Uncertainty budgeting for range calibration

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Abstract

The Guide to the Expression of Uncertainty (GUM) in Measurement established a general procedure to evaluate measurement uncertainty. The Guide covers only the evaluation of a single result or a set of results. Modern measurement instruments and procedures operate over a wide range of values. Therefore in practice a calibration procedure is needed that is valid for this range. It should include an evaluation of uncertainty associated with the calibration results and for the subsequent measurements performed with the calibrated instrument. Traditionally regression is used for this purpose. In this talk we will discuss the weaknesses of the regression approach and suggest an alternative to overcome the weaknesses.

1 Introduction

Measurements are applied to a variety of different quantities of interest and different methods and principles are in use. A very common cause-and-effect structure for many measurements is shown in Fig. 1. Often the quantity of interest (the measurand) is an input to a measuring device (instrument or procedure) and an indication (or observation) is the output. A typical example is a digital voltmeter which has input connectors for the quantity DC voltage and a digital display to indicate the value.

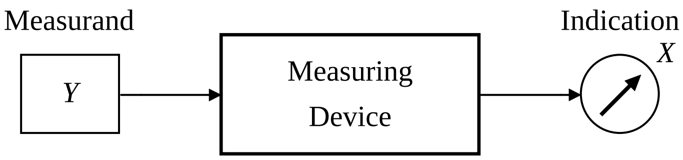

Figure 1: Cause-and-effect structure for many measurements.

We are interested in situations where a measuring device corresponding to the structure in Fig. 1 is used for a range of values of the measurand Y. We begin with the simple case where the response of the measuring device within the calibration range is linear and we discuss the calibration and the uncertainty budget. Traditionally a linear regression is used in these cases. We apply a linear regression to the simple case to demonstrate the limitations of this approach. We introduce the deviation from linearity in the model to deal with the limitation and combine it with a 2-point calibration. Based on a set of calibration points we develop an uncertainty budget for a real measuring device and discuss its limitations. In the last section we discuss how the calibration concept can be extended to deal with non-linear measuring devices.

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2 Calibrating a linear measuring device

When the response of the measuring device (indication X) is a linear function of the measurand Y, we use the term linear measuring device. The corresponding cause-and-effect structure is shown in Fig. 2. A detailed description about the modelling of measurements can be found in [1]. Perfectly linear measuring devices rarely exist in practice. It can never be known that the response of a real measuring device is exactly linear. Therefore a real device is considered linear if the deviation of the response from a perfectly linear response is smaller than a known limit and we use the term limit for the deviation from linearity for it.

When the limit for the deviation from linearity is large almost any device can be considered linear. Thus the setting of this limit has an influence on the quality of the measurement results when the device is used and it should therefore have an influence on the corresponding uncertainty associated with the results. The details are discussed in section 2.2.

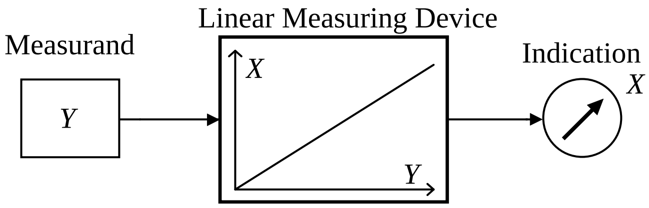

Figure 2: Cause-and-effect structure for a linear measuring device.

A special case of a linear measuring device is a device where the output of device for a zero input is zero. We will call this a zero adjusted linear measuring device. In this case the indication (or output) X of the device is proportional to the input Y. When the measuring device is used to measure an unknown measurand Y_i we observe the indication X_i. We need a function to calculate an estimate for the measurand based on the known indication X_i and the known inverse of the response of the measuring device. If the response is linear and zero adjusted then the inverse of the response is also linear and zero adjusted. The measurement function is:

$$Y_i = K_{\text{cal}} \cdot X_i \, , \tag{1}$$

where K_{cal} is the calibration factor. In general the measurement function for a linear measuring device is

$$Y_i = K_{\text{cal}} \cdot X_i - Y_{\text{zero}} \, , \tag{2}$$

where Y_{zero} is the negative of the known value of the measurand Y which produces an indication of zero.

Traditionally a linear regression is used [2, 3, 4] to calibrate a linear measuring device and to find appropriate values for K_{cal} and Y_{zero}.

2.1 Calibration using a linear regression

Suppose we have a linear measuring device corresponding to the cause-and-effect structure in Fig. 2 and we have a set of N known realisations of the measurand Y with different values $Y_{\text{cal},i}$ and the associated observed values of the indication $X_{\text{cal},i}$. We can use a linear regression to calculate K_{cal} and Y_{zero}:

$$K_{\text{cal}} = \frac{N \sum_{i=1}^{N} Y_{\text{cal},i}^2 - \left(\sum_{i=1}^{N} Y_{\text{cal},i} \right)^2}{N \sum_{i=1}^{N} X_{\text{cal},i} \cdot Y_{\text{cal},i} - \sum_{i=1}^{N} X_{\text{cal},i} \cdot \sum_{i=1}^{N} Y_{\text{cal},i}} \text{ and } Y_{\text{zero}} = \frac{K_{\text{cal}} \sum_{i=1}^{N} X_{\text{cal},i} - \sum_{i=1}^{N} Y_{\text{cal},i}}{N} \, . \tag{3}$$

