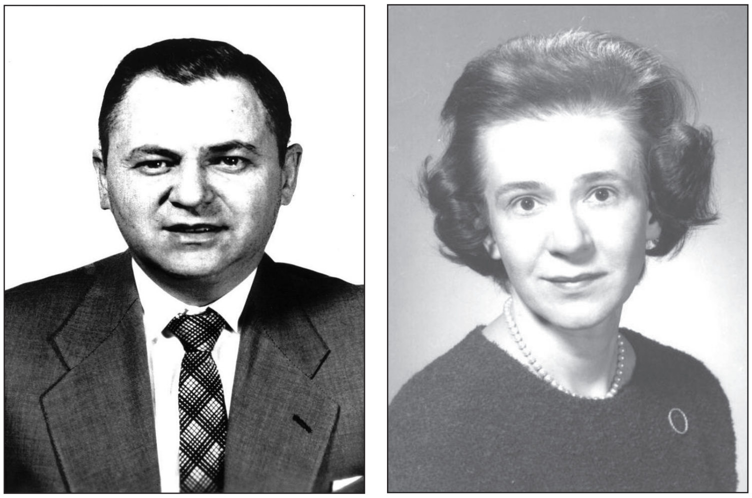

Figure 1. Milton Abramowitz (left) passed away in 1958, just nineteen months after the *Handbook* project was initiated. Irene Stegun (right) then undertook stewardship of the project. Stegun passed away in 2008.

The culmination of that project was the release in May 2010 of the online and freely available NIST Digital Library of Mathematical Functions (DLMF) (`http://dlmf.nist.gov/`) and its print companion, the *NIST Handbook of Mathematical Functions* [3]; see Figure 3.

Mathematical Content

The mathematical content of the DLMF[3] is organized into thirty-six chapters; see Table 1. The first three chapters are methodological in nature; they provide some essential definitions and background needed for the analysis and computation of special functions. Particular care is taken with

[3]*We use DLMF to refer to both the online handbook and its print companion.*

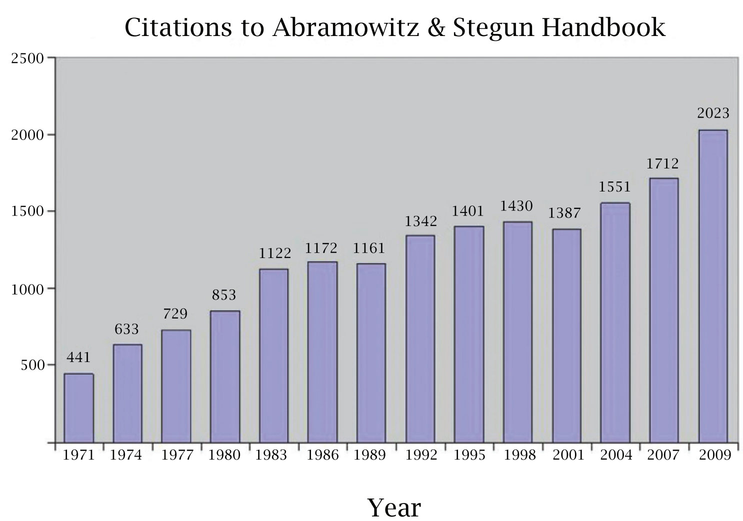

Figure 2. Citations to the NBS *Handbook of Mathematical Functions*, M. Abramowitz and I. Stegun, eds., 1964. Source: *Web of Science*.

topics that are not dealt with sufficiently thoroughly in the literature for our purposes. These include, for example, multivalued functions of complex variables, for which new definitions of branch points and principal values are supplied; the Dirac delta function, which is introduced in a more readily comprehensible way for mathematicians; numerically satisfactory solutions of differential and difference equations; and numerical analysis for complex variables. In addition, there is a comprehensive account of the great variety of analytical methods that are used for deriving and applying the extremely important asymptotic properties of the special functions, including exponentially powerful re-expansions of remainder terms, and also double asymptotic properties.

