A Toolbox for Designing and Analyzing Phase-Shifting Interferometry Algorithms with Characteristic Polynomials

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Abstract: Many of the recent advances in understanding of phase-shifting algorithms have yet to be incorporated into the software for commercial phase-shifting interferometers. A toolbox, for the open-source computer algebra system Maxima, simplifies analysis and design of arbitrary phase-shifting algorithms and can help incorporate them in commercial software.

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1. Phase-shifting interferometry

The confluence of optical interferometry and digital data processing in the 1970's [1] gave rise to the technique of phase shifting interferometry that is now widely used for measuring the form of precision surfaces. The digital detection with linear CCD array detectors and computer processing of interferometric irradiance maps (fringes) has enabled phase measurements with uncertainties close to 2π/1000 [2]. In phase-shifting interferometry, using a two-beam interferometer, a set of irradiance maps is measured and between each measurement the phase between the two beams is incremented (shifted) by a known, constant amount. The phase difference between the two beams at each detector pixel can then be calculated from the set of sampled intensity measurements. This discovery has resulted in a large body of work on improved phase-shifting algorithms (PSA) for phase measurements.

2. Phase measurements and characteristic polynomials – a brief summary

The description of PSAs by characteristic polynomials was introduced by Y. Surrel [3]. Most existing phase shifting algorithms can be described in a systematic way using characteristic polynomials. Characteristic polynomials can also be used to design new phase shifting algorithms with specific desired characteristics. A fundamental relationship linking the frequency response of a phase shifting algorithm with phase increment δ to the roots of its characteristic polynomial on the unit circle in the complex plane is:

$$P(e^{i\delta\nu}) = F_a(\nu) + i\,F_b(\nu) \quad . \tag{1}$$

$F_a(\nu)$ and $F_b(\nu)$ are the real and imaginary parts of the phase-shifting algorithm's spectral transfer function as a function of the normalized frequency ν. P is a complex polynomial called the characteristic polynomial of the algorithm [3]. For an algorithm that is sensitive at normalized frequency κ, i.e. $F_a(\kappa)$ and $F_b(\kappa)$ are not zero, the polynomial can be written in either factorized or expanded form

$$P(z) = \prod_{\substack{m=-N \\ m\neq\kappa}}^{N} (z - e^{i\nu\delta}) = \sum_{k=0}^{2N-1} c_k z^k \quad , \tag{2}$$

where N is the number of intensity samples $I_k = I(\phi_0 + k\delta)$, and the variable z is on the unit circle in the complex plane. The complex coefficients $c_k = a_k + ib_k$ of the expanded characteristic polynomial are needed to determine the phase:

$$\phi = arg\left(\sum_{k=0}^{2N-1} c_k I_k\right) = \arctan\left(\frac{\sum_{k=0}^{2N-1} b_k I_k}{\sum_{k=0}^{2N-1} a_k I_k}\right) \quad . \tag{3}$$

3. An example: the 7-sample, π/2 phase shift algorithm (de Groot [4], Phillion [5])

A well known phase shifting algorithm by de Groot [4], which is available in several commercial phase shifting interferometry software packages, is the 7-sample algorithm with 90° phase shift. It is defined by the hermitean characteristic polynomial

$$P(z) = i(z-1)(z+1)(z+i)^4 \quad . \tag{4}$$

The roots of the characteristic polynomial in Eq. 4 on the unit circle in the complex plane are shown in the

