SEISMIC APPLICATIONS OF NONLINEAR RESPONSE SPECTRA BASED ON THE THEORY OF MODAL ANALYSIS

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ABSTRACT

A fast nonlinear response spectra analysis algorithm based on the theory of modal analysis and superposition is proposed to overcome the drawbacks of using the time-consuming nonlinear response history analysis in seismic design. Because linear modal analysis has found great acceptance in performance-based seismic engineering, it is here extended to the nonlinear domain by using the force analogy method that links the global responses with local inelasticity of the structure. Geometric nonlinearity is incorporated into the analysis by modifying the initial stiffness matrices to consider gravity load effects. By ignoring geometric stiffness update, the theory of modal analysis and superposition is easily incorporated into the proposed algorithm. Numerical simulation is performed to demonstrate the accuracy of the algorithm in capturing both the maximum global and local responses.

Keywords: Nonlinear modal analysis, force analogy method, state space method, geometric nonlinearity, response spectra analysis.

1. INTRODUCTION

Simple analysis tools are often used in structural design to calculate the demands, and linear response spectra analysis (LRSA) based on square-root-of-the-sum-of-the-squares (SRSS) is one of the simple tools for estimating the seismic demand in designing structures constructed in seismically active regions. Chopra (2007) has documented the history and evolution of LRSA over the past decades. However, when subjected to a major earthquake, structures often respond in the nonlinear domain because the seismic demand will exceed its corresponding capacity by design. In this case, LRSA, as the named suggested, faces the limitation of being unable to capture the nonlinear behavior, making the analysis method impractical.

To overcome the limitation of LRSA in predicting nonlinear response, the use of nonlinear response spectra analysis (NRSA) has been proposed in the past with two schools of thoughts. One is the substitute-structure method (Shibata and Sozen 1976), where the response spectra remain linear but the period and damping of the structure are adjusted to achieve the targeted nonlinear responses.

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Because there is no closed-form relationship between pseudo-acceleration and displacement in a nonlinear system, different adjustment factors must be used in this method to achieve different response quantities. On the other hand, Newmark and Hall (1982) proposed nonlinear response spectra, where linear spectra shapes are adjusted to reflect the nonlinear behavior of the structure. The adjustment factors are developed based on ductility, which is an unknown quantity without first conducting any calculation to determine the seismic demand. Therefore, NRSA have never been fully developed, and seismic design today is still largely based on linear elastic procedures.

In this paper, a simple NRSA tool based on adjusting the response spectra by adopting yield displacement as the nonlinear parameter is proposed. Numerical simulation is performed to demonstrate the accuracy and efficiency of this tool in capturing both maximum global and local responses in comparison to those obtained using the extensive nonlinear response history analysis.

2. FORCE ANALOGY METHOD

The detailed derivation of the force analogy method has been presented in Wong and Yang (1999) and it is briefly summarized here. Let the total displacement $\mathbf{x}(t)$ at each degree of freedom (DOF) be represented as the summation of the elastic displacement $\mathbf{x}'(t)$ and the inelastic displacement $\mathbf{x}''(t)$:

$$\mathbf{x}(t) = \mathbf{x}'(t) + \mathbf{x}''(t) \tag{1}$$

Similarly, let the total moment $\mathbf{m}(t)$ at the plastic hinge locations (PHLs) of a moment-resisting frame be separated into elastic moment $\mathbf{m}'(t)$ and inelastic moment $\mathbf{m}''(t)$:

$$\mathbf{m}(t) = \mathbf{m}'(t) + \mathbf{m}''(t) \tag{2}$$

The displacements in equation (1) and the moments in equation (2) are related by the equations:

$$\mathbf{m}'(t) = \mathbf{K}'^{T}\mathbf{x}'(t) \quad , \qquad \mathbf{m}''(t) = -(\mathbf{K}'' - \mathbf{K}'^{T}\mathbf{K}^{-1}\mathbf{K}')\mathbf{\Theta}''(t) \tag{3}$$

