Waveform metrology and a quantitative study of regularized deconvolution

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***Abstract*—We present methodology and preliminary results of a Monte-Carlo simulation to perform a quantified analysis of regularized deconvolution in the context of full waveform metrology. We analyze the behavior of different regularized inversion methods with varying dimensionless parameters that serve as indicators of the problem difficulty, including: the ratio of input pulse duration to that of the system impulse response, signal to noise ratio, pulse shape, and/or the ratio of high frequency roll-off of input to system response. We characterize the waveform estimates in terms of pulse parameters and total error, comparing different Tikhonov deconvolution algorithms as well as commonly used heuristic approaches. We present a quantitative comparison of the relative merits of the different procedures, and compare the numerical performance with asymptotic analyses.**

I. Introduction

Waveforms constitute the currency of modern, data-intensive communications.[1] As such, and as data-rates and waveform complexity increase, the problem of waveform measurement is receiving increased interest and relevance.

A *waveform* is defined as, “A representation of a signal, for example, a graph, plot, oscilloscope presentation, discrete time series, equation, or table of values”, where a *signal* is, “[a] physical phenomenon that is a function of time” [1]. Predominantly engineers have employed *waveform parameter metrology* to characterize signals of interest. Under this paradigm pulse parameters must first be defined; see for example the IEEE standard [1]. Waveform parameter metrology then consists of techniques for estimating the central values of these parameters and their associated uncertainties from measured waveforms. Parameter estimation techniques differ in the ways they account for the finite response time (“finite bandwidth”) of the measurement device. One example is the root-sum-of-squares (RSS) estimation of the transition duration (see [2]).

The characterization of a waveform by a collection of waveform parameters entails a collapse of information; subtleties of waveform shape are necessarily lost. For example, there are infinitely many step-like functions with any given transition duration. When these steps are measured as part of an eye-mask test, some of these functions may pass while others fail. Increasingly, the complexity of waveform aberrations that engineers must control requires more detailed measurement of underlying waveform shapes. The loss of information entailed by waveform parameter metrology is no longer acceptable and a new paradigm is required.

At NIST we have responded to this need by developing a new waveform measurement capability referred to as *full waveform metrology*. This paradigm has as its goal an estimate of the central value and associated uncertainty of the entire waveform as a function of time. To do so requires that correlations between uncertainties in the time-domain be accounted for, as these correlations, although small in magnitude in the time domain, can add in phase to produce strong signatures in the frequency domain (for example, see [3], [4], and [5]). In principle the full waveform metrology paradigm is desirable as a universal referent; any waveform parameter analysis may be derived from a full waveform measurement combined with uncertainty propagation rules, as in [5] and [6]. However, the increased measurement and computational requirements of the full waveform paradigm incur a significantly higher burden on waveform measurement analysis.

One of the more delicate analysis procedures is the deconvolution of the measurement system response from the measured waveform. Deconvolution is a common problem arising in signal processing, image analysis, as well as several biomedical applications. As is well known, deconvolution is ill-posed and requires regularization to control noise amplification. The literature on regularization is large. It includes discussion of diverse regularized inversion frameworks as well as competing strategies for selecting the parameters appearing in these frameworks. Furthermore, asymptotic analysis exists where limits are taken with respect to large sample size, small noise, or combination of both. From the perspective of quantitative analysis of realistic, non-asymptotic measurement scenarios, much of this literature is inadequate. Studies between competing regularization frameworks or parameter selection strategies are rarely attempted. This fact has been noted in, for example, [7] and [8]. In the few cases where comparisons are attempted, the analysis is often not quantitative or the sample sizes are insufficient to make statistical claims about relevant strengths and weaknesses. [2] Concerning the theoretical asymptotic analysis, much of this is of limited

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[2]A notable exception is the recent quantitative comparison of parameter selection strategies in the context of optical topography for biomedical applications [9]. The quantitative, simulation-based approach taken in that work is similar to ours albeit for a different application entailing, in turn, different constraints and accuracy goals.

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quantitative value. In our laboratory practice the signal to noise ratio (SNR) is between 10^2 and 10^5, and the dimensionality of the problem is typically between 500 and 5000. Although we can use the asymptotic arguments as heuristics, they are generally not suitable for reliable quantitative guidance in our measurement context.