To investigate the effectiveness of the linear regression[2] in determining K_{cal}, we look at the sensitivity coefficients

$$\frac{\partial K_{\text{cal}}}{\partial X_{\text{cal},j}} = -K_{\text{cal}} \cdot c \cdot \left(Y_{\text{cal},j} - \overline{Y}_{\text{cal}} \right) \text{ and } \frac{\partial K_{\text{cal}}}{\partial Y_{\text{cal},j}} = 2 \cdot c \cdot \left(Y_{\text{cal},j} - \overline{Y}_{\text{cal}} \right) - K_{\text{cal}} \cdot c \cdot \left(X_{\text{cal},j} - \overline{X}_{\text{cal}} \right) \tag{4}$$

$$\text{with } c = \frac{1}{\sum_{i=1}^{N} X_{\text{cal},i} \cdot Y_{\text{cal},i} - N \cdot \overline{X}_{\text{cal}} \cdot \overline{Y}_{\text{cal}}}, \ \overline{X}_{\text{cal}} = \frac{1}{N} \sum_{i=1}^{N} X_{\text{cal},i} \text{ and } \overline{Y}_{\text{cal}} = \frac{1}{N} \sum_{i=1}^{N} Y_{\text{cal},i} .$$

If the deviation from linearity is small then we can approximate the difference of the value $X_{\text{cal},j}$ from the average of all X_{cal} values in Equ. (4) by

$$X_{\text{cal},j} - \overline{X}_{\text{cal}} \approx \frac{Y_{\text{cal},j} - \overline{Y}_{\text{cal}}}{K_{\text{cal}}} \tag{5}$$

which leads to

$$\frac{\partial K_{\text{cal}}}{\partial Y_{\text{cal},j}} \approx c \cdot \left(Y_{\text{cal},j} - \overline{Y}_{\text{cal}} \right). \tag{6}$$

The sensitivity coefficient of a calibration point P_j : ($Y_{\text{cal},j}$, $X_{\text{cal},j}$) with respect to K_{cal} is proportional to the distance in the Y- and X-direction of that point from the average of the calibration points.

A similar investigation leads to the conclusion that calibration points closest to the origin have the largest sensitivity coefficient with respect to Y_{zero}. In addition Y_{zero} is also sensitive to changes in calibration points at the other end of the calibration range because of their influence on K_{cal}.

As a consequence K_{cal} is always determined by the calibration points which are far away from the centre of all calibration points and Y_{zero} is mostly determined by the points close to the origin. If the other points have a minor role in determining K_{cal} or Y_{zero} what is the purpose of measuring them? We will discuss this in Section 2.2.

Usually the state of knowledge about the calibration points is incomplete. Therefore we introduce the uncertainties associated with the calibration points. We can use the linear regression equations in Equ. (3) as a model equation (measurement function) and propagate the uncertainties accordingly. The sensitivities can be calculated with Equ. (4). Usually the $Y_{\text{cal},j}$ are not independent from each other because they are derived from a common standard. In such cases we need to include the correlation between them. Tab. 1 shows some example data points together with a given standard uncertainty in parenthesis. The data is considered uncorrelated.

Table 1: Example data points for a linear regression. The standard uncertainty is given in parenthesis.

j	1	2	3	4	5
$Y_{\text{cal},j}$	9.00(5)	3.00(5)	6.00(5)	9.00(5)	12.00(5)
$X_{\text{cal},j}$	0.000(8)	0.800(8)	1.800(8)	2.600(8)	3.400(8)

[2] In statistical text books, X is usually the known or independent quantity and Y is the unknown and dependent quantity. Therefore the equations presented here look different but are essential equal to the equations used in statistics.

Using Equ. (3) as the model of evaluation (measurement function) we propagate the uncertainty according to the GUM [5] by using the software package GUM Workbench [6] and we calculate the uncertainty budget for K_{cal} shown in Tab. 2. The main contributions are highlighted in grey.

Table 2: Uncertainty budget for the slope K_{cal} based on a regression and the data from Tab. 1.

Quantity	Value	Standard uncertainty	Sensitivity coefficient	Uncertainty contribution	Contribution Coefficient
$X_{\mathrm{cal},1}$	0.0	$8.0\cdot10^{-3}$	0.81	$6.5\cdot10^{-3}$	9.5 %
$Y_{\mathrm{cal},1}$	0.0	0.050	-0.23	-0.012	30.5 %
$X_{\mathrm{cal},2}$	0.80000	$8.0\cdot10^{-3}$	0.41	$3.2\cdot10^{-3}$	2.4 %
$Y_{\mathrm{cal},2}$	3.0000	0.050	-0.11	$-5.4\cdot10^{-3}$	6.6 %
$X_{\mathrm{cal},3}$	1.80000	$8.0\cdot10^{-3}$	$19\cdot10^{-21}$	$150\cdot10^{-24}$	0.0 %
$Y_{\mathrm{cal},3}$	6.0000	0.050	-0.011	$-540\cdot10^{-6}$	0.0 %
$X_{\mathrm{cal},4}$	2.60000	$8.0\cdot10^{-3}$	-0.41	$-3.2\cdot10^{-3}$	2.4 %
$Y_{\mathrm{cal},4}$	9.0000	0.050	0.11	$5.7\cdot10^{-3}$	7.3 %
$X_{\mathrm{cal},5}$	3.40000	$8.0\cdot10^{-3}$	-0.81	$-6.5\cdot10^{-3}$	9.5 %
$Y_{\mathrm{cal},5}$	12.0000	0.050	0.24	0.012	31.9 %
K_{cal}	3.4884	0.0211			

As it can be expected from the previous discussion only two of the five points contribute significantly to the uncertainty (uncertainty contribution coefficient >10 %). It should be noted that the calibration Point 3 close to the centre of all points has virtually no influence on the value or the uncertainty of K_{cal}. The uncertainty of K_{cal} is only dependent on the uncertainty of the data points and not on the quality of fit of the data to a straight line.

2.2 Deviation from linearity

Often a chi-square test is used to check the quality of the fit [7]. We do not believe that this is a useful approach especially if the uncertainty associated with the $X_{\mathrm{cal},j}$ and $Y_{\mathrm{cal},j}$ is not dominated by distortion from noise. The chi-square test checks if the data is consistent with a statistical model. When the reproducibility of the measuring device is very good a statistical model is not advisable since the deviation from linearity is largely deterministic.