The remaining thirty-three chapters each address a particular class of functions. These chapters each have a similar organization. The first section provides a brief description of the notation that is used and its relation to alternate notations. In most cases we have adopted notations in standard use. But we have deviated from this in a few cases where we feel existing notations have serious drawbacks. For example, for the hypergeometric function we often use the notation $\mathbf{F}(a,b;c;z)$ in place of the more conventional ${}_2F_1(a,b;c;z)$ or $F(a,b;c;z)$. This is because $\mathbf{F}$ is akin to the notation used for Bessel functions, inasmuch as $\mathbf{F}$ is an entire function of each of its parameters a, b, and c. This results in fewer restrictions and simpler equations. Other examples are: (a) the notation for the Ferrers functions—also known as associated Legendre functions on the cut—for which existing notations can easily be confused with those for other associated Legendre functions; (b) the spherical Bessel functions for which existing notations are unsymmetric and inelegant; and (c) elliptic integrals for which both Legendre's forms and the more recent symmetric forms are treated fully. Of these, the latter is remarkably powerful: the newer symmetric integrals achieve extensive reductions in numbers of transformations when compared with the classical Legendre forms of the elliptic integrals, from eighty-one to three, for example.

Following the notation section comes the meat of the chapter: an enumeration of mathematical properties, such as the defining differential equation, special values, periods, poles, zeros, elementary identities, series representations, integral representations, transformations, relations to other functions, asymptotic expansions, derivatives, integrals, sums, and products. All of these are presented in the telegraphic style characteristic of the Abramowitz and Stegun handbook. One noteworthy new feature of the DLMF is that references are provided that connect every formula to a proof. Not only is this a means of verification, but it also allows researchers to learn proof techniques that may help them derive variations on the formulae.

In each function chapter a section entitled Graphics provides a set of 2D and 3D views of the functions designed to convey visually their essential features. Complex-valued functions are displayed by showing a 3D surface representing the modulus of the function; the surface is color coded to convey information about the phase (a 4D effect); see Figure 3.

Each function chapter also has sections on mathematical and physical applications. The point here is not to provide a complete or exhaustive treatment of applications but instead to provide references to representative applications to give the reader an appreciation of their wealth and diversity.

Finally, each chapter ends with a section on computation. Here, rather than provide formal algorithms that can be transformed into code, the DLMF provides a set of hints on fruitful numerical approaches with references. Included are indications of key approximations that are particularly relevant to computation.

Some of the progress in the mathematics of special functions over the past fifty years can be seen by comparing the DLMF with the Abramowitz and Stegun handbook (A&S). Fully half the content of A&S was tables of numerical function values. None of these are present in the DLMF, though the volume of technical content (measured in printed pages) is similar. The technical information for classic special functions has been greatly expanded in the DLMF. For example, the topic of Airy functions, which occupies barely five pages as a subsection of the chapter on Bessel Functions of Fractional Order in A&S, has become a first-class chapter of twenty-one pages in the printed DLMF. As another example, the treatment of orthogonal polynomials has expanded from twenty to fifty pages.

In addition, a variety of functions absent in A&S receive full treatment in the DLMF. These include generalized hypergeometric functions, the Meier *G*-function, *q*-hypergeometric functions, multidimensional theta functions, Lamé functions, Heun functions, Painlevé transcendents, functions of matrix argument, and integrals with coalescing saddles. The last, for example, includes so-called diffraction catastrophes, which arise in the connection between ray optics and wave optics. Applications include the study of rainbows, twinkling starlight, and the focusing of sunlight by rippling water.

Online-only Content

The online version of the DLMF provides additional technical content in comparison with its print counterpart. For example, in some cases additional instances in a series of formulae, such as weights and nodes of higher-order Gauss quadrature formulae, are available online.