In this work we present preliminary results of a simulation-based study designed for quantitative analysis and comparison of regularized deconvolution strategies for full waveform metrology. For compatibility with waveform parameter metrology, we present quantitative analysis of the effects of deconvolution on waveform parameters. The design variables of our study include dimensionless ratios such as the ratio of input pulse duration to system impulse response duration, signal to noise ratio, as well as variation in pulse shapes. These ratios are varied systematically over a range of realistic values. We characterize the estimates of the deconvolved input waveform in terms of the duration and amplitude, and in terms of a full waveform error norm. We compare estimates obtained by different analytical procedures, including use of the as-measured waveform, the root sum of squares "rule of thumb", and by Tikhonov deconvolution with two different regularization parameter selectors. Finally, we draw conclusions regarding the relative merits of the different procedures.

II. Waveform Metrology

A. Convolution and noise model

We investigate the problem of using a calibrated system to measure the impulse response function of another measurement device under test. We model *ideal* waveform measurement as a convolution,

$$y_*(t) = \int_0^\infty a(s)x_*(t-s)\mathrm{d}s.$$

Here a is the system response function of the calibrated device, and x_* and y_* are, respectively, the input waveform and ideal measurement for device under test. The asterisk subscripts on the waveforms denote ideal quantities. In practice we measure waveforms at N discrete times, for example $\mathbf{y}_* := (y_*(t_0), \ldots, y_*(t_{N-1}))^T$. In the present analysis we assume equi-spaced times, $t_n := t_0 + n\Delta t$, and negligible jitter.

For the numerical experiments below we replace the *continuous, linear* convolution above with a *discrete, periodic* approximation. More precisely, the integral kernel is replaced by a matrix operator

$$\mathbf{y}_* = \mathbf{A}\mathbf{x}_*, \tag{1}$$

where the matrix $\mathbf{A}$ is circulant. In our context, the convolution kernel $a(t)$ is an oscilloscope response function and is designed to approximate a delta function subject to engineering constraints. As such, $a(t) = 0$ for $t < 0$, and then exhibits a primary lobe followed by decaying oscillations. In addition, the ideal waveforms $x_*(t)$ are impulse-like as well. Under these conditions the circulant (periodized) measurement model is acceptable. Modifications to accommodate step-like waveforms are discussed in [4].

In practice, x_* is unknown and the measured quantity is contaminated by noise:

$$\mathbf{y} = \mathbf{A}\mathbf{x}_* + \sigma\mathbf{n}. \tag{2}$$

We model the measurement noise vector $\sigma\mathbf{n}$ as an multivariate Gaussian random variable with mean zero and, in this work, constant diagonal covariance $\sigma^2\mathbf{I}$.

B. Waveform Parameters and Waveforms

In this work we study deconvolution effects on estimation of the *pulse duration at half maximum* or, more simply, the *pulse duration*. This parameter is estimated from a measured waveform as follows. The half-max value is defined as half of the maximum of the sampled waveform values. (Alternative estimates of the waveform amplitude were considered; the differences were negligible for our purpose.) By use of sampled values of the waveform to generate local approximations, the two times at which the waveform is estimated to take on the half-max values are determined [5]. The difference of these times defines our estimate of the pulse duration at half maximum. We abbreviate the pulse duration by $\tau(\lambda) = \tau(\mathbf{x}(\lambda))$for a given waveform $\mathbf{x}(\lambda)$ depending on a parameter λ. Intuitively, deconvolution increases in difficulty as the "speed" of the input waveform becomes comparable to that of the measurement device. The ratio

$$\mathcal{T} = \tau(\mathbf{x}_*)/\tau(\mathbf{a}) \tag{3}$$

serves as one of the design variables in our study.

We model the impulse-like waveforms a and x_* using n-pole low-pass filters, e.g., Butterworth, Chebyshev, Bessel-Thompson. Descriptions of these filters can be found in standard textbooks, see for example [10]. The number of poles n describes the high-frequency roll-off of the magnitudes of these transfer functions in the Fourier domain and will serve as one of the design parameters for the Monte Carlo study. By convention, we scale the amplitudes of the impulse response functions to have unit total integral, $\int a(t)\mathrm{d}t = 1$. This corresponds to their role as approximate delta functions with high-frequency attenuation.