We use an additional quantity $\delta_{\mathrm{lin}}(Y)$ in the model to cope with the deviation from linearity. Fig. 3 shows the cause-and-effect structure of a linear measuring device including $\delta_{\mathrm{lin}}(Y)$. The actual value of $\delta_{\mathrm{lin}}(Y)$ is dependent on the measured value Y.

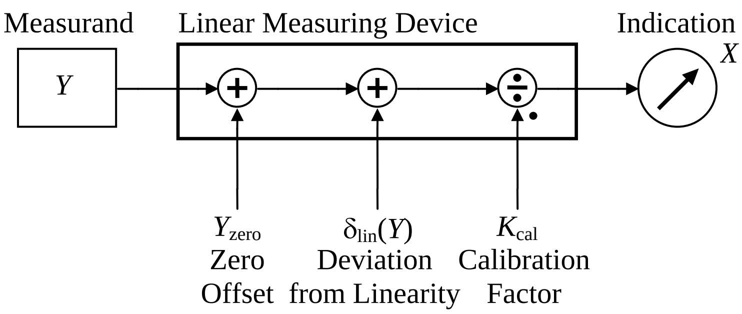

Figure 3: Cause-and-effect structure of a linear measuring device with a deviation from linearity.

From the cause-and-effect structure in Fig. 3 we can derive the measurement model

$$\frac{Y + Y_{\mathrm{zero}} + \delta_{\mathrm{lin}}(Y)}{K_{\mathrm{cal}}} = X \,, \tag{7}$$

where Y_{zero} is the zero offset, $\delta_{lin}(Y)$ is the deviation from linearity and K_{cal} is the calibration factor.

We can use the measurement model (7) to derive the measurement function for an unknown measurand Y_i

$$Y_i = X_i \cdot K_{cal} - \delta_{lin}(Y)_i - Y_{zero}\,. \tag{8}$$

In practice, the exact value of $\delta_{lin}(Y)_i$ is unknown. But it should be possible to state an upper limit ε_{lin} as a guardband for the deviation from linearity so that

$$\left|\delta_{lin}\left(Y\right)_i\right| < \varepsilon_{lin} \text{ for all } i. \tag{9}$$

In these cases the limit ε_{lin} can be converted to an uncertainty using a rectangular distribution.

2.3 Two point calibration

As demonstrated with the sensitivity analysis in Section 2.1 and the uncertainty budget in Tab. 2, the value of the slope evaluated by a linear regression is determined mostly by the extreme calibration points. Therefore we use a much simpler 2-point calculation to evaluate the calibration factor K_{cal} and zero offset Y_{zero}.

We use two extreme points for the calculation P_l : ($Y_{cal,l}$, $X_{cal,l}$) and P_u : ($Y_{cal,u}$, $X_{cal,u}$). The lower point is often the origin. The upper point is usually chosen close to the end of the range to minimize the influence on the uncertainty of K_{cal}. Since the two points define the slope of the linear measuring device response, the value of $\delta_{lin}(Y)$ for these points is zero. Using the measurement model from Equ. (7) and applying it to both calibration points, we get:

$$K_{cal} = \frac{Y_{cal,u} - Y_{cal,l}}{X_{cal,u} - X_{cal,l}} \text{ and } Y_{zero} = K_{cal} \cdot X_{cal,l} - Y_{cal,l}\,. \tag{10}$$

Note that Equ. (10) is a special case of Equ. (3) for $N = 2$.

Tab. 3 shows the uncertainty budget for K_{cal} using P_1: ($X_{cal,1}$,$Y_{cal,1}$) and P_5: ($X_{cal,5}$,$Y_{cal,5}$) from Tab. 1 as calibration points. Compared with the uncertainty budget in Tab. 2 we get a very similar value and uncertainty for K_{cal} but with a much simpler model equation.

Table 3: Uncertainty budget for the slope K_{cal} based on a 2-point calculation.

Quantity	Value	Standard Uncertainty	Sensitivity Coefficient	Uncertainty Contribution	Contribution Coefficient
$X_{cal,1}$	0.0	$8.00 \cdot 10^{-3}$	1.0	$8.3 \cdot 10^{-3}$	12.1 %
$Y_{cal,1}$	0.0	0.0500	-0.29	-0.015	37.9 %
$X_{cal,5}$	3.40000	$8.00 \cdot 10^{-3}$	-1.0	$-8.3 \cdot 10^{-3}$	12.1 %
$Y_{cal,5}$	12.0000	0.0500	0.29	0.015	37.9 %
K_{cal}	3.5294	0.0239			

Especially when the uncertainty of the calibration points is not dominated by noise but by systematic effects in Y_{cal}, it is not advisable to apply a regression. A two point calibration together with an additional term to cope with the possible deviation from linearity is more effective in those cases.

2.4 Limited resolution

Measuring devices always have a limited resolution. Since most of the present measuring devices use a digital display, we discuss this case here. Fig. 4 shows the cause-and-effect structure of a linear measuring device with a deviation from linearity and a limited resolution.

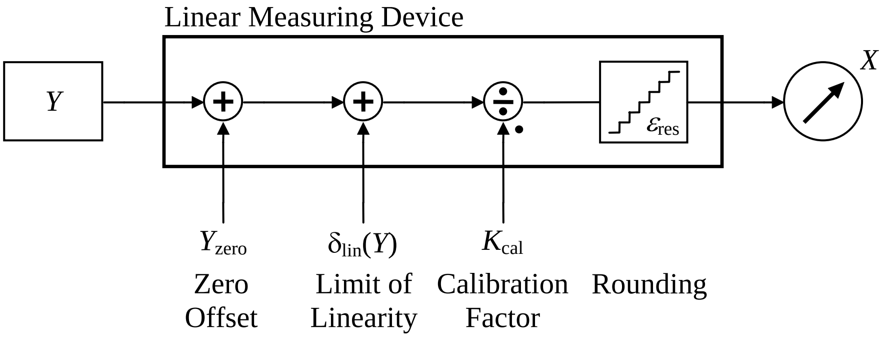

Figure 4: Cause-and-effect structure of a linear measuring device with a limited resolution.