The online DLMF also has a much larger collection of visualizations, and all 3D visualizations there are interactive [4]. Surface plots are used to display the essential properties of complex-valued functions in the DLMF. See Figure 4, for example, where we display the modulus of the psi function, $|\psi(x + iy)|$ for $-4 < x < 4$ and $-3 < y < 3$. Here the colors merely encode the value of the modulus itself, i.e., they are redundant. However, by clicking on this plot, DLMF users can summon up an interactive 3D visualization in which this can be changed. Users can have the colors correspond to the complex phase, either by a continuous mapping or by a four-color mapping based on quadrant (as in Figure 3). One can also rotate, zoom, and pan the view, change the axes' scaling, etc. So, for example, by viewing from the top, rescaling the z axis to zero, and selecting the continuous phase-color map, one obtains a color density plot of the phase. Finally, a tool is provided to generate a plane parallel to each of the three axes and viewing its intersection with the surface, either with a static plane or as a movie with the plane sweeping through the figure.

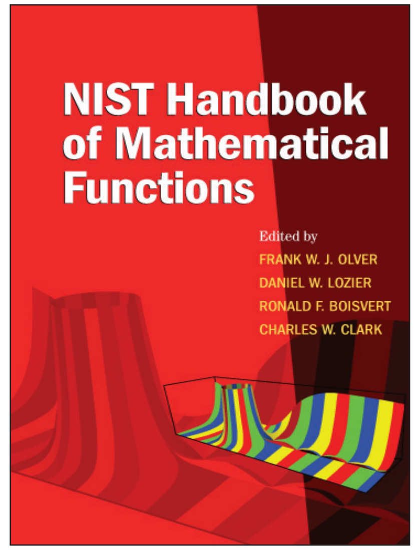

Figure 3. The print companion to the NIST Digital Library of Mathematical Functions.

The DLMF provides its interactive 3D content in two separate formats: Virtual Reality Modeling Language (VRML) [5] and Extensible 3D (X3D) [6], a newer standard. Eventually both may be replaced by WebGL [7], an emerging standard that is gaining much support. Viewing such 3D interactive content requires a browser plugin, and, given the evolving and somewhat chaotic landscape for this technology, support for various browsers and operating systems is quite uneven. The DLMF Help page provides suggestions for how to obtain free plugins for common systems and browsers.

Another substantial collection of online-only content is information about software for the special functions. In each chapter, a Software section lists research articles describing software development efforts associated with functions covered in the chapter. In addition, a table of links is provided to help users identify available software for each function (`http://dlmf.nist.gov/software/`). Open-source software, software associated with published books, and commercial software are all included. We hope to make this list exhaustive and up-to-date; we invite suggestions for additions.

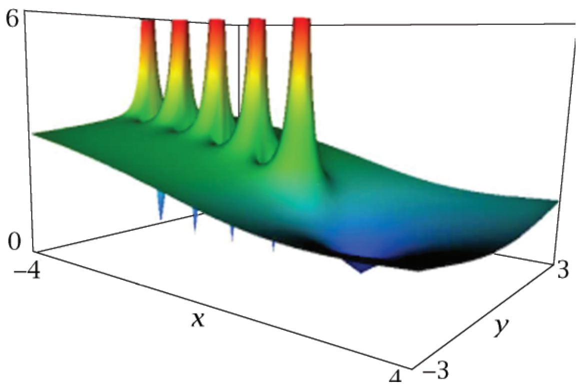

Figure 4. $|\psi(x+iy)|$, the modulus of the complex psi function.

The online DLMF also provides a rich set of tools that enhance its use as an interactive reference work. For example, each equation has a link providing the following additional information (metadata):

- The name of each symbol in the equation linked to its definition. This is particularly useful when the equation is encountered as the result of an online search.
- Reverse links to places in the DLMF where the equation is cited.
- A URL that can be used to refer to the equation.
- Alternate encodings of the equation: TeX, MathML, png (image).

The online references have a similarly rich set of links associated with them:

- Links to reviews in the AMS's MathSciNet and Zentralblatt's MATH database.
- Links to full text of articles online.
- Links to software associated with the reference.
- Reverse links to places in the DLMF where the reference is cited.

To ensure that reference links remain stable, the DLMF uses digital object identifiers (DOIs) [8] rather than URLs to refer to online versions of published articles. Developed and maintained by a consortium of publishers, DOIs have emerged as the de facto standard article identifier for the Web. DOIs are resolved to current URLs by presenting them to the resolver service at `http://dx.doi.org/`.