For a nominal measured waveform $\mathbf{y}_* = \mathbf{A}\mathbf{x}_*$, we define the signal-to-noise ratio in a mean-square sense

$$\mathcal{S} := \frac{\|\mathbf{y}_*\|^2}{E\Big(\|\sigma\mathbf{n}\|^2\Big)} = \frac{\|\mathbf{y}_*\|^2}{N\sigma^2}, \tag{4}$$

where the norm is the usual Euclidean norm in $\mathbf{R}^N$. Note that this definition, in conjunction with our scaling convention for $a(t)$, implies that $\mathcal{S}$ would scale with $\tau(\mathbf{x}_*)$ if the amplitude of x_* were fixed. To separate these dependencies we scale x_* so that $\|\mathbf{y}_*\|^2 = 1$, leading to

$$\mathcal{S} = \frac{1}{N\sigma^2}. \tag{5}$$

The ratio $\mathcal{S}$ is an indicator of the potential for instability of inversion and is another design variable in this work.

III. Deconvolution and Regularization

A. Parametric inversion

If the sole concern is for a characterization of the input waveform in terms of its pulse duration, a "rule of thumb" inversion exists. Namely, given estimates of $\tau(y)$ and $\tau(a)$ (presumably the former is measured by the experiment and the later is known as a result of a calibration measurement) the RSS estimate of $\tau(x_*)$ is defined by

$$\tau_{RSS} := \sqrt{\tau(y)^2 - \tau(a)^2}. \quad (6)$$

Equation (6) is exact, assuming that $x_*(t)$ and $a(t)$ are Gaussian waveforms, i.e., of the form $\exp\big(-(t-\mu)^2/\alpha^2)\big)$ for some μ and α. To our knowledge, the example of two Gaussians is the *only case* for which the RSS estimation of τ is exact. For more general waveform shapes (6) serves as an estimate, and as pointed out in [2] it can be unacceptably poor. As RSS estimation and, more generally, RSS-like "rules-of-thumb" are common in waveform applications (cf. [11]), we present it as part of our analysis. As is clear from [2] and amplified in the analysis below, the RSS approach can be incorrect even in a qualitative sense. We advise against using this type of analysis absent detailed experimentation to determine its suitability. An alternative, RSS-like approach is discussed in [12].

B. Tikhonov equations

We measure $\mathbf{y}$ and wish to invert (2) to estimate $\mathbf{x}$ in the presence of noise. As $\mathbf{A}$ is circulant, it is diagonalized by the discrete Fourier transform matrix (DFT) to yield the complex diagonal matrix $\hat{\mathbf{A}} = \mathrm{diag}(\hat{\mathrm{a}}_1, \ldots, \hat{\mathrm{a}}_\mathrm{N})$. In this basis (discrete complex exponentials), the system (2) is equivalent to

$$\hat{\mathbf{y}} = \hat{\mathbf{A}}\hat{\mathbf{x}}_* + \sigma\mathbf{n}. \quad (7)$$

If for all entries $|a_j| \neq 0$, then $\hat{\mathbf{A}}$ is invertible (as is $\mathbf{A}$). More generally, one may always define the least-squares solution. For the problems we consider, this least-squares solution provides an unacceptable estimate for $\mathbf{x}_*$, as the operator $\mathbf{A}$ is ill-conditioned ($|\hat{a}_j|$ is small for some j) causing noise amplification to dominate the least-squares inversion. This is common for inverse problems and the common solution is to introduce some form of regularization into the inversion.

Among the various regularization frameworks we consider that attributed to Tikhonov. In brief, given a penalty operator $\mathbf{L}$ and a scalar value λ, the least-squares normal equations are replaced by a regularized counterpart:

$$\mathbf{x}(\lambda) := \Big(\mathbf{A}^*\mathbf{A} + \lambda^2\mathbf{L}^*\mathbf{L}\Big)^{-1}\mathbf{A}^*\mathbf{y}. \quad (8)$$

For $\lambda = 0$ one observes that $\mathbf{x}(\lambda)$ is the least-squares solution. There are several ways of deriving (8) that highlight different interpretations, see [8].

In this work we consider regularized inversion of the Tikhonov system assuming a smoothness penalty, $\mathbf{L} = \mathbf{D}_2$, where $\mathbf{D}_2$ is the periodized second-difference operator. As with the convolution kernel, periodization allows for simultaneous diagonalization of $\mathbf{A}$ and $\mathbf{L}$ by the suitably dimensioned DFT matrix. We implement the diagonalization by use of a fast Fourier transform summation. We also investigated results for $\mathbf{L} = \mathbf{I}$, but space does not permit discussion of them here.

C. Regularization parameter selection

For given $\mathbf{A}$ and penalty function $\mathbf{L}$, the Tikhonov equations contain a free regularization parameter λ. The development of *selectors*–strategies for selecting λ–has resulted in a vast literature. Periodically, reviews are attempted, see for example, [13] and the popular Matlab toolbox [14]. However, we find that the quantitative assessment of the various schemes in the literature is insufficient for our purposes. The ample proposed algorithms with meager numerical support serves as a driver for the present Monte Carlo study.

Selectors can be broadly categorized as methods that require prior knowledge of the noise level or its point-by-point expectation, and methods that do not. As representatives respectively, we include the *discrepancy principle* (D) and the *L-curve*(L) method in our simulations. Definitions and details of these two methods may be found in several texts, for example [7] and [8]. The selector method will be indicated by subscripts, e.g. λ_D and λ_L.

As a benchmark against which to evaluate other methods, we define the optimal λ_{opt} as the λ that minimizes the normalized waveform error $e(\lambda) := \|\mathbf{x}(\lambda) - \mathbf{x}_*\|/\|\mathbf{x}_*\|$. For a fixed penalty $\mathbf{L}$ this is the minimum full waveform error that can be achieved with Tikhonov regularization.

IV. Numerical Study

We simulated several different combinations of n-pole response functions for $a(t)$ and $x_*(t)$. Here we report on the use of a 4^{th} order Butterworth impulse response with duration of 5.28 ps for a, and a 2^{nd} order Bessel-Thompson impulse response for $x_*(t)$. The waveforms are sampled at $N = 512$ points at intervals of $\Delta t = 5\text{ns}/4096$. Note that at this sampling rate, all waveforms are sampled at a rate greater than or equal to approximately 8.7 samples per characteristic cycle.[3] At this rate of oversampling we assume that convolution and deconvolution discretization effects are negligible.

The dimensionless ratios $\mathcal{T}$ and $\mathcal{S}$ ((3) and (5)) are the design variables for the Monte Carlo analysis. Informally, inversion difficulty scales inversely with these parameters; measurements involving larger values of $\mathcal{S}$ and $\mathcal{T}$ should be inverted more easily. The values we use are

$\log_{10}(\mathcal{S})$	2	3	4	5
$\mathcal{T}$	1	2	3	5

For each combination of $\mathcal{S}$ and $\mathcal{T}$ we generate 1000 noisy waveforms according to (2). To analyze the quantitative errors by use of, for example, the discrepancy principle as a selection

[3] Informally, we define the characteristic cycle as twice τ and compute $2 \cdot 5.28 \cdot 4096/5000 \approx 8.7$.

strategy for λ, we examine the error metrics

$$\Delta e_D := \overline{\left(\frac{\|\mathbf{x}_D - \mathbf{x}_*\|}{\|\mathbf{x}_*\|}\right)} \cdot 100$$
$$\Delta \tau_D := \frac{\overline{\tau_D} - \tau_*}{\tau_*} \cdot 100 \tag{9}$$

The overline refers to averaging over all 1000 noise instances. A similar set of metrics is defined for the parameter selection algorithms λ_{opt} and λ_L. In addition to mean values, we report standard deviations as we are concerned with the stability of the numerical inversion.

V. Results

A. Regularization parameter selection algorithms

The mean and standard deviation of the regularization parameters $\overline{\lambda_L}$, $\overline{\lambda_D}$, and $\overline{\lambda_{opt}}$ are tabulated in Table I. For fixed $\mathcal{T}$, we find that the mean values of λ for all three selectors scale with σ. This could be anticipated from statistically based derivations of the Tikhonov equations, for example see [8]. More surprising is that for fixed $\mathcal{S}$, generally the value of λ determined for the three different methods *increases* with increasing $\mathcal{T}$. Originally our qualitative expectation was that λ would correlate with problem difficulty. If so, then relative to a fixed characteristic time of the measurement device $\tau(a)$, a slower input waveform is presumably more easily inverted and λ would decrease with increasing $\mathcal{T}$. Clearly this idea must be revisited in light of our results. Note below that the waveform error is consistent with expectations, e.g., Δe decreases as a function of $\mathcal{S}$ and $\mathcal{T}$.

Quantitatively, we find that for low signal to noise ($\mathcal{S} = 10^2$) the mean values of λ_L and λ_D agree to within about 20 % but are about a factor of 3 larger than λ_{opt}. Thus for noisy signals there is little difference between selectors, and both are over-damped with respect to the optimal inversion. As the signal increases relative to noise ($\mathcal{S} = 10^5$) we find that the situation changes: $\overline{\lambda_L} \approx 2\overline{\lambda_{opt}}$ and $\overline{\lambda_D} \approx 4\overline{\lambda_{opt}}$. Comparing the differences between λ_L and λ_D relative to their variances (see Table I), we conclude that in this high-signal case the two selectors are statistically different. As before, both are over-damped with respect to the optimal selector. Over-damping of the discrepancy principle has been noted previously, see for example [7] and [8]. However, under-damping with use of the L-curve selector has been observed in numerical experiments (cf. [7]) and is predicted by analysis. Hence, the over-damped nature of the L-curve selector observed here is unexpected and perhaps draws attention to shortcomings of existing analysis.

To elaborate, a theorem of Bakushinskii states that a selector that does not explicitly depend on σ, must necessarily diverge in the limit $\sigma \to 0$ [15]. The argument is indirect and couched in the language of Banach spaces–infinite-dimensional, normed linear spaces. The L-curve does not depend explicitly on σ, and therefore some have argued that it follows from Bakushinskii's result that this selector diverges [16]. This has been the source of some discussion in the literature, engaging both L-curve opponents and advocates. In particular, the analysis of [17] suggests that the signature of the divergence predicted by Bakushinskii is indicated by extreme *under-damping* in the limit $\sigma \to 0$. This stands in stark contrast to what we observe in our study: λ_L is slightly over-damped but otherwise near optimal, and λ_D (which does explicitly depend on $\mathcal{S}$) performs worse. The analysis presented in [17] demonstrates the divergence for "smooth" problems with a norm penalty, $\mathbf{L} = \mathbf{I}$. Notwithstanding the ambiguity as to the precise definition of smooth, our numerical results are based on the second difference penalty, $\mathbf{L} = \mathbf{D}_2$, which may account for the discrepancy. In summary, existing asymptotic analyses require refinement to be useful as *a priori* predictors of performance.

Closer inspection of Table I suggests the optimal λ follows a scaling law. Performing linear least squares analysis in log-log space, we find that λ_{opt} can be well approximated by

$$\tilde{\lambda}(\mathcal{T}, \sigma) := 3.8640\mathcal{T}^{1.4883}\sigma^{1.001}. \tag{10}$$

The relative residual errors all lie in the interval $[-1.8\%, 2.5\%]$. Additional simulations reveal that the scaling law (10) persists when the filter order of a and x_* is changed, although the values of the coefficients and exponents change, and the range of $\mathcal{T}$ and $\mathcal{S}$ over which the law holds changes as well. This scaling law is unexpected and suggests that a deeper analysis should be performed.

B. Waveform parameter estimation.

Errors in the estimated pulse durations are shown in Table II. One point that immediately stands out is the poor performance of the RSS correction. Conventional wisdom (suggested from analysis of Gaussian waveforms) states that finite bandwidth effects of a measurement system will cause the measured waveform to be "slower" than the true input signal. In Table II we show the error that would ensue from using the observed $\tau(y)$ as an estimate for underlying $\tau(x_*)$. When the time scale of x_* is directly comparable to the measurement system $\mathcal{T} = 1$, in fact we find that $\tau(y)$ is approximately 30 % higher than $\tau(x_*)$, which qualitatively corresponds to this conventional wisdom. Quantitatively, we observe that the RSS correction consistently overestimates the effects of the convolution and thereby over-corrects. The situation changes dramatically as $\mathcal{T}$ increases. For all values of $\mathcal{T} \geq 2$ we find that the measured waveform is *faster* than the true input signal. In this case the RSS correction, by further subtracting a value from $\tau(y)$, increases the error. Indeed, our results indicate that doing nothing at all can be an improvement over using the RSS heuristic. Of course this statement depends strongly on the waveform shape, as the composition law for Gaussians is a well-known fact. The more accurate statement, and one which supports the full waveform metrology measurement paradigm, is that the suitability of applying heuristic corrections must be checked on a case-by-case basis.

Turning to the estimated pulse durations of the deconvolved waveforms, we find that deconvolution with regularization uniformly improves the estimation of τ over the measured waveform. This is to be expected. Nevertheless it is reassuring to find that the instability of the underlying deconvolution

TABLE I

MEAN VALUE OF λ (STANDARD DEVIATION). THE OPTIMAL λ IS THE VALUE MINIMIZING $\|\mathbf{x}_\lambda - \mathbf{x}_*\|^2$. THIS IS COMPARED TO THE L-CURVE AND MOROZOV DISCREPANCY SELECTION ALGORITHMS (SEE TEXT).

$\mathcal{T}$ \ $\log_{10}(\mathcal{S})$		2	3	4	5
1	Optimal	0.396 (0.060)	0.121 (0.016)	0.038 (0.005)	0.012 (0.001)
	L-Curve	1.312 (0.130)	0.341 (0.032)	0.086 (0.007)	0.022 (0.002)
	Discrepancy	1.464 (0.243)	0.450 (0.062)	0.139 (0.017)	0.043 (0.004)
2	Optimal	1.092 (0.171)	0.342 (0.050)	0.109 (0.014)	0.035 (0.004)
	L-Curve	2.956 (0.266)	0.879 (0.078)	0.242 (0.019)	0.065 (0.004)
	Discrepancy	3.517 (0.637)	1.191 (0.179)	0.386 (0.051)	0.123 (0.015)
3	Optimal	1.968 (0.312)	0.626 (0.093)	0.198 (0.026)	0.063 (0.008)
	L-Curve	4.972 (0.467)	1.553 (0.131)	0.443 (0.037)	0.123 (0.008)
	Discrepancy	5.832 (1.152)	2.073 (0.335)	0.694 (0.097)	0.224 (0.029)
5	Optimal	4.216 (0.667)	1.326 (0.208)	0.422 (0.061)	0.135 (0.017)
	L-Curve	10.915 (1.430)	3.192 (0.274)	0.959 (0.080)	0.273 (0.020)
	Discrepancy	11.067 (2.597)	4.069 (0.730)	1.430 (0.217)	0.473 (0.065)

problem is controlled by regularization. It is notable that we do not find significant differences between the two selectors. This observation somewhat belies the immense literature devoted to the study of such algorithms.

C. Full waveform error.

In Table III we show the percent residual error of the entire waveform. The full waveform error of the measured waveform decreases as the input signal slows with respect to the measurement system. However, this decrease is very slow. Tikhonov regularized deconvolution improves on this and the effectiveness of the correction depends on the signal-to-noise ratio. We note that Δe_L is consistently smaller than Δe_D. However, it is more instructive to observe that the differences between using the L-curve and the Morozov Discrepancy Principle to select λ are not terribly striking.

VI. CONCLUSION

We describe a numerical study designed to quantify and compare the accuracy of competing algorithms for waveform parameter estimation and deconvolution. We find that the commonly-used RSS heuristic for estimating pulse duration can be qualitatively incorrect, to say nothing of its quantitative performance. Our results indicate that the instability associated with deconvolution can be reliably controlled by regularization. Even more, perhaps undermining the large literature devoted to the problem, parameter selection strategy does not radically alter the results. Finally, we observe that the optimal parameter follows a scaling law with very low residuals. We emphasize that these results require more investigation to include a greater range of waveform shapes. Even more subtle is to introduce inconsistency in the measurement model, i.e., the convolution kernel is itself a perturbation of a nominal value. We will report on these investigations in the future.

REFERENCES

[1] IEEE Instrumentation and Measurement Society, "IEEE Standards on transitions, pulses, and related waveforms," Institute Electrical and Electronics Engineers, Tech. Rep., 2003.

[2] C. Mittermayer and A. Steininger, "On the determination of dynamic errors for rise time measurement with an oscilloscope," *IEEE Trans. on Instruments and Measurement*, vol. 48, no. 6, pp. 1103–1107, Dec. 1999.

[3] D. F. Williams, A. Lewandowski, T. S. Clement, C. M. Wang, P. D. Hale, J. M. Morgan, D. A. Keenan, and A. Dienstfrey,"Covariance-based uncertainty analysis of the NIST electro-optic sampling system," *IEEE Trans. Microwave Theory Tech.*, vol. 54, no. 3, pp. 481-491, Jan. 2006.

[4] P. D. Hale, A. Dienstfrey, J. C. M. Wang, D. F. Williams, A. Lewandowski, D. A. Keenan, and T. Clement, "Traceable waveform calibration with a covariance-based uncertainty analysis," *IEEE Trans. on Instruments and Measurement*, vol. 58, no. 10, pp. 3554–3568, Oct. 2009.

[5] P. D. Hale and C. M. J. Wang, "Calculation of pulse parameters and propagation of uncertainty," *IEEE Trans. on Instruments and Measurement*, vol. 58, no. 3, pp. 639–648, Mar. 2009.

[6] M. Cox, M. Dainton, P. Harris, and N. Ridler, "The evaluation of uncertainties in the analysis of calibration data," in *Instrumentation and Measurement Technology Conference, 1999. IMTC/99. Proceedings of the 16th IEEE*, vol. 2, Venice, Italy, May 24–May 29 1999, pp. 1093–1098.

[7] P. C. Hansen, *Rank-Deficient and Discrete Ill-Posed Problems: Numerical Aspects of Linear Inversion*, ser. Monographs on Mathematical Modeling and Computation. Philadelphia, PA: Society for Industrial and Applied Mathematics, 1998.

[8] C. R. Vogel, *Computational Methods for Inverse Problems*, ser. Frontiers in Applied Mathematics. Philadelphia, PA: Society for Industrial and Applied Mathematics, 2002.

[9] T. Correia, A. Gibson, M. Schweiger, and J. Hebden, "Selection of regularization parameter for optical topography," *Journal of Biomedical Optics*, vol. 14, no. 034044, p. 11 pages, May/June 2009.

[10] W.-K. Chen, *Passive and Active Filters*. John Wiley and Sons, 1986.

[11] H. Johnson and M. Graham, *High-speed Digital Design: A Handbook of Black Magic*. Prentice Hall, 1993.

[12] N. G. Paulter and D. R. Larson, "Pulse parameter uncertainty analysis," *Metrologia*, vol. 39, pp. 143–155, 2002.

[13] A. Neumaier, "Solving ill-conditioned and singular linear systems: A tutorial on regularization," *SIAM Review*, vol. 40, no. 3, pp. 636–666, 1998.

[14] P. C. Hansen, "Regularization tools version 4.0 for Matlab 7.3," *Numerical Algorithms*, vol. 46, pp. 189–194, 2007.

[15] A. B. Bakushinskii, "Remarks on choosing a regularization parameter using the quasi-optimality and ratio criterion," *USSR Computational Mathematics and Mathematical Physics*, vol. 24, no. 4, pp. 181–182, 1984.

[16] C. R. Vogel, "Non-convergence of the L-curve regularization parameter selection method," *Inverse Problems*, vol. 12, pp. 535–547, 1996.

[17] M. Hanke, "Limitations of the L-curve method in ill-posed problems," *BIT*, vol. 36, no. 2, pp. 287–301, 1996.

TABLE II

PERCENT MEAN RELATIVE ERROR (STANDARD DEVIATION) IN ESTIMATED PULSE DURATION, $\Delta\tau$ (9). HEURISTIC PARAMETER ESTIMATION TECHNIQUES ARE COMPARED TO DECONVOLUTION WITH TIKHONOV REGULARIZATION USING DIFFERENT PARAMETER SELECTION ALGORITHMS.

$\mathcal{T}$ \ $\log_{10}(\mathcal{S})$		2	3	4	5
1	Measured	30.25 (1.70)	30.43 (0.57)	30.43 (0.18)	30.43 (0.06)
	RSS	-16.57 (2.66)	-16.27 (0.90)	-16.26 (0.28)	-16.26 (0.09)
	Optimal	5.94 (2.43)	-1.68 (1.72)	-5.49 (1.31)	-4.41 (0.77)
	L-Curve	18.83 (1.43)	4.74 (1.02)	-3.31 (0.86)	-5.44 (0.59)
	Discrepancy	20.20 (3.07)	7.07 (1.70)	-0.97 (0.86)	-5.40 (0.41)
2	Measured	-7.29 (1.65)	-6.77 (0.56)	-6.69 (0.18)	-6.69 (0.06)
	RSS	-22.01 (1.96)	-21.38 (0.67)	-21.29 (0.22)	-21.28 (0.07)
	Optimal	-4.55 (1.60)	-3.32 (0.64)	-3.24 (0.67)	-1.63 (0.55)
	L-Curve	-3.38 (1.14)	-4.33 (0.54)	-3.33 (0.26)	-2.73 (0.33)
	Discrepancy	-2.46 (1.39)	-4.59 (0.52)	-3.33 (0.22)	-3.35 (0.20)
3	Measured	-8.72 (2.36)	-7.84 (0.83)	-7.68 (0.28)	-7.67 (0.09)
	RSS	-15.07 (2.53)	-14.13 (0.89)	-13.95 (0.30)	-13.94 (0.10)
	Optimal	-3.16 (1.15)	-2.34 (0.86)	-0.74 (0.47)	-0.56 (0.30)
	L-Curve	-2.74 (0.99)	-3.20 (0.40)	-1.79 (0.34)	-0.52 (0.19)
	Discrepancy	-2.49 (1.01)	-3.17 (0.36)	-2.47 (0.37)	-0.81 (0.19)
5	Measured	-4.84 (2.39)	-3.31 (0.89)	-2.92 (0.31)	-2.83 (0.10)
	RSS	-6.98 (2.45)	-5.41 (0.91)	-5.01 (0.31)	-4.92 (0.11)
	Optimal	-1.13 (1.25)	-0.65 (0.54)	-0.61 (0.36)	-0.20 (0.25)
	L-Curve	-1.96 (0.97)	-0.81 (0.42)	-0.64 (0.20)	-0.48 (0.16)
	Discrepancy	-1.84 (1.02)	-1.03 (0.45)	-0.61 (0.17)	-0.62 (0.12)

TABLE III

PERCENT MEAN RELATIVE ERROR (STANDARD DEVIATION) IN FULL WAVEFORM ESTIMATION Δe (9). ERROR IN USING THE MEASURED WAVEFORM (NO DECONVOLUTION), IS COMPARED TO THE MINIMAL ERROR POSSIBLE AND DIFFERENT PARAMETER SELECTION ALGORITHMS.

$\mathcal{T}$ \ $\log_{10}(\mathcal{S})$		2	3	4	5
1	Measured	29.17 (0.39)	27.81 (0.13)	27.67 (0.04)	27.66 (0.01)
	Optimal	20.01 (0.76)	15.05 (0.55)	11.39 (0.39)	8.62 (0.28)
	L-Curve	23.33 (0.44)	17.13 (0.32)	12.50 (0.24)	9.12 (0.18)
	Discrepancy	23.80 (1.06)	18.07 (0.70)	13.71 (0.46)	10.38 (0.30)
2	Measured	15.37 (0.38)	12.21 (0.13)	11.85 (0.04)	11.81 (0.01)
	Optimal	10.17 (0.42)	7.28 (0.29)	5.43 (0.20)	4.09 (0.14)
	L-Curve	11.58 (0.29)	8.19 (0.16)	5.95 (0.12)	4.35 (0.09)
	Discrepancy	12.06 (0.68)	8.72 (0.40)	6.54 (0.26)	4.93 (0.17)
3	Measured	12.06 (0.35)	7.53 (0.13)	6.91 (0.04)	6.85 (0.01)
	Optimal	7.02 (0.32)	4.74 (0.20)	3.48 (0.14)	2.61 (0.09)
	L-Curve	7.94 (0.27)	5.31 (0.11)	3.82 (0.08)	2.79 (0.06)
	Discrepancy	8.30 (0.56)	5.65 (0.29)	4.18 (0.18)	3.14 (0.12)
5	Measured	10.51 (0.32)	4.59 (0.12)	3.49 (0.04)	3.36 (0.01)
	Optimal	4.75 (0.26)	2.83 (0.12)	1.99 (0.08)	1.47 (0.06)
	L-Curve	5.56 (0.34)	3.16 (0.08)	2.19 (0.05)	1.59 (0.03)
	Discrepancy	5.64 (0.52)	3.36 (0.20)	2.38 (0.11)	1.77 (0.07)