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Rectangular distribution whose end-points are not exactly known: curvilinear trapezoidal distribution

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Abstract

The state of knowledge concerning a quantity about which scant specific information is available is often represented by a rectangular probability distribution on some interval (Z_1, Z_2) specified by scientific judgment. Often, the end-points Z_1 and Z_2 are not exactly known. If the state of knowledge about the left end-point Z_1 can be represented by a rectangular distribution on the interval (a, c) and the state of knowledge about the right end-point Z_2 can be represented by a rectangular distribution on the interval (d, b), where $a \le c \le d \le b$, then the resulting probability distribution looks like a trapezoid whose sloping sides are curved. We can refer to such a probability distribution as curvilinear trapezoidal distribution. Depending on the values of a, c, d, and b, the curvilinear trapezoidal distribution may be asymmetric. We describe the probability density function (pdf) and the moments of a curvilinear trapezoidal distribution which arises from inexactly known end-points of a rectangular distribution. In particular, we give compact algebraic expressions for the expected value and the variance. Then we discuss how random numbers from such a distribution may be generated. We compare the curvilinear trapezoidal distribution which arises from inexact end-points with the corresponding trapezoidal distribution whose sloping sides are straight. We also compare the curvilinear trapezoidal distribution which arises from inexactly known end-points with the curvilinear trapezoidal distribution which arises when the mid-point of a rectangular distribution is known (fixed), the half-width is not exactly known, and the state of knowledge about the half-width may be represented by a rectangular distribution.

Keywords: Rectangular distribution, Trapezoidal distribution, Uncertainty in measurement

1. Introduction

Rectangular distributions are often used in metrology to represent the state of knowledge about a quantity for which scant specific information is available. The best assessments based on scientific judgment of the minimum and the maximum possible values of the quantity are set as the end-points of the rectangular distribution. Thus the end-points are inexactly known. The extent of knowledge concerning the end points may be different. For example, one of the end-points may be a known boundary for the quantity value, such as zero, and the other end-point may not be exactly known. Thus, only one end-point may not be exactly known. In this paper, we describe a probability distribution which represents the state of knowledge about a quantity when the available information about one or both end-points of a rectangular distribution may be represented by other (narrower) rectangular distributions.

Suppose the end-points of a rectangular distribution on an interval (Z_1, Z_2) are inexactly known, the state of knowledge about Z_1 may be represented by a rectangular distribution on a specified interval (a, c), and the state of knowledge about Z_2 may be represented by a rectangular distribution on a specified interval (d, b), where $a \le c \le d \le b$. In section 2, we show that the resulting distribution looks like a trapezoid whose sloping sides are curved. We can refer to such a distribution as curvilinear trapezoidal distribution. Depending on the values of a, c, d, and b, the curvilinear trapezoidal distribution may be asymmetric. We describe the probability density function (pdf) and the moments of a curvilinear trapezoidal distribution which arises when one or both end-points of a rectangular distribution are inexactly known. In particular, we give compact algebraic expressions for the expected value and the variance. We will discuss how random numbers from such distributions may be generated. In section 3, we compare the curvilinear trapezoidal distribution which arises from inexactly known end-points with the corresponding trapezoidal distribution whose sloping sides are straight. In section 4, we compare the curvilinear trapezoidal distribution which arises from inexactly known end-points with the curvilinear trapezoidal distribution which arises when the mid-point of a rectangular distribution is known (fixed), the half-width is not exactly known, and the state of knowledge about the half-width may be represented by a rectangular distribution. Concluding remarks appear in section 5.

2. Curvilinear trapezoidal distribution arising from inexactly known end-points

We will first discuss the case where both end-points of a rectangular distribution are inexactly known then discuss the cases where only one-end point is not exactly known.