Math Search

While the DLMF does have an extensive index, the main way in which one finds reference information online is through search engines. Existing search technology is largely based upon analysis of words in documents. As a reference work written in a "telegraphic" style, there is just not a wealth of words in the DLMF for a search engine to index. As a result, it was necessary to develop our own math-oriented search engine. The DLMF search engine is based on Lucene [8], a full-featured open-source text-based search engine maintained by the Apache project. To enable math search this tool was augmented with additional data and processing layers [10, 11].

The query interface is a single text box. Query terms can be English text or mathematical expressions. The latter can be expressed in a LaTeX-like form, or using notation used in common computer algebra systems. So, for example, the Bessel function $K_n(z)$ can be referred to as $\mathsf{K_n(z)}$, $\mathsf{BesselK_n(z)}$, `BesselK(n,z)`, or `BesselK[n,z]`. To enable alternative terminologies and notations in queries each equation is augmented behind the scenes with a number of additional metadata terms which can be used in matches. Examples of DLMF queries are given in Table 2.

To match mathematical expressions, the underlying search engine should understand elementary arithmetic rules like commutative and associative laws. It should also realize that a search for $\sin(x)$ should probably also return expressions with $\sin(y)$, etc. Providing a search engine with such mathematical smarts remains a challenge. Instead, we approximate this using query relaxation. If a user's search does not result in any matches, the system will allow matches with successively larger numbers of intervening symbols or matches with symbols in any order.

Finally, search results must be delivered in order of perceived relevance. Unfortunately, standard relevance metrics used in ranking, where the frequency of occurrence of terms and the size of matching documents are the primary factors, are inadequate for finding objects in a math reference. Instead, the nature of the matching object and the query terms carry considerably more weight. For example, equations which serve as definitions should rank higher, followed by theorems, and so on. Also, query terms that are special function names are given more weight than variable names such as x or t.

The page of search results delivered by the DLMF is a list of matched objects, which can include displayed equations, figures, tables, section titles or text, and references. Matched portions of equations and text are highlighted to help the user understand why the item was matched. Clicking on any item provides a link to see the object in context.

Figure 5. DLMF editors and developers at NIST included the following (l. to r.): Abdou Youssef, Brian Antonishek, Daniel Lozier, Marjorie McClain, Bruce Miller, Bonita Saunders, Charles Clark, Frank Olver, and Ronald Boisvert.

The DLMF search engine is but a tentative first step in the emerging area of math search. Many opportunities for fruitful progress in this field remain.

Math on the Web

Displaying mathematical content on the Web presents its own set of challenges. For the DLMF we have decided to embrace MathML [12], the open community standard for representing mathematics developed by the World Wide Web Consortium (W3C). MathML provides a number of advantages. It allows equations to be scaled automatically when one scales the text size in the browser. It enables accessibility features to be deployed; for example, a MathML browser plugin is available that recites equations. MathML also has the potential to support semantics-preserving exchange of mathematical content.[4] One example is cut-and-paste of equations between Web pages, text editors, and computation systems or other types of interoperability between computer algebra systems.

MathML presents some practical problems, however. First, it is not yet supported by all browsers. It is supported natively in Firefox, but in Internet Explorer only with a specialized (free) third-party plug-in. Opera has some MathML support, but has problems rendering pages with particularly complex notation like the DLMF. Other browsers, such as Safari and Chrome, do not currently provide MathML support. Because of this, the DLMF is delivered in two forms: with MathML and with equations as images. The DLMF server identifies the requesting browser and sends the most appropriate form (though this can be overridden by the user).