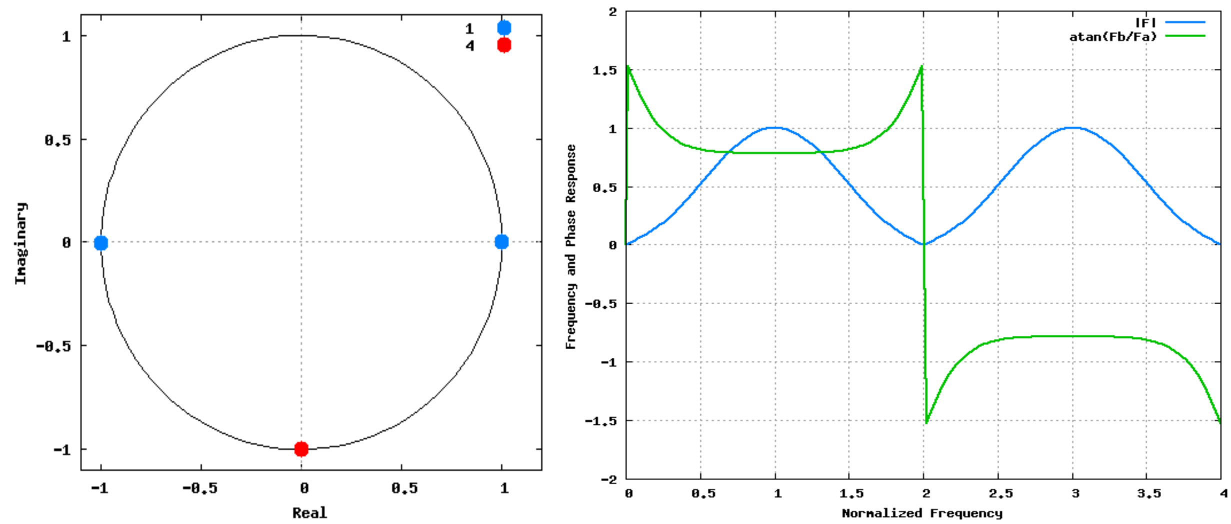

Figure 1: Characteristic diagram (left) and spectral response (right) of the 7-sample, 90° de Groot phase-shifting algorithm.

characteristic diagram in Fig. 1. This diagram shows location of the roots in the complex plane and their multiplicity. When the characteristic polynomial in Eq. 4 is expanded, and the a_k and b_k coefficients are calculated, the well know phase formula of the 7-sample, 90° phase step algorithm derived by de Groot [4] is obtained:

$$\phi = \arctan\left(\frac{-I_0+7I_2-7I_4+I_6}{-4I_1+8I_3-4I_5}\right) \quad . \tag{5}$$

The algorithm defined by the characteristic polynomial in Eq. 4 is identical to the P=7 algorithm for phase step 90° by Phillion [5]. A very closely related algorithm is obtained, when the averaging procedure introduced by Schmit and Creath [6] for deriving phase-shifting algorithms is extended to 7 samples. The resulting 7A algorithm has the same properties as the 7-sample algorithms in references [5,6], with the exception that it measures $-\phi$. For the evaluation of the performance of a PSA like the one shown in Fig. 2. on the right is very useful. The diagram shows the sensitivity of the 7-sample de Groot PSA and its phase response as a function of normalized frequency for the first and second alias. The phase response at frequency 1 is stationary and very flat. As a result, the algorithm is not sensitive to phase shift “calibration errors”, because phase step errors (sampling errors) have little effect on the calculated phase. The algorithm is insensitive to all even harmonics in the irradiance signal because the sensitivity is zero at the 2^{nd} harmonic. It is, however, not well suited for non-sinusoidal fringes with high harmonic content because it is sensitive at all odd harmonics.

4. Wavelength-shifting interferometry

The phase of a light wave with wavelength λ in an interferometer resonator of length L is $\phi = 2\pi L/\lambda + \phi_0$. A phase shift can therefore be effected either by changing the resonator length L or by changing the wavelength λ. Wavelength shifting interferometry is most useful when several overlapping fringe systems from resonators with different lengths are encountered. This is the case, for example, when the flatness error of a plane-parallel window is measured with a Fizeau interferometer. The beams reflected from both the front surface and the back surface will form interference fringes with the reference beam. In addition, there will be a third fringe system due to the interference of the beams from the front and back surfaces. Even more fringe systems may have to be considered when multiple reflections cannot be neglected. Fig. 2 shows a flatness error measurement of the “front” surface of a plane parallel window with a Fizeau interferometer using a 15-sample, 45° phase step PSA. The algorithm was designed to filter out the fringes due to the back surface reflection.

5. The toolbox

The phase-shifting algorithm (PSA) toolbox was developed for the open source computer algebra system Maxima [7], which is available for all common computing platforms. It consists of a set of Maxima functions for finding the roots of characteristic polynomials, calculating the coefficients of a PSA from the characteristic polynomial, and for calculating statistical properties of phase shifting algorithms. Functions are provided to plot characteristic diagrams and the frequency response of phase-shifting algorithms. The plotting functions rely on the worksheet interface to