where $\mathbf{\Theta}''(t)$ is the plastic rotation at the PHLs, $\mathbf{K}$ is the global stiffness matrix, $\mathbf{K}'$ is the stiffness matrix relating the plastic rotations at the PHLs and the forces at the DOFs, and $\mathbf{K}''$ is the stiffness matrix relating the plastic rotations with the corresponding moments at the PHLs. The relationship between plastic rotation $\mathbf{\Theta}''(t)$ and inelastic displacement $\mathbf{x}''(t)$ is:

$$\mathbf{x}''(t) = \mathbf{K}^{-1}\mathbf{K}'\mathbf{\Theta}''(t) \tag{4}$$

Substituting the two equations in equation (3) into equation (2) and making use of equations (1) and (4), then rearranging the terms gives the governing equation of the force analogy method:

$$\mathbf{m}(t) + \mathbf{K}''\mathbf{\Theta}''(t) = \mathbf{K}'^{T}\mathbf{x}(t) \tag{5}$$

3. MODAL ANALYSIS WITH GEOMETRIC NONLINEARITY

Two nonlinear effects must be considered in performing analysis with geometric nonlinearity. First is the reduction in local stiffness of the structural members due to the presence of axial load in the columns (i.e., P-δ effect). This can be done by modifying the stiffness matrices $\mathbf{K}$, $\mathbf{K}'$, and $\mathbf{K}''$ defined in equation (3). However, the axial force in the column members varies in a dynamic analysis, resulting in time-varying stiffness matrices $\mathbf{K}(t)$, $\mathbf{K}'(t)$, and $\mathbf{K}''(t)$. Let $\mathbf{K}_o$, $\mathbf{K}'_o$, and $\mathbf{K}''_o$ represent the global stiffness matrix at time zero, where only gravity load is applied on the column members. It follows that

$$\mathbf{K}(t) = \mathbf{K}_o + \mathbf{K}_g(t) \quad , \quad \mathbf{K}'(t) = \mathbf{K}'_o + \mathbf{K}'_g(t) \quad , \quad \mathbf{K}''(t) = \mathbf{K}''_o + \mathbf{K}''_g(t) \tag{6}$$

where $\mathbf{K}_g(t)$, $\mathbf{K}'_g(t)$, and $\mathbf{K}''_g(t)$ are the change in stiffness matrices due to the change in axial load on the column members during dynamic loading.

The second nonlinear effect comes in when lateral force $\mathbf{F}_f(t)$ is induced due to lateral displacement of the entire structure (i.e., P-Δ effect). This effect can be modeled using a P-Δ column in a two-dimensional analysis. The relationship between this lateral force $\mathbf{F}_f(t)$ and the total displacement of the structure $\mathbf{x}(t)$ can be written in the form:

$$\mathbf{F}_f(t) = \mathbf{K}_f \mathbf{x}(t) \tag{7}$$

where $\mathbf{K}_f$ is a function of the gravity loads on the P-Δ column and the corresponding story height, but it is not a function of time.

The equation of motion after considering both P-δ and P-Δ effects becomes

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}(t)\mathbf{x}'(t) = -\mathbf{M}\ddot{\mathbf{g}}(t) + \mathbf{F}_f(t) \tag{8}$$

where $\mathbf{M}$ is the $n \times n$ mass matrix, $\mathbf{C}$ is the $n \times n$ damping matrix, $\dot{\mathbf{x}}(t)$ is the $n \times 1$ velocity response at each DOF, $\ddot{\mathbf{x}}(t)$ is the $n \times 1$ acceleration response at each DOF, and $\ddot{\mathbf{g}}(t)$ is the $n \times 1$ ground acceleration vector, where each term relates to the direction of the corresponding DOF. Replacing the elastic displacement $\mathbf{x}'(t)$ in equation (8) by the difference between total displacement $\mathbf{x}(t)$ and inelastic displacement $\mathbf{x}''(t)$ through rearranging the terms in equation (1), and substituting equation (7) into the resulting equation, it follows that