2.1 *Both end-points inexactly known*

Suppose the probability distribution of a variable X is rectangular on the interval (Z_1, Z_2), where both end-points Z_1 and Z_2 are inexactly known. Then the pdf of X given $Z_1 = z_1$ and $Z_2 = z_2$ is

$$f_{X|Z_1,Z_2}(x \mid z_1, z_2) = \frac{1}{z_2 - z_1} \quad \text{if} \quad z_1 \le x \le z_2 . \tag{1}$$

Suppose the probability distribution of Z_1 can be represented by a rectangular distribution on an interval (a, c), where $a \le c$. Then the pdf of Z_1 is

$$f_{Z_1}(z_1) = \frac{1}{c - a} \quad \text{if} \quad a \le z_1 \le c . \tag{2}$$

Suppose the probability distribution of Z_2 can be represented by a rectangular distribution on an interval (d, b), where $a \le c \le d \le b$. Then the pdf of Z_2 is

$$f_{Z_2}(z_2) = \frac{1}{b-d} \quad \text{if} \quad d \le z_2 \le b. \tag{3}$$

Then, as discussed in the appendix 1, the unconditional pdf $f(x)$ of X is

$$f(x) = \begin{cases} \dfrac{(b-a)\ln(b-a)-(d-a)\ln(d-a)-(b-x)\ln(b-x)+(d-x)\ln(d-x)}{(c-a)(b-d)} & \text{if } a \le x \le c \\ \dfrac{(b-a)\ln(b-a)-(d-a)\ln(d-a)-(b-c)\ln(b-c)+(d-c)\ln(d-c)}{(c-a)(b-d)} & \text{if } c \le x \le d \\ \dfrac{(b-a)\ln(b-a)-(x-a)\ln(x-a)-(b-c)\ln(b-c)+(x-c)\ln(x-c)}{(c-a)(b-d)} & \text{if } d \le x \le b \end{cases} \tag{4}$$

The unconditional pdf $f(x)$ of X given in (4) can also be expressed as

$$f(x) = \begin{cases} \dfrac{1}{(c-a)(b-d)} \ln \dfrac{(b-a)^{(b-a)}(d-x)^{(d-x)}}{(d-a)^{(d-a)}(b-x)^{(b-x)}} & \text{if } a \le x \le c \\ \dfrac{1}{(c-a)(b-d)} \ln \dfrac{(b-a)^{(b-a)}(d-c)^{(d-c)}}{(d-a)^{(d-a)}(b-c)^{(b-c)}} & \text{if } c \le x \le d. \\ \dfrac{1}{(c-a)(b-d)} \ln \dfrac{(b-a)^{(b-a)}(x-c)^{(x-c)}}{(b-c)^{(b-c)}(x-a)^{(x-a)}} & \text{if } d \le x \le b \end{cases} \tag{5}$$

The pdf (4) has four parameters a, c, d, and b. As indicated in figure 1, a plot of the pdf $f(x)$ looks like a trapezoid whose sloping sides are curved. The parameters a and b represent the end-points of the pdf and the intermediate parameters c and d identify the flat part of the pdf (4). We will refer to the probability distribution represented by the pdf (4) as Curvilinear Trapezoid (a, c, d, b). For comparison, figure 1 also displays the corresponding Trapezoid (a, c, d, b) with straight sloping sides.

As shown in the appendix 2, the k-th moment $E(X^k)$ of the pdf $f(x)$ given in (4) is

$$E(X^k) = \frac{1}{(c-a)(b-d)} \frac{1}{k+1} \sum_{j=0}^{k} \frac{c^{j+1}-a^{j+1}}{j+1} \frac{b^{k-j+1}-d^{k-j+1}}{k-j+1}, \tag{6}$$

for $k = 1, 2, \ldots$. In particular, the expected value $E(X)$ and the variance $V(X)$ of the pdf $f(x)$ are, respectively,

$$E(X) = \frac{a+b+c+d}{4}, \tag{7}$$

and

$$V(X) = \frac{(r+2s+t)^2}{48} + \frac{(r^2+t^2)}{36}, \tag{8}$$

where $r = (c - a)$, $s = (d - c)$, and $t = (b - d)$. The parameter $r = (c - a)$ is the width of the left curvilinear part, $s = (d - c)$ is the width of the middle flat part, and $t = (b - d)$ is the width of the right curvilinear part of the pdf (4). The standard deviation $S(X)$ of the pdf $f(x)$ is square-root of the variance $V(X)$. We note that even though the pdf $f(x)$ of Curvilinear Trapezoid (a, c, d, b) has seemingly complicated form, the expected value and the variance have very simple forms.