1. Algebraic and Analytic Methods
2. Asymptotic Approximations
3. Numerical Methods
4. Elementary Functions
5. Gamma Function
6. Exponential, Logarithmic, Sine, and Cosine Integrals
7. Error Functions, Dawson's and Fresnel Integrals
8. Incomplete Gamma and Related Functions
9. Airy and Related Functions
10. Bessel Functions
11. Struve and Related Functions
12. Parabolic Cylinder Functions
13. Confluent Hypergeometric Functions
14. Legendre and Related Functions
15. Hypergeometric Function
16. Generalized Hypergeometric Functions and Meijer G-Function
17. q-Hypergeometric and Related Functions
18. Orthogonal Polynomials
19. Elliptic Integrals
20. Theta Functions
21. Multidimensional Theta Functions
22. Jacobian Elliptic Functions
23. Weierstrass Elliptic and Modular Functions
24. Bernoulli and Euler Polynomials
25. Zeta and Related Functions
26. Combinatorial Analysis
27. Functions of Number Theory
28. Mathieu Functions and Hill's Equation
29. Lamé Functions
30. Spheroidal Wave Functions
31. Heun Functions
32. Painlevé Transcendents
33. Coulomb Functions
34. $3j$, $6j$, $9j$ Symbols
35. Functions of Matrix Argument
36. Integrals with Coalescing Saddles

Table 1. Chapters of the DLMF.

The second issue is that MathML is a language for computers, not humans, and hence authoring in MathML is a challenge. See Table 3 for an illustration. Authors are more comfortable writing in LaTeX, and hence this is used as the input language for the DLMF. From this we wished to generate each of XHTML + MathML, HTML + images, and a traditional printed handbook. Generating MathML from LaTeX source is far from trivial, especially if one wants to represent mathematical semantics in the MathML. Such information is simply not available in the LaTeX.

A LaTeX to MathML translator was developed at NIST to confront this problem [13]. LaTeXML is a Perl program that completely mimics the processing of TeX but generates XML rather than dvi.

A postprocessor converts the XML into other formats such as HTML or XHTML, with options

[4] *MathML comes in two flavors: Presentation Markup and Content Markup. The former is focused only on proper display, while the latter includes some semantic-preserving markup. Most current implementations only support Presentation Markup.*

Query	Interpretation
`Gamma(?) = int`	$\Gamma(x) = \int$, for any x. ? is a single-character wildcard.
`int_$^$ BesselJ`	Any definite integral containing a *J* Bessel function. $ is a multicharacter wildcard, which will match "infinity" and "pi/2", for example
`sum_infinity^infinity BesselJ`	Any sum with limits -∞ to ∞ containing the *J* Bessel function
`zeta contour integral`	Text search for contour integrals of the zeta function

Table 2. Examples of DLMF queries.

to convert the math into MathML (currently only presentation) or images. The system relies on additional declarations and a variety of heuristics to perform the generation. For example, a style file was developed for the DLMF project that contains special LaTeX markup for common objects such as the special functions and common operators such as derivatives. The discipline imposed by such declarations greatly improves the reliability of the generated MathML without seriously inconveniencing authors. While it was developed specifically for the DLMF project, LaTeXML is a general-purpose tool and has been applied to much broader collections of scientific publications [14].

Putting It All Together

No one institution has the resources and technical expertise to carry out a project like this one alone. NIST is fortunate to have had the cooperation of a large number of experts worldwide in the assembly of the technical information contained in the DLMF. The DLMF chapters were assembled by twenty-nine external authors working under contract to NIST. NIST was responsible for editing the material so as to achieve the necessary depth and breadth of coverage and to ensure a uniform style of presentation. In addition, a set of twenty-five independent validators were enlisted to check the accuracy of the technical content. The extensive validation process is another feature that distinguishes the DLMF from A&S. Development of the DLMF website and its interactive content was performed by NIST staff. Overall responsibility for the project was vested in four NIST principal editors (F. Olver, D. Lozier, R. Boisvert, and C. Clark). They were advised by a group of external associate editors.

A project as complex as this one (both technically and organizationally) takes time. The initial plans for the project were laid out at a workshop held at NIST in 1997. So the DLMF has been some twelve years in the making. The development of A&S itself took about ten years. It is to their credit that NBS/NIST management had the patience to see these efforts through such a long and exacting process.

The Future

NIST is committed to the continued maintenance and development of the DLMF. We already have had lots of suggestions for improvements and extensions. New chapters on probability functions and computer algebra are under consideration. We are also considering bringing back tables, but, rather than static tables, tables generated on demand with certified correct values (quite a challenge!).

We do hope that the DLMF provides a firm foundation for extending the legacy of Abramowitz and Stegun far into the twenty-first century.

Math	TeX	MathML Presentation Markup
$\mathrm{Ln}\, z = \int_1^z \frac{dt}{t}$	`\[\mathop{\mathrm{Ln}\/} \nolimits z=\int _{1}^{z} \frac{dt}{t}\]`	<m:math display="block"> <m:mrow> <m:mrow> <m:mi>Ln</m:mi> <m:mo>⁡</m:mo> <m:mi>z</m:mi> </m:mrow> <m:mo>=</m:mo> <m:mrow> <m:msubsup> <m:mo>∫</m:mo> <m:mn>1</m:mn> <m:mi>z</m:mi> </m:msubsup> <m:mfrac> <m:mrow> <m:mi mathvariant="normal">ⅆ</m:mi> <m:mo>⁡</m:mo> <m:mi>t</m:mi> </m:mrow> <m:mi>t</m:mi> </m:mfrac> </m:mrow> </m:mrow> </m:math>

Table 3. Representing equations in TeX and MathML.

Acknowledgements

The DLMF is the result of efforts by more than fifty editors, authors, validators, and developers. A complete list can be found on the DLMF website at `http://dlmf.nist.gov/about/staff`.

References

[1] M. Abramowitz and I. Stegun, eds., *Handbook of Mathematical Functions, with Formulas, Graphs and Tables,* Applied Mathematics Series 55, National Bureau of Standards, Washington, DC, 1964.

[2] R. F. Boisvert and D. W. Lozier, Handbook of mathematical functions, in *A Century of Excellence in Measurements, Standards, and Technology—A Chronicle of Selected NBS/NIST Publications, 1901–2000* (D. Lide, ed.), NIST Special Publication 958, 2000, 135–139.

[3] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, eds., *NIST Handbook of Mathematical Functions*, Cambridge University Press, 2010.

[4] Q. Wang and B. Saunders, Web-based 3D visualization in a digital library of mathematical functions, in *Proceedings of the 10th International Conference on 3D Web Technology*, Bangor, Wales, UK, 2005, 151–157.

[5] A. Ames, D. R. Nadeau, and J. L. Moreland, *VRML 2.0 Sourcebook*, Wiley, 1996.

[6] D. Brutzman and L. Daly, *X3D: Extensible 3D Graphics for Web Authors*, Morgan Kaufmann, 2007.

[7] *WebGL—OpenGL ES 2.0 for the Web*, Khronos Group, `http://www.khronos.org/webgl/`.

[8] *The DOI System*, International DOI Foundation, `http://www.doi.org/`.

[9] *Lucene*, The Apache Software Foundation, `http://lucene.apache.org/`.

[10] A. Youssef, Information search and retrieval of mathematical contents: Issues and methods, in *Proceedings of the ISCA 14th International Conference on Intelligent and Adaptive Systems and Software Engineering*, Toronto, July 20–22, 2005, 100–105.

[11] ______, Methods of relevance ranking and hit-content generation in math search, in *Proceedings of the 6th International Conference on Mathematical Knowledge Management*, Hagenberg, Austria, June 27–30, 2007, 393–406.

[12] *Mathematical Markup Language (MathML)*, The World Wide Web Consortium (W3C), `http://www.w3.org/Math/`.

[13] B. Miller, *LaTeXML, a LaTeX to XML Converter*, `http://dlmf.nist.gov/LaTeXML/`.

[14] H. Stamerjohanns, M. Kohlhase, D. Ginev, C. David, and B. Miller, Transforming large collections of scientific publications to XML, *Mathematics in Computer Science* **3** (2010), 299-307.

Note: Certain commercial products are identified in this paper in order to describe the work adequately. Such identification is not intended to imply recommendation or endorsement by the National Institute of Standards and Technology, nor is it intended to imply that the materials or equipment identified are necessarily the best available for the purpose.

All graphic images in this article are courtesy of NIST.