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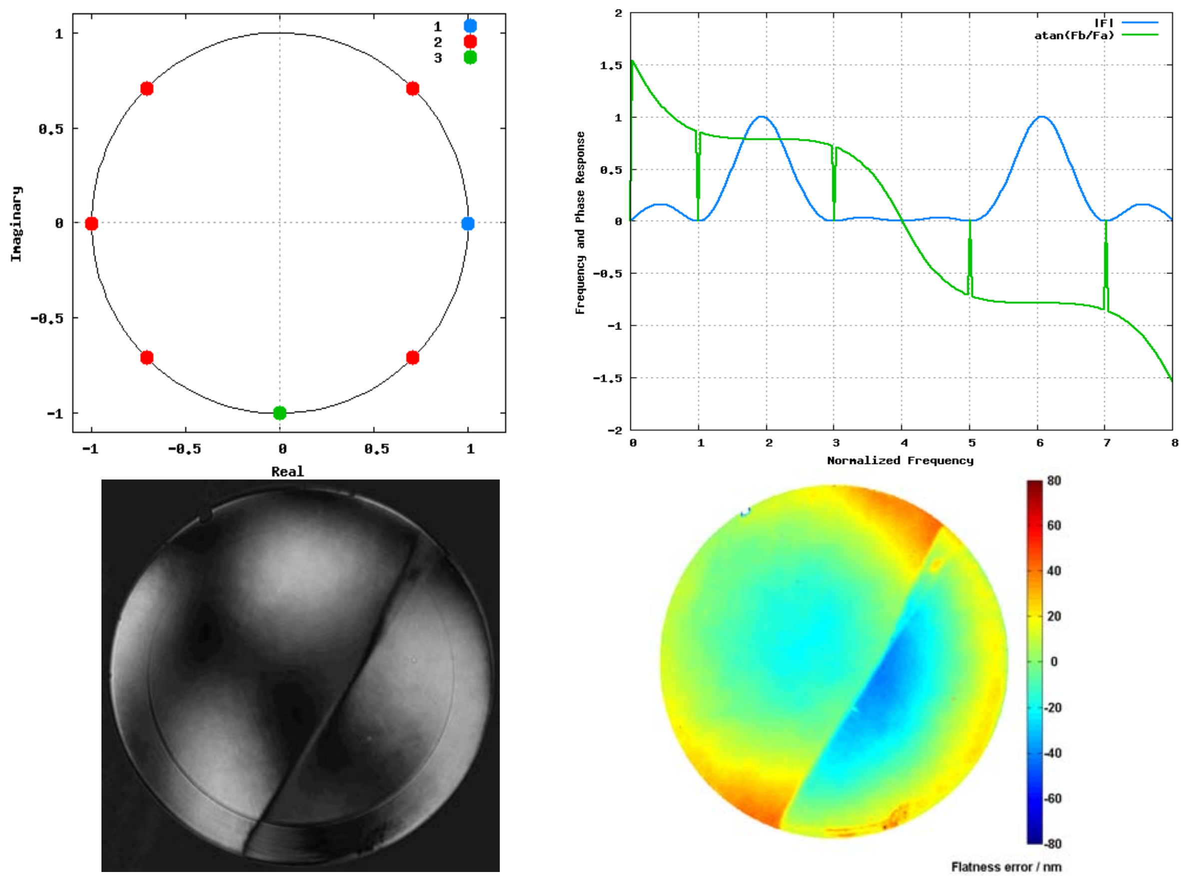

Figure 2: 15-sample, 45° PSA and its spectral response (top row), and fringes and flatness error of the front surface of a plane-parallel window, measured with this PSA (bottom row).

Maxima provided by wxmaxima [8]. We hope that the PSA toolbox will make it easier for suppliers of commercial phase-shifting interferometry software to incorporate new phase-shifting algorithms into their software and that the ease of analyzing and designing algorithms will, over time, result in more useful selections of algorithms available to users of these software packages. A better selection of PSA would also make commercial phase-shifting interferometry software more suitable for wavelength-shifting interferometry. The PSA toolbox may also be useful to educators who wish to illustrate the characteristics of different PSA to their students. The PSA toolbox is in the public domain and is available via e-mail from the author of this paper.

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[1] J. H. Bruning, D. R. Harriott, J. E. Gallagher, D. P. Rosenfeld, A. D. White, and D. J. Brangaccio, *Digital wavefront measuring interferometer for testing optical surfaces and lenses,* Appl. Opt. **13**, 2693-2703 (1974)
[2] J. Schwider, R. Burow, K.-E. Elssner, J. Grzanna, R. Spolaszyk, and K. Merkel, *Digital wavefront measuring interferometry: some systematic error sources,* Appl. Opt. 22, 3421-3432 (1983)
[3] Y. Surrel, *Fringe Analysis,* in P. K. Rastogi (Ed.), Photomechanics, Topics in Appl. Phys. **77**, 52-102 (2000)
[4] P. de Groot, *Derivation of algorithms for phase-shifting interferometry using the concept of a data sampling window,* Appl. Opt. **34**, 4723-4730 (1995)
[5] D. W. Phillion, *General methods for generating phase-shifting interferometry algorithms,* Appl. Opt. **36**, 8098-8115 (1997)
[6] J. Schmit and K. Creath, *Extended averaging technique for derivation of error-compensating algorithms in phase-shifting interferometry,* Appl. Opt. **34**, 3810-3819 (1995)
[7] http://maxima.sourceforge.net
[8] http://wxmaxima.sourceforge.net