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}_o\mathbf{x}(t) = -\mathbf{M}\ddot{\mathbf{g}}(t) + \mathbf{K}_f\mathbf{x}(t) - \mathbf{K}_g(t)\mathbf{x}(t) + \mathbf{K}(t)\mathbf{x}''(t) \tag{9}$$

To simplify equation (9), let

$$\mathbf{K}_e = \mathbf{K}_o - \mathbf{K}_f \tag{10}$$

where $\mathbf{K}_e$ represents the elastic stiffness of the entire structure that has incorporated the geometric nonlinear effect due to gravity loads. Pre-multiplying equation (4) by the stiffness matrix $\mathbf{K}(t)$ gives $\mathbf{K}(t)\mathbf{x}''(t) = \mathbf{K}'(t)\mathbf{\Theta}''(t)$, and substituting this result into the last term of equation (9) gives

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}_e\mathbf{x}(t) = -\mathbf{M}\ddot{\mathbf{g}}(t) - \mathbf{K}_g(t)\mathbf{x}(t) + \mathbf{K}'(t)\mathbf{\Theta}''(t) \tag{11}$$

To transform the response to the modal coordinates, which is required in any response spectra analysis, let the modal displacement $\mathbf{q}(t)$ be the $r \times 1$ vector and related to $\mathbf{x}(t)$ by the equation

$$\mathbf{x}(t) = \mathbf{\Phi}\mathbf{q}(t) \tag{12}$$

where $\mathbf{\Phi}$ is the $n \times r$ modal matrix computed based on elastic stiffness $\mathbf{K}_e$, and r is the total number of modes to be considered in the analysis. Now substituting equation (12) into equation (11) gives

$$\mathbf{M\Phi\ddot{q}}(t) + \mathbf{C\Phi\dot{q}}(t) + \mathbf{K}_e\mathbf{\Phi q}(t) = -\mathbf{M\ddot{g}}(t) - \mathbf{K}_g(t)\mathbf{x}(t) + \mathbf{K}'(t)\mathbf{\Theta}''(t) \tag{13}$$

Pre-multiplying equation (13) by $\mathbf{\Phi}^T$, it follows that

$$\mathbf{\Phi}^T\mathbf{M\Phi\ddot{q}}(t) + \mathbf{\Phi}^T\mathbf{C\Phi\dot{q}}(t) + \mathbf{\Phi}^T\mathbf{K}_e\mathbf{\Phi q}(t) = -\mathbf{\Phi}^T\mathbf{M\ddot{g}}(t) - \mathbf{\Phi}^T\mathbf{K}_g(t)\mathbf{x}(t) + \mathbf{\Phi}^T\mathbf{K}'(t)\mathbf{\Theta}''(t) \tag{14}$$

Assuming that the damping matrix $\mathbf{C}$ exhibits proportional damping characteristics, the matrix multiplications on the left side of equation (14) become

$$\mathbf{M}_d\mathbf{\ddot{q}}(t) + \mathbf{C}_d\mathbf{\dot{q}}(t) + \mathbf{K}_d\mathbf{q}(t) = -\mathbf{\Phi}^T\mathbf{M\ddot{g}}(t) - \mathbf{\Phi}^T\mathbf{K}_g(t)\mathbf{x}(t) + \mathbf{\Phi}^T\mathbf{K}'(t)\mathbf{\Theta}''(t) \tag{15}$$

where $\mathbf{M}_d = \mathbf{\Phi}^T\mathbf{M\Phi}$, $\mathbf{C}_d = \mathbf{\Phi}^T\mathbf{C\Phi}$, and $\mathbf{K}_d = \mathbf{\Phi}^T\mathbf{K}_e\mathbf{\Phi}$ are the diagonal modal mass, modal damping, and modal stiffness matrices, respectively. Equation (15) can be expressed in long form:

$$\begin{aligned} &\begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m_r \end{bmatrix} \begin{bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \\ \vdots \\ \ddot{q}_r(t) \end{bmatrix} + \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c_r \end{bmatrix} \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \vdots \\ \dot{q}_r(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} k_1 & 0 & \cdots & 0 \\ 0 & k_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & k_r \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_r(t) \end{bmatrix} = -\begin{bmatrix} \boldsymbol{\varphi}_1^T \\ \boldsymbol{\varphi}_2^T \\ \vdots \\ \boldsymbol{\varphi}_r^T \end{bmatrix} \mathbf{M\ddot{g}}(t) - \begin{bmatrix} \boldsymbol{\varphi}_1^T \\ \boldsymbol{\varphi}_2^T \\ \vdots \\ \boldsymbol{\varphi}_r^T \end{bmatrix} \mathbf{K}_g(t)\mathbf{x}(t) + \begin{bmatrix} \boldsymbol{\varphi}_1^T \\ \boldsymbol{\varphi}_2^T \\ \vdots \\ \boldsymbol{\varphi}_r^T \end{bmatrix} \mathbf{K}'(t)\mathbf{\Theta}''(t) \end{aligned} \tag{16}$$

where the subscripts $1,\ldots,r$ correspond to the associated modal parameters and modal responses. This gives r coupled modal equations of the form:

$$m_i\ddot{q}_i(t) + c_i\dot{q}_i(t) + k_iq_i(t) = -\boldsymbol{\varphi}_i^T\mathbf{M\ddot{g}}(t) - \boldsymbol{\varphi}_i^T\mathbf{K}_g(t)\mathbf{x}(t) + \boldsymbol{\varphi}_i^T\mathbf{K}'(t)\mathbf{\Theta}''(t) \quad i = 1,\ldots,r \tag{17}$$

4. NONLINEAR RESPONSE SPECTRA ANALYSIS

In equation (17), geometric nonlinearity due to gravity loads has already been fully incorporated in the calculation $k_i = \boldsymbol{\varphi}_i^T\mathbf{K}_e\boldsymbol{\varphi}_i$, while the term $\boldsymbol{\varphi}_i^T\mathbf{K}_g(t)\mathbf{x}(t)$ addresses the change in geometric nonlinear effects during dynamic loading. This nonlinear term typically has a small effect on the overall structural response, and therefore an assumption is made to ignore the geometric stiffness update by setting $\mathbf{K}_g(t) = \mathbf{0}$. With no geometric update being done, it follows accordingly that

$$\mathbf{K}_g'(t) = \mathbf{K}_g''(t) = \mathbf{0} \quad , \quad \mathbf{K}'(t) = \mathbf{K}_o' \quad , \quad \mathbf{K}''(t) = \mathbf{K}_o'' \tag{18}$$

Now consider material nonlinearity in the last term of equation (17). Another assumption is made here to disregard this term by modifying the response spectra used in determining the maximum responses. In linear response spectra analysis (LRSA), two parameters used in defining the response spectra shapes are periods and damping ratios. When nonlinear effects are considered, one additional parameter is needed to define the nonlinear behavior of the system. Here, yield displacement is chosen as the additional parameter because it is a structural property and is independent on the characteristics of earthquake ground motions. The yield displacement of each mode D_{yi}, where $i = 1,...,r$, can be calculated by making use of equations (3) and (18):

$$\mathbf{m}'(t) = \mathbf{K}_o'^T \mathbf{x}'(t) \tag{19}$$

If the structure is responding elastically and purely in the ith mode, the displacement pattern takes the shape of the ith mode up to yielding when the first plastic hinge is formed. At this time,

$$\mathbf{x}'(t) = \mathbf{x}(t) = \boldsymbol{\varphi}_i D_{yi} \tag{20}$$

In addition, $\mathbf{m}'(t) = \mathbf{m}(t)$ up to yielding. Substituting equation (20) into equation (19) gives

$$\mathbf{m}(t) = \left(\mathbf{K}_o'^T \boldsymbol{\varphi}_i\right) D_{yi} \tag{21}$$

The objective is to scan through the moment values at all the PHLs to determine what D_{yi} value will first cause any moment to reach its corresponding moment capacity. Once this is done, following the same procedure for all the other modes produces all r values of D_{yi}. Elastic-plastic behavior is assumed for the SDOF system for simplicity. In summary, equation (17) becomes

$$m_i \ddot{q}_i(t) + c_i \dot{q}_i(t) + k_i q_i(t) = -\boldsymbol{\varphi}_i^T \mathbf{M}\ddot{\mathbf{g}}(t) \quad i = 1,...,r \tag{22}$$

In a two-dimensional analysis, equation (22) reduces to

$$m_i \ddot{q}_i(t) + c_i \dot{q}_i(t) + k_i q_i(t) = -m_i \Gamma_{ix} \ddot{g}_x(t) \quad i = 1,...,r \tag{23}$$

where Γ_{ix} is the modal participation factor of the ith mode.

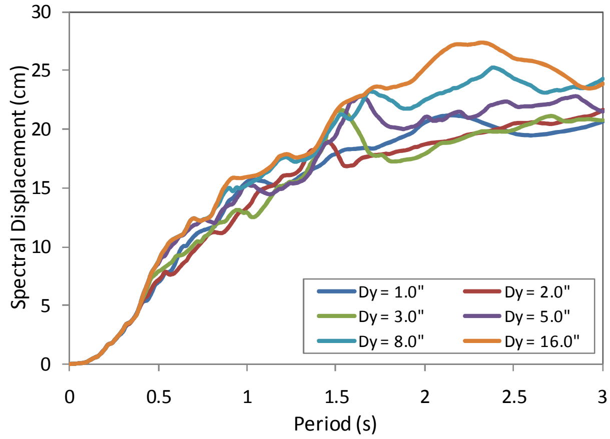
5. NUMERICAL SIMULATION

To demonstrate the accuracy of the algorithm, 13 earthquake time histories were extracted from the FEMA P-695 document (2009). Using different yield displacement levels and an elastic-plastic model for the stiffness of the system, 3% damped nonlinear mean response spectra are generated and shown in Figures 1(a) to 1(c). Here, 3% damping is chosen instead of commonly-used 5% because hysteretic damping is directly considered in both material and geometric nonlnearities.

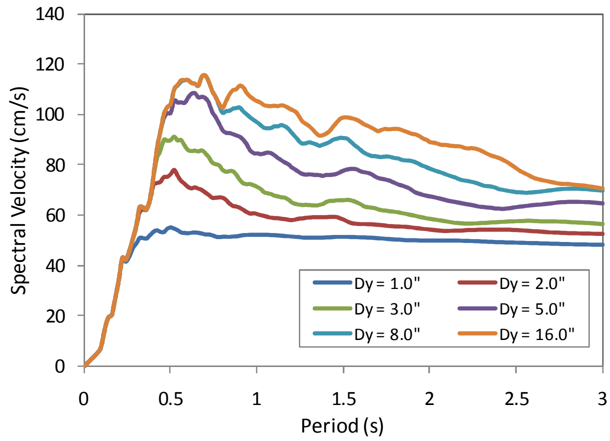
Consider the 16-story moment-resisting frame as shown in Figure 1(d), let the mass be 318.7 Mg on each of the 15 floors and 239.9 Mg on the roof. Gravity loads on the P-Δ column of 2,989 kN are applied on each of the 15 floors and 2,242 kN is applied at the roof level. The damping is assumed to be 3% in all modes. A total of 224 PHLs are identified as shown in Figure 1(d), all of which are assumed to exhibit elastic-plastic behavior with moment capacity m_y equal to the corresponding member's plastic moment at yield:

$$m_y = \sigma_y Z \tag{24}$$

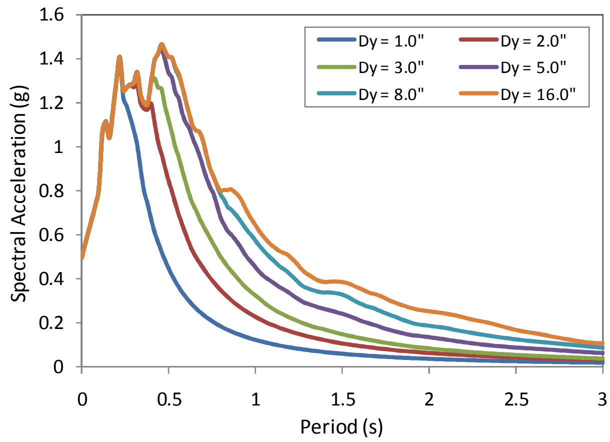
where $\sigma_y = 344.7$ MPa. All beams are subjected to a 14.01 kN/m uniform gravity loads, while interaction between axial force and moment capacity is ignored in the columns. The force analogy method is used to relate the local plastic hinge responses with the global displacement responses.



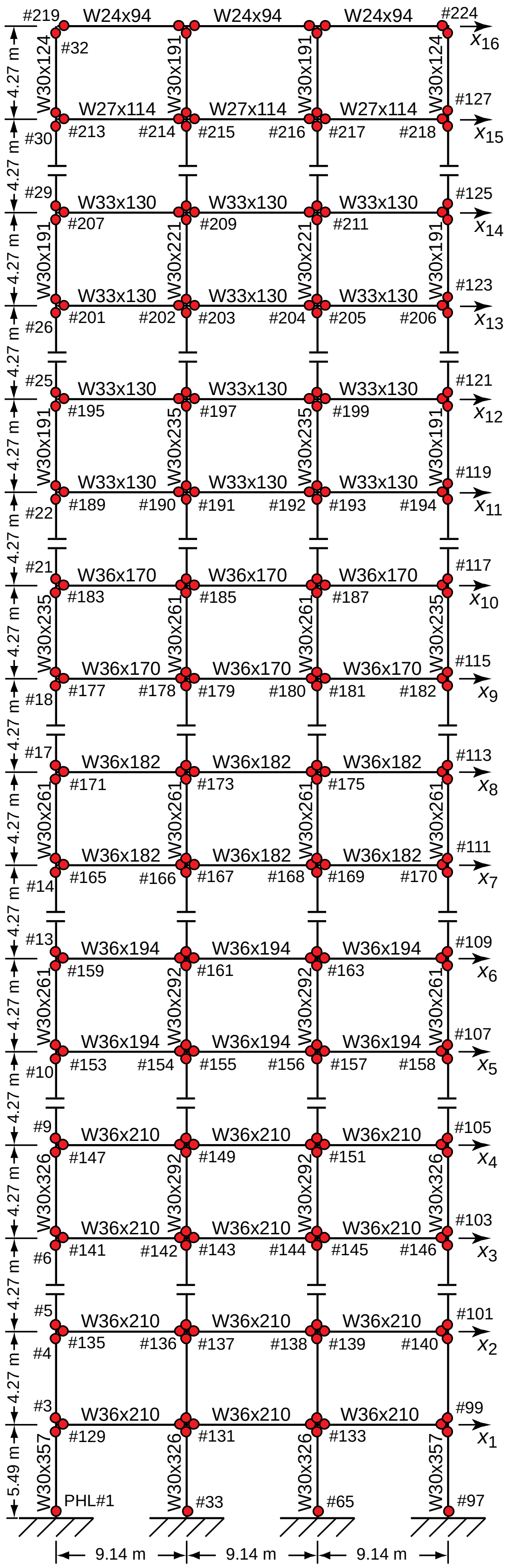
(a) Nonlinear spectral displacement



(b) Nonlinear spectral velocity



(c) Nonlinear spectral acceleration



(d) Sixteen-story moment-resisting frame

Figure 1: 3% damped nonlinear response spectra and two-dimensional structural model.

Nonlinear response history analysis (NRHA) is first conducted. By subjecting the 16-story frame to each of the 13 earthquake time histories that were previously used to generate the response spectra in Figure 1 with an amplification factor of 3.5, the mean of NRHA, mean minus one standard deviation (μ–σ), and mean plus one standard deviation (μ+σ) of the maximum global responses are presented in Figure 2 and the maximum local beam plastic rotation responses are presented in Figure 3.

Nonlinear response spectra analysis (NRSA) is then performed on the frame. Table 1 summarizes the periods, yield displacements, and modal participation factors of the first 9 elastic modes. By subjecting the frame to the 3% damped nonlinear mean response spectra as shown in Figure 1 with the same amplification factor of 3.5 on the earthquake ground motions, the maximum global and local beam responses based on NRSA are presented in Figures 2 and 3, respectively. In addition, linear response spectra analysis (LRSA) global results are also plotted in Figure 2 as a comparison. Results show that NRSA has reasonable accuracy in predicting the maximum responses.

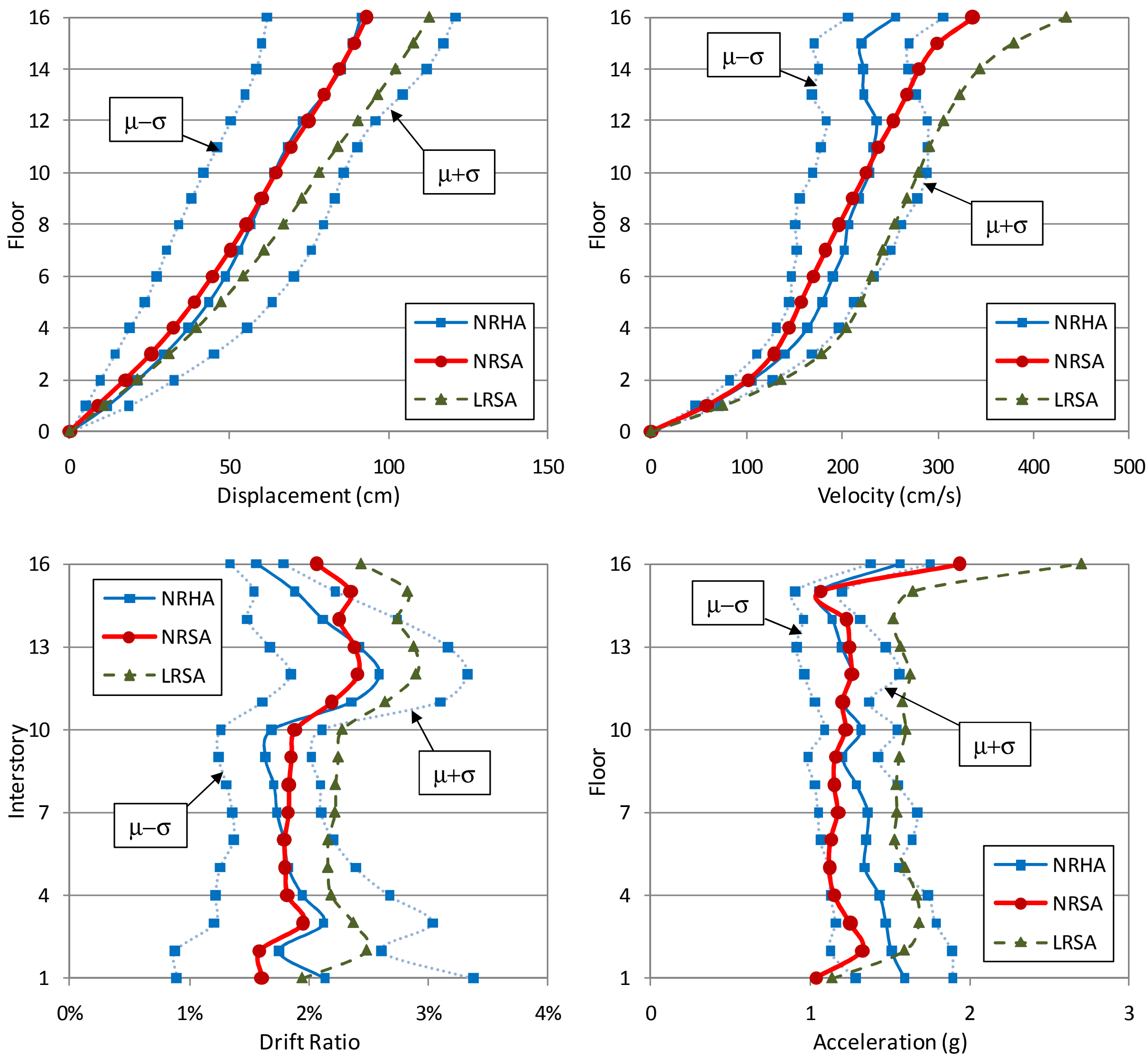


Figure 2: Comparisons of 16-story global responses between NRHA, NRSA, and LRSA.

Table 1: Parameters used in NRSA for the 16-story frame

	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6	Mode 7	Mode 8	Mode 9
Period (s)	2.9598	1.0833	0.6350	0.4468	0.3360	0.2613	0.2115	0.1736	0.1470
Γ_{ix}	0.8037	0.0969	0.0455	0.0144	0.0154	0.0058	0.0069	0.0028	0.0031
D_{yi}	10.6504	0.8808	0.4256	0.1276	0.1476	0.0590	0.0611	0.0218	0.0235

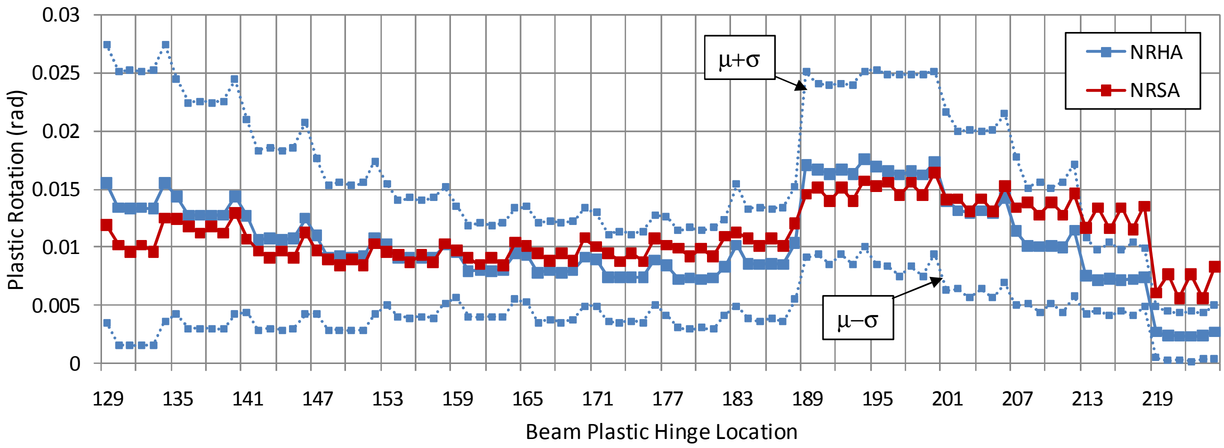


Figure 3: Comparison of 16-story plastic hinge responses between NRHA and NRSA.

6. CONCLUSION

A fast nonlinear response spectra analysis algorithm that incorporates both material and geometric nonlinearities was presented. It was observed that an additional parameter that considers material nonlinearity based on yield displacement is needed in defining the response spectra. By including *P*-δ and *P*-Δ effects due to gravity loads only in the initial stiffness while updating the geometric stiffness during dynamic loading is ignored, no additional parameter is needed in defining the response spectra to account for geometric nonlinearity of the structure. Numerical simulation showed that this treatment of geometric and material nonlinearities is accurate when comparing both global and local NRSA responses based on SRSS with those obtained using the nonlinear response history analysis.

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