The derived measurement model for the measuring device in Fig. 4 is

$$\mathrm{round}\left(\frac{Y+Y_{zero}+\delta_{lin}(Y)}{K_{cal}}, \varepsilon_{res}\right)=X\ , \tag{11}$$

where the function round() is rounding the indication to the number of digits of the display expressed as a limit of resolution ε_{res} which is equal to half of the last digit.

The limited resolution is a non-linearity in the transfer function of the measuring device. Furthermore a digital quantisation (rounding) introduces discontinuities which make the mathematical inversion of the measurement model impossible. Therefore we use the common solution of adding an additional quantity δX_{res} with a rectangular distribution to the indication which represents the resolution:

$$\frac{Y+Y_{zero}+\delta_{lin}(Y)}{K_{cal}}=X+\delta X_{res}\text{ , with }\left|\delta X_{res}\right|\leq\varepsilon_{res}\ . \tag{12}$$

From Equ. (12) we derive the measurement function for the measurement of an unknown measurand Y_i

$$Y_i=\left(X_i+\delta X_{res,i}\right)\cdot K_{cal}-\delta_{lin}(Y)_i-Y_{zero} \tag{13}$$

and for the calibration of K_{cal} and Y_{zero}:

$$K_{cal}=\frac{Y_{cal,u}-Y_{cal,l}}{\left(X_{cal,u}+\delta X_{res,u}\right)-\left(X_{cal,l}+\delta X_{res,l}\right)}\text{ and }Y_{zero}=K_{cal}\cdot\left(X_{cal,l}+\delta X_{res,l}\right)-Y_{cal,l}\ . \tag{14}$$

3 Example with real measurement data

We want to apply the calibration concept developed in Section 2 to calibrate and use a pressure sensor. Although in practical cases one needs to use proper units for the quantities, we do not use units here to simplify the discussion.

3.1 Calibration

Tab. 4 shows the measurement data for eleven calibration points. We assume that the linearity of the pressure generator which generates the known pressures $Y_{cal,j}$ is much better than the linearity of the device under calibration. Therefore we can ignore the uncertainty of the $Y_{cal,j}$ in the context of the evaluation of the deviation from linearity. Fig. 5 shows a plot of the response of the measuring device together with a plot of the deviation from linearity (residuals [4]).

Table 4: Measured data observed for the measuring device (pressure sensor).

Point, j	0	1	2	3	4	5	6	7	8	9	10
$Y_{\text{cal},i}$	0.0	2.0	4.0	6.0	8.0	10.0	12.0	14.0	16.0	18.0	20.0
$X_{\text{cal},i}$	0.0000	0.2039	0.4080	0.6120	0.8160	1.0201	1.2242	1.4283	1.6325	1.8367	2.0410

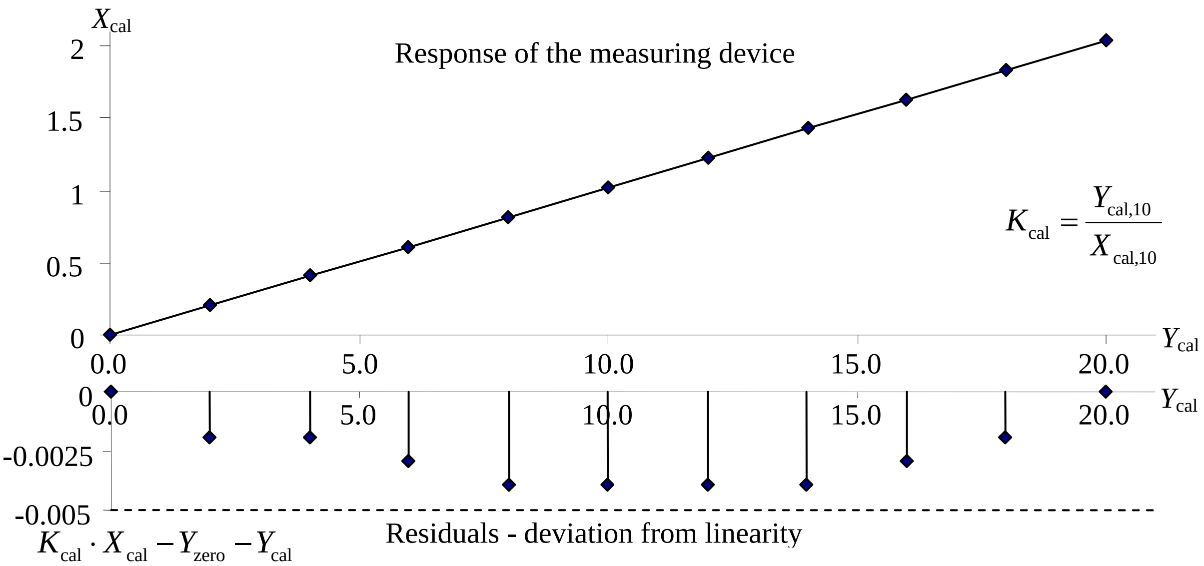

Figure 5: Response of the measuring device and residuals based on the data in Tab. 4.

We use Point 0 and Point 10 to calculate K_{cal} and Y_{zero} from Equ. (14). The limit of resolution is $\varepsilon_{\text{res}} = 0.00005$. While the zero point can be generated with a negligible uncertainty, the uncertainty in $u(Y_{\text{cal,10}})$ cannot be neglected and it is $u(Y_{\text{cal,10}}) = 2.3\cdot10^{-3}$. Repeated observations $X_{\text{cal},j}$ do not change and are therefore treated as constants. This is in practice often the case when industrial sensors are calibrated and the stability of the measurement system and the standards is much better than the resolution. Tab. 5 shows the uncertainty budget for the evaluation of the calibration factor K_{cal}.

Table 5: Uncertainty budget for K_{cal} based on a 2-point calibration with Point 0 and 10 from Tab. 4.

Quantity	Value	Standard Uncertainty	Sensitivity Coefficient	Uncertainty Contribution	Contribution Coefficient
$Y_{\text{cal,10}}$	20.00000	$2.30\cdot10^{-3}$	0.49	$1.1\cdot10^{-3}$	97.1 %
$X_{\text{cal,10}}$	2.041				
$\delta X_{\text{res,10}}$	0.0	$28.9\cdot10^{-6}$	-4.8	$-140\cdot10^{-6}$	1.5 %
$\delta X_{\text{res,0}}$	0.0	$28.9\cdot10^{-6}$	4.8	$140\cdot10^{-6}$	1.5 %
K_{cal}	9.79912	$1.14\cdot10^{-3}$			

A similar evaluation leads to a value of $Y_{\text{zero}} = 0.0$ and an uncertainty of $u(Y_{\text{zero}}) = 280\cdot10^{-6}$. Because Point 0 is used in both evaluations K_{cal} and Y_{zero} are correlated with $r(K_{\text{cal}},Y_{\text{zero}}) = 0.12$.

The residuals in Fig. 5 show that the measuring device has a significant non-linearity which could be corrected. We will handle the correction later in Section 4. In an industrial context, it is not always feasible to apply such a correction. Therefore we will increase the uncertainty instead. We specify an upper limit for the deviation from linearity ε_{lin} as 0.005 and use the residuals to prove that

$$\left| K_{\text{cal}} \cdot X_{\text{cal},j} - Y_{\text{zero}} - Y_{\text{cal},j} \right| < \varepsilon_{\text{lin}} \quad \text{for all calibration points } (j = 1 \ldots 9). \tag{15}$$

Since we do not know for which measured values the sensor will be subsequently used, we use a rectangular distribution for $\delta_{\text{lin}}(Y)$. We could improve the uncertainty in this special case by establishing a negative expectation value for $\delta_{lin}(Y)$ of half the largest residual, but it would make the use of the sensor more complicated since the user must apply this correction to every measured value.

The calibration provides the following information to the user for the subsequent measurements:

- The calibration factor K_{cal} which contains the link to the stated reference for traceability.
- The zero offset Y_{zero} and the correlation $r(K_{cal},Y_{zero})$ which are often omitted in case the device can be zero adjusted by the user later before the measurement.
- A limit of the deviation from linearity ε_{lin} to allow the measuring device to be used over the specified range.

3.2 Measuring an unknown measurand

After we have calibrated the measuring device we can use it to measure different unknown measurands within the calibrated range. We use the indications X_{cal} found during the calibration as examples for observations of unknown measurands. We use Equ. (13) as a measurement function to evaluate the results. Tab. 6 shows the uncertainty budget for a measurement in the middle of the calibrated range.

Table 6: Uncertainty budget for a measurement with the measuring device

Quantity	Value	Standard Uncertainty	Sensitivity Coefficient	Uncertainty Contribution	Contribution Coefficient
Y_{zero}	0.0	$28.9\cdot10^{-6}$	-1.0	$-29\cdot10^{-6}$	0.0 %
K_{cal}	9.79912	$1.14\cdot10^{-3}$	1.0	$1.2\cdot10^{-3}$	13.8 %
X_5	1.0201				
$\delta X_{res,5}$	0.0	$28.9\cdot10^{-6}$	9.8	$280\cdot10^{-6}$	0.8 %
$\delta_{lin}(Y)_{,5}$	0.0	$2.89\cdot10^{-3}$	-1.0	$-2.9\cdot10^{-3}$	85.3 %
Y_5	9.99608	$3.12\cdot10^{-3}$			

Table 7: Results with their associated expanded uncertainties over the calibrated range of the measuring device.

Quantity	Value	Exp. Uncertainty	Coverage factor
Y_0	0.0	$5.8\cdot10^{-3}$	2.00
Y_1	1.9980	$5.8\cdot10^{-3}$	2.00
Y_2	3.9980	$5.9\cdot10^{-3}$	2.00
Y_3	5.9971	$6.0\cdot10^{-3}$	2.00
Y_4	7.9961	$6.1\cdot10^{-3}$	2.00
Y_5	9.9961	$6.2\cdot10^{-3}$	2.00
Y_6	11.9961	$6.4\cdot10^{-3}$	2.00
Y_7	13.9961	$6.6\cdot10^{-3}$	2.00
Y_8	15.9971	$6.9\cdot10^{-3}$	2.00
Y_9	17.9980	$7.1\cdot10^{-3}$	2.00
Y_{10}	20.0000	$7.4\cdot10^{-3}$	2.00

As expected, the contribution because of the limited linearity $\delta_{lin}(Y)$ dominates the uncertainty budget. Tab. 7 shows the results with the associated expanded uncertainties for different values of the measurand over the calibrated range of the measuring device. It should be noted that since the uncertainty budget in Tab. 6 is dominated by the quantity introduced to cope with the limited linearity which has a rectangular distribution assigned to it, the distribution of the Y_i is not normal but more trapezoidal. The relative influence of this contribution is larger for smaller values of the measurand leading to an overestimation of the coverage factor by about 15% for small measurands around zero and down to 3% for measurands at the higher end of the calibrated range. For the midrange value in Tab. 6 the coverage factor in Tab. 7 is about 10% too large.

It would be possible to improve the expanded uncertainty by applying a better coverage factor found for example by Monte Carlo simulation [8]. In the next section we will instead improve the uncertainty by correcting for the non-linearity and thereby reducing the contribution of $\delta_{\text{lin}}(Y)$.

Fig. 6 shows the performance of the measuring device over the calibration range as uncorrected deviations $\Delta Y = Y - Y_{\text{cal}}$ with their expanded uncertainties ($k_{p95} = 2.0$), treating the Y_{cal} as constants. The expanded uncertainty interval always covers the value zero.

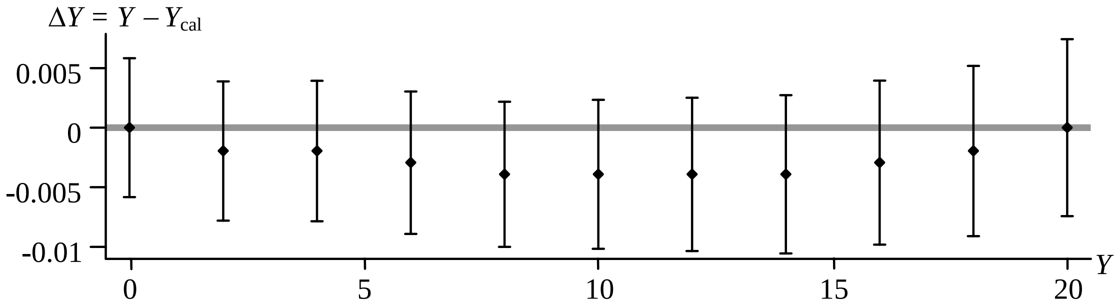

Figure 6: Performance of the measuring device over the full range with uncertainty bars ($k_{p95} = 2.0$)

The uncertainty of the results Tab. 7 is very similar. Therefore the use of the measuring device could be simplified for the user by applying the same value for the uncertainty to all values of *Y* over the calibration range. The indication *X* should be multiplied with a calibration factor $K_{\text{cal}} = 9.799$ to calculate the value of an unknown measurand *Y* and the expanded uncertainty can be predicted with $U_{\text{pred}}(Y) = 7.4 \cdot 10^{-3}$ ($k_{\text{p95}} = 2.0$).

4 Non linear measuring device

The measuring device we have used as an example in Section 3 shows a significant deviation from linearity which is the dominating uncertainty component if it is not corrected (Tab. 6). The non-linearity of these pressure sensors is quite common and usually a third degree polynomial is used to correct for it.

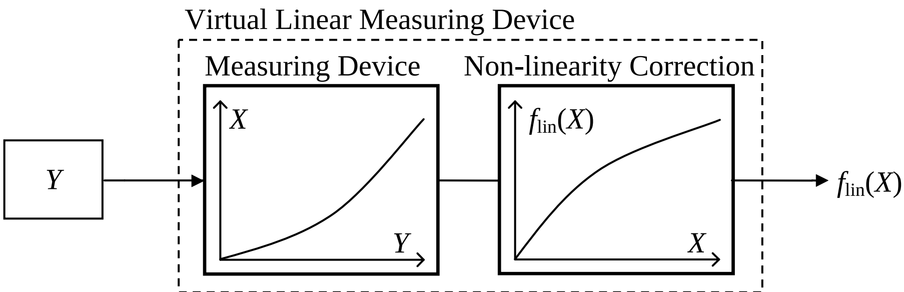

Figure 7: Cause-and-effect structure of a linearized measuring device.

For the specific sensor used as an example here, the following function has been found to be useful as a correction:

$$f_{\text{lin}}(X) = a + b \cdot X + c \cdot X^2 + d \cdot X^3, \qquad (16)$$

where $a = 0$, $b = 9.806027$, $c = -2.251197 \cdot 10^{-3}$, $d = -5.753193 \cdot 10^{-4}$. The parameters *a*, *b*, *c* and *d* have been found by regression, but it does not matter how we find a useful correction function.

Fig. 7 shows the cause-and-effect structure of a non-linear measuring device being corrected by a correction function. The combination of the measuring device together with the function to correct for non-linearity can be treated as a new virtual linear measuring device which should be more linear than the measuring device without correction and it can be calibrated following the procedure discussed in Section 2. The measurement model can be derived from Equ. (12) by replacing the right side by the corrected observation with $f_{\text{lin}}()$ (Equ. (16)) and we get:

$$\frac{Y + Y_{\text{zero}} + \delta_{\text{lin}}(Y)}{K_{\text{cal}}} = f_{\text{lin}}\left(X + \delta X_{\text{res}} \right). \qquad (17)$$

From Equ. (17) we derive the measurement function for the measurement of an unknown measurand Y_i

$$Y_i = f_{\text{lin}}\left(X_i + \delta X_{\text{res},i} \right) \cdot K_{\text{cal}} - \delta_{\text{lin}}(Y)_i - Y_{\text{zero}} \qquad (18)$$

and for the calibration, the evaluation model for K_{cal} and Y_{zero}:

$$K_{\text{cal}} = \frac{Y_{\text{cal},u} - Y_{\text{cal},l}}{f_{\text{lin}}\left(X_{\text{cal},u} + \delta X_{\text{res},u} \right) - f_{\text{lin}}\left(X_{\text{cal},l} + \delta X_{\text{res},l} \right)} \text{ and } Y_{\text{zero}} = K_{\text{cal}} \cdot f_{\text{lin}}\left(X_{\text{cal},l} + \delta X_{\text{res},l} \right) - Y_{\text{cal},l}\,. \qquad (19)$$

The key question is whether or not there is an uncertainty involved with the function $f_{\text{lin}}()$ which need to be included in the uncertainty budget for Y_i, K_{cal} or Y_{zero}? Since $f_{\text{lin}}()$ has been established based on uncertain data, the parameters *a*, *b*, *c* and *d* are also uncertain. The consequence is that there is more than one set of values for the parameters *a*, *b*, *c* and *d* which would be a useful correction for the non-linearity. We are free to pick one of the many. After we have picked one set of values, the values we used are fixed. They become part of the sensor and we can treat them as if they would be hardwired to the sensor. After we have fixed the parameters *a*, *b*, *c* and *d*, the function $f_{\text{lin}}()$ is completely defined and known and there is no uncertainty left about it. Applying a fixed and known function to the data does not introduce any uncertainty. But it might change the sensitivities or modify the distribution of the result and therefore should be part of the measurement model as shown in Equ. (17).

Table 8: Uncertainty budget for the slope K_{cal} based on a 2-point calibration using the correction function from Equ. (16).

Quantity	Value	Standard Uncertainty	Sensitivity Coefficient	Uncertainty Contribution	Contribution Coefficient
$Y_{\text{cal},10}$	20.00000	$2.30 \cdot 10^{-3}$	0.050	$120 \cdot 10^{-6}$	97.1 %
$X_{\text{cal},10}$	2.041				
$\delta X_{\text{res},10}$	0.0	$28.9 \cdot 10^{-6}$	-0.49	$-14 \cdot 10^{-6}$	1.5 %
$\delta X_{\text{res},0}$	0.0	$28.9 \cdot 10^{-6}$	0.49	$14 \cdot 10^{-6}$	1.5 %
K_{cal}	1.000008	$117 \cdot 10^{-6}$			

Tab. 8 shows the uncertainty budget for K_{cal} using the measurement function in Equ. (19) with the correction function $f_{\text{lin}}()$ for the non-linearity.

Comparing the uncertainty budgets in Tab. 5 and Tab. 8 it should be noted that the value of the calibration factor K_{cal} changed from 9.99608 to 1.000008. This is due to the fact that $f_{\text{lin}}()$ not only improves the linearity but also does a linear scaling because $b = 9.806027$. But the uncertainty contribution coefficients do not change and also the relative uncertainty $u(K_{\text{cal}})/|K_{\text{cal}}|$ is the same. Using $f_{\text{lin}}()$ has in our case no influence on the value or the uncertainty of Y_{zero}. Therefore we use $Y_{\text{zero}} = 0.0$, $u(Y_{\text{zero}}) = 280 \cdot 10^{-6}$ and $r(K_{\text{cal}}, Y_{\text{zero}}) = 0.12$, the same as for the uncorrected measuring device.

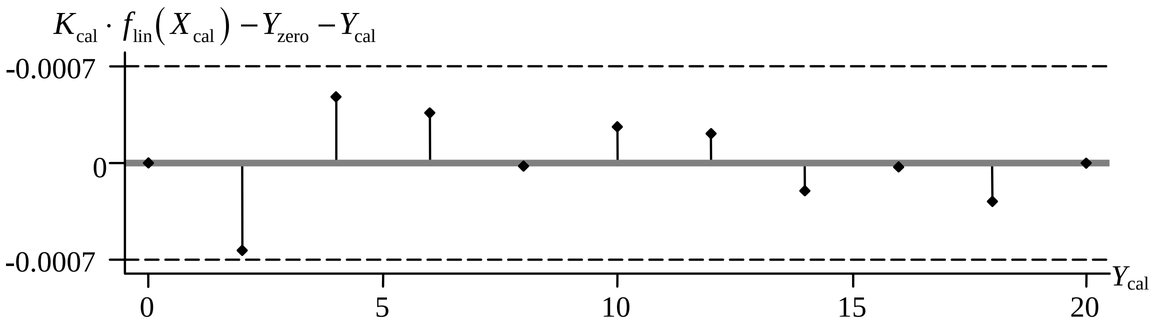

Figure 8: The residuals based on the data in Tab. 4 after applying the correction function $f_{lin}()$ from Equ. (16).

Fig. 8 shows a plot of the residual after applying the correction function $f_{lin}()$ from Equ. (16) to the calibration points in Tab. 4. The upper limit for the deviation from linearity can be reduced to $\varepsilon_{lin} = 0.0007$ which is almost a magnitude better than for the uncorrected measuring device.

Table 9: Uncertainty budget for a measurement with using the correction from Equ. (16).

Quantity	Value	Standard Uncertainty	Sensitivity Coefficient	Uncertainty Contribution	Contribution Coefficient
X_5	1.0201				
$\delta X_{res,5}$	0.0	$28.9 \cdot 10^{-6}$	9.8	$280 \cdot 10^{-6}$	5.0 %
K_{cal}	1.000008	$117 \cdot 10^{-6}$	10	$1.2 \cdot 10^{-3}$	82.4 %
Y_{zero}	0.0	$283 \cdot 10^{-6}$	-1.0	$-280 \cdot 10^{-6}$	2.5 %
$\delta_{lin}(Y)_{,5}$	0.0	$404 \cdot 10^{-6}$	-1.0	$-400 \cdot 10^{-6}$	10.1 %
Y_5	10.00025	$1.27 \cdot 10^{-3}$			

Tab. 9 shows the uncertainty budget for a measurement with the measuring device in the middle of the calibrated range using the correction from Equ. (16). It should be noted that the uncertainty contribution of the term coping with the limited linearity has dropped to a level where it is almost insignificant. The uncertainty budget is now dominated by the uncertainty of the calibration factor K_{cal} which is dominated by the uncertainty of the calibration standard $Y_{cal,10}$. To improve the uncertainty further a better calibration standard would be needed.

Table 10: Results with their associated expanded uncertainties over the calibrated range of the measuring device using the correction from Equ. (16).

Quantity	Value	Exp. Uncertainty	Coverage factor
Y_0	0.0	$990 \cdot 10^{-6}$	2.00
Y_1	1.9994	$1.2 \cdot 10^{-3}$	2.00
Y_2	4.0005	$1.4 \cdot 10^{-3}$	2.00
Y_3	6.0004	$1.8 \cdot 10^{-3}$	2.00
Y_4	8.0000	$2.1 \cdot 10^{-3}$	2.00
Y_5	10.0003	$2.5 \cdot 10^{-3}$	2.00
Y_6	12.0002	$3.0 \cdot 10^{-3}$	2.00
Y_7	13.9998	$3.4 \cdot 10^{-3}$	2.00
Y_8	16.0000	$3.8 \cdot 10^{-3}$	2.00
Y_9	17.9997	$4.3 \cdot 10^{-3}$	2.00
Y_{10}	20.0000	$4.7 \cdot 10^{-3}$	2.00

Table 10 shows the results with an expanded uncertainty covering the full calibrated range. Using the correction Equ. (16) the uncertainty improves in the lower range by about a factor of 5 while the improvement at the upper end of the range is less than factor of 2. The reason is that for values of Y larger than 4, the calibration factor K_{cal} is the biggest contributor to the uncertainty with a sensitivity coefficient equal to the value of Y.

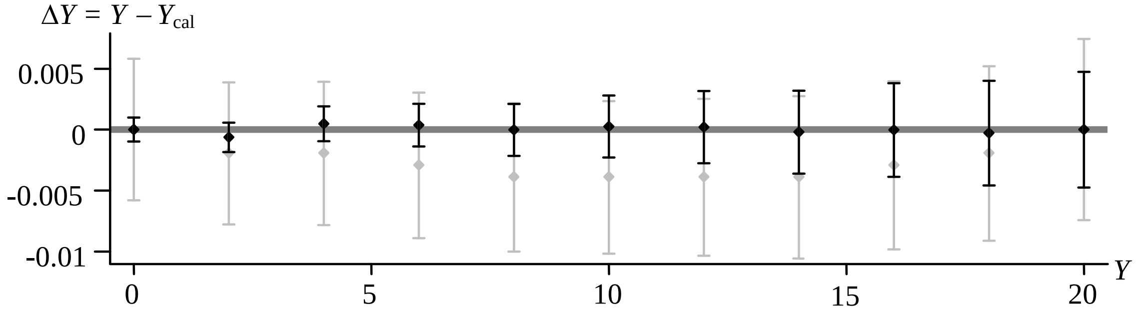

Figure 9: Performance of the measuring device over the full range with uncertainty bars (k_{p95} = 2.0), using the correction for non-linearity from Equ. (16). The performance of the uncorrected device is plotted in grey.

Fig. 9 shows the performance of the corrected measuring device over the calibration range as deviations $\Delta Y = Y - Y_{cal}$ with their expanded uncertainties (k_{p95} = 2.0), treating the Y_{cal} as constants. The performance of the uncorrected measuring device is plotted in grey to allow a direct comparison. It should be noted that the expanded uncertainty for the corrected measuring device is increasing with the value of the measurand Y.

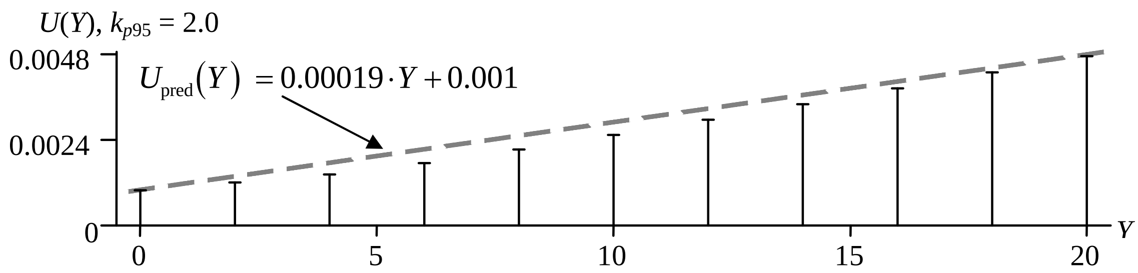

Figure 10: Expanded uncertainty (k_{p95} = 2.0) as a function of the measurand Y.

Fig. 10 shows the expanded uncertainty (k_{p95} = 2.0) as a function of the measured value from Tab. 10. Based on the evaluated uncertainties for different values of Y we can specify an envelope function $U_{pred}(Y) = 0.00019 \cdot Y + 0.001$ which characterizes the prediction of the expanded uncertainty for subsequent measured values determined with the measuring device.

The use of the measuring device can be simplified by specifying that the result should be calculated from the indication X by using the equation

$$Y = 1.000008 \cdot \left(9.806027 \cdot X - 2.251197 \cdot 10^{-3} \cdot X^2 - 5.753193 \cdot 10^{-4} \cdot X^3\right) \tag{20}$$

while the expanded uncertainty should be predicted using the specification

$$U_{pred}(Y) = 0.00019 \cdot Y + 0.001, \ (k_{p95} = 2.0). \tag{21}$$

It should be noted that this is only possible if the same coverage factor is used over the full range.

5 Conclusions

To calibrate a measuring device for the use over a range of values it is useful to assume linearity of the device response and introduce a quantity to cope with the possible deviation from linearity. A linear regression can be used to evaluate the calibration factor and the zero offset but this is not advisable especially if the uncertainty of the calibration is not dominated by noise effects. Instead, a simple 2-point calibration at the ends of the calibration range should be used. It is still necessary to perform a linearity study over the full calibration range to investigate the possible deviation from linearity which always needs to be included in the uncertainty budget when measuring with the measuring device.

To simplify the use of a measuring device a significant but small non-linearity may not be corrected. But this will lead to a larger uncertainty when using the device.

A fixed correction function of the indication for non-linearity does not contribute to the measurement uncertainty. Therefore any kind of function (or method) can be used to correct for non-linearities as long as it results in a fixed function which improves the overall linearity. After the linearity correction is established the measuring device plus correction can be treated as a linear device and should be calibrated with a 2-point calibration and an investigation of the deviation from linearity should be carried out. The same data might be used for establishing the correction and an upper limit on the deviation from linearity if a sufficient number of data points are used.

It is important that the same correction function (or method) be used to improve the linearity during calibration and subsequent use of the measuring device for unknown measurands.

If the coverage factor is constant over the calibration range, it is possible to specify a simple envelope function to predict the expanded uncertainty for later measurement values in the calibration range.

Disclaimer

The software GUM Workbench is identified in this paper in order to specify the calculation procedure adequately. Such identification is not intended to imply recommendation or endorsement by the National Institute of Standards and Technology, nor is it intended to imply that the software identified is necessarily the best available for the purpose.

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