Suppose $\{u_1, \ldots, u_n\}$, $\{v_1, \ldots, v_n\}$, and $\{w_1, \ldots, w_n\}$, are three independent sets of random numbers obtained by a random number generator from a rectangular distribution on the interval [0, 1]. Define $z_{1i} = a + (c - a) \times u_i$, then $\{z_{11}, \ldots, z_{1n}\}$ is a set of n random numbers from a rectangular distribution on the interval (a, c) for the left end-point Z_1. Define $z_{2i} = d + (b - d) \times v_i$, then $\{z_{21}, \ldots, z_{2n}\}$ is a set of n random numbers from a rectangular distribution on the interval (d, b) for the right end-point Z_2, where $a \leq c \leq d \leq b$. Now define $x_i = z_{1i} + (z_{2i} - z_{1i}) \times w_i$, then $\{x_1, \ldots, x_n\}$ is a set of n random numbers from a curvilinear trapezoidal distribution with parameters a, c, d, and b, which arises when the end-points of a rectangular distribution are inexactly known and their probability distributions may be represented by independent rectangular distributions.

2.2 *Only left end-point inexactly known*

Suppose the probability distribution of a variable X_1 is rectangular on the interval (Z_1, b), where the right end-point is known to be b, the left end-point Z_1 is inexactly known, and the state of knowledge about Z_1 can be represented by a rectangular distribution on an interval (a, c), where $a \leq c \leq b$. Then the unconditional pdf $f_1(x)$ of X_1 can be obtained by taking the limit of the pdf (4) as the point d tends to the right end-point b.

When determining the limits we need to use the *L'Hospital's Rule* for indeterminate forms which states the following. If the limits of each of the two functions $\phi_1(z)$ and $\phi_2(z)$ are zero but the limit of $[\phi_1'(z) / \phi_2'(z)]$ is finite, where $\phi_1'(z)$ and $\phi_2'(z)$ are the first order derivatives of $\phi_1(z)$ and $\phi_2(z)$ respectively, then $\lim [\phi_1(z) / \phi_2(z)] = \lim [\phi_1'(z) / \phi_2'(z)]$. The rule extends to second and higher order derivatives.

Thus the pdf $f_1(x)$ of X_1 is

$$f_1(x) = \begin{cases} \dfrac{1}{(c-a)} \ln \dfrac{(b-a)}{(b-x)} & \text{if } a \le x \le c \\ \dfrac{1}{(c-a)} \ln \dfrac{(b-a)}{(b-c)} & \text{if } c \le x \le b \end{cases}. \tag{9}$$

The pdf (9) has three parameters a, c, and b. The shape of pdf $f_1(x)$ is indicated figure 2. The parameters a and b represent the end-points and the parameters c and b identify the flat part of the pdf (9). We will refer to the probability distribution represented by the pdf (9) as Curvilinear Trapezoid (a, c, b) with only left-side curving.

The k-th moment $E(X_1^k)$, expected value $E(X_1)$, and variance $V(X_1)$ of the pdf $f_1(x)$ are determined by taking the limits of (6), (7), and (8) as d tends to b and $t = (b - d)$ tends to zero. Thus

$$E(X_1^k) = \frac{1}{(c-a)} \frac{1}{k+1} \sum_{j=0}^{k} b^{k-j} \left(\frac{c^{j+1} - a^{j+1}}{j+1} \right), \tag{10}$$

for $k = 1, 2, \ldots$,

$$E(X_1) = \frac{a + 2b + c}{4}, \tag{11}$$

and

$$V(X_1) = \frac{(r+2s)^2}{48} + \frac{r^2}{36}, \tag{12}$$

where $r = (c - a)$ and $s = (b - c)$.

Suppose $\{u_1, \ldots, u_n\}$ and $\{w_1, \ldots, w_n\}$, are two independent sets of random numbers from a rectangular distribution on the interval [0, 1]. Define $z_{1i} = a + (c - a) \times u_i$, then $\{z_{11}, \ldots, z_{1n}\}$ is a set of n random numbers from a rectangular distribution on the interval (a, c) for the left end-point Z_1. Define $x_i = z_{1i} + (b - z_{1i}) \times w_i$, then $\{x_1, \ldots, x_n\}$ is a set of n random numbers from the pdf (9) with parameters a, c, and b.

2.3 *Only right end-point inexactly known*

Suppose the probability distribution of a variable X_2 is rectangular on the interval (a, Z_2), where the left end-point is known to be a, the right end-point Z_2 is inexactly known, and the state of knowledge about Z_2 can be represented by a rectangular distribution on an interval (d, b), where $a \le d \le b$. Then the unconditional pdf $f_2(x)$ of X_2 can be obtained by taking the limit of the pdf (4) as the point c tends to the left end-point a. Thus the pdf $f_2(x)$ of X_2 is

$$f_2(x) = \begin{cases} \dfrac{1}{(b-d)} \ln \dfrac{(b-a)}{(d-a)} & \text{if } a \le x \le d \\ \dfrac{1}{(b-d)} \ln \dfrac{(b-a)}{(x-a)} & \text{if } d \le x \le b \end{cases}. \tag{13}$$

The pdf (13) has three parameters a, d, and b. The shape of pdf $f_2(x)$ is indicated in figure 3. The parameters a and b represent the end-points and the parameters a and d identify the flat part of the pdf (13). We will refer to the probability distribution represented by the pdf (13) as Curvilinear Trapezoid (a, d, b) with only right-side curving.

The k-th moment $E(X_2^k)$, expected value $E(X_2)$, and variance $V(X_2)$ of the pdf $f_2(x)$ are determined by taking the limits of (6), (7), and (8) as c tends to a and $r = (c - a)$ tends to zero. Thus

$$E(X_2^{\,k}) = \frac{1}{(b-d)} \frac{1}{k+1} \sum_{j=0}^{k} a^j \left(\frac{b^{k-j+1} - d^{k-j+1}}{k-j+1} \right), \tag{14}$$

for $k = 1, 2, \ldots$,

$$E(X_2) = \frac{2a+b+d}{4}, \tag{15}$$

and

$$V(X_2) = \frac{(2s+t)^2}{48} + \frac{t^2}{36}, \tag{16}$$

where $s = (d - a)$ and $t = (b - d)$.

Suppose $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$, are two independent sets of random numbers from a rectangular distribution on the interval [0, 1]. Define $z_{2i} = d + (b - d) \times v_i$, then $\{z_{21}, \ldots, z_{2n}\}$ is a set of n random numbers from a rectangular distribution on the interval (d, b) for the right end-point Z_2. Define $x_i = a + (z_{2i} - a) \times w_i$, then $\{x_1, \ldots, x_n\}$ is a set of n random numbers from the pdf (13) with parameters a, d, and b.

Note 1: If we take the limit as the point c tends to the point a and the point d tends to the point b, then the pdf (4) reduces to

$$f(x) = \frac{1}{(b-a)} \quad \text{if } a \le x \le b, \tag{17}$$

We recognize (17) as the pdf of a rectangular distribution on the interval (a, b). Thus as one would expect, if the left end-point Z_1 is the known value a, and the right end-point Z_2 is the known value b, then the pdf (4) reduces to the pdf of a rectangular distribution on the interval (a, b). If we take the limit of the k-th moment (6), the expected value (7), and the variance (8) of the pdf (4) as the point c tends to the point a and the point d tends to the point b, then we obtain

$$E(X^k) = \frac{1}{k+1}\sum_{j=0}^{k} a^j b^{k-j} = \frac{b^{k+1} - a^{k+1}}{(b-a)(k+1)}, \tag{18}$$

for $k = 1, 2, \ldots$,

$$E(X) = \frac{a+b}{2}, \tag{19}$$

and

$$V(X) = \frac{(b-a)^2}{12}. \tag{20}$$

As one would expect, the expressions(18), (19), and (20) are the k-th moment, for $k = 1, 2, \ldots$, the expected value, and the variance of a rectangular distribution on the interval (a, b) [1].

3. Comparison with trapezoidal distribution

We will discuss trapezoidal distributions corresponding to the curvilinear trapezoidal distributions discussed in sections 2.1, 2.2, and 2.3.

3.1 *Trapezoidal distribution with both sides sloping*

As discussed in [2, section 2], the pdf of a variable Y having the trapezoidal distribution Trapezoid (a, c, d, b) is

$$
g(y) = \begin{cases} \dfrac{2}{(b-a+d-c)} \dfrac{(y-a)}{(c-a)} & \text{if} \quad a \le y \le c \\ \dfrac{2}{(b-a+d-c)} & \text{if} \quad c \le y \le d \, . \\ \dfrac{2}{(b-a+d-c)} \dfrac{(b-y)}{(b-d)} & \text{if} \quad d \le y \le b \end{cases} \tag{21}
$$

The k-th moment $E(Y^k)$ of the pdf $g(y)$ is

$$
E(Y^k) = \frac{1}{(k+2)(k+1)} \frac{2}{(b-a+d-c)} \left(\frac{b^{k+2} - d^{k+2}}{b-d} - \frac{c^{k+2} - a^{k+2}}{c-a} \right), \tag{22}
$$

for $k = 1, 2, \ldots$ [2]. In particular, the expected value and the variance of the pdf $g(y)$ are, respectively,

$$
E(Y) = \frac{1}{3(b-a+d-c)} \left(\frac{b^3 - d^3}{b-d} - \frac{c^3 - a^3}{c-a} \right), \tag{23}
$$

and

$$
V(Y) = \frac{(r+2s+t)^2}{48} + \frac{(r^2+t^2)}{24} - \frac{(r^2-t^2)^2}{144(r+2s+t)^2}, \tag{24}
$$

where $r = (c - a)$, $s = (d - c)$, and $t = (b - d)$ [2, Table 1]. The standard deviation $S(Y)$ of the variable Y is the square-root of the variance $V(Y)$.

The GUM [3, section 4.3.9] suggests that in some cases a trapezoidal distribution is a more realistic expression of the state of knowledge than a rectangular distribution. Therefore, we compare the expected value and the variance of the curvilinear trapezoidal distribution having the pdf (4) with the expected value and the variance of the corresponding trapezoidal distribution having the pdf (21) with the same parameters a, c, d, and b. The difference $E(Y) - E(X)$ (equation (23) minus equation (7)) between the expected values of trapezoidal and curvilinear trapezoidal distributions is

$$
E(Y) - E(X) = -\frac{(c-a)^2 - (b-d)^2}{12(b-a+d-c)} = -\frac{r^2 - t^2}{12(r+2s+t)}. \tag{25}
$$

The difference $V(Y) - V(X)$ (equation (24) minus equation (8)) between the variances of trapezoidal and curvilinear trapezoidal distributions is

$$V(Y) - V(X) = \frac{(r^2 + t^2)}{72} - \frac{(r^2 - t^2)^2}{144(r + 2s + t)^2}. \tag{26}$$

Metrology Example 3.1:
Suppose scant specific information is available concerning a quantity X involved in measurement. Further suppose that the best estimate of the quantity value is μ and the minimum and maximum possible quantity values are estimated to be, respectively $\mu - D_1$ and $\mu + D_2$, where μ is regarded as known but D_1 and D_2 are inexactly known. In the absence of additional knowledge, suppose the state of knowledge about X is represented by a rectangular distribution on the interval $(\mu - D_1, \mu + D_2)$. Suppose the inexact state of knowledge about D_1 is represented by a rectangular distribution on some interval $(\delta_1 - \varepsilon_1, \delta_1 + \varepsilon_1)$ with expected value δ_1 and half-width ε_1, where $\delta_1 \geq \varepsilon_1 > 0$, and suppose the inexact state of knowledge about D_2 is represented by a rectangular distribution on some interval $(\delta_2 - \varepsilon_2, \delta_2 + \varepsilon_2)$ with expected value δ_2 and half-width ε_2, where $\delta_2 \geq \varepsilon_2 > 0$.

Let us consider the case $\mu = 0$, $\delta_1 = \delta_2 = 1$, $\varepsilon_1 = 0.25$, and $\varepsilon_2 = 0.5$. A corollary of section 2.1 is that the unconditional state of knowledge distribution of X is Curvilinear Trapezoid (a, c, d, b) with pdf $f(x)$ given in (4), where $a = \mu - \delta_1 - \varepsilon_1 = -1.25$, $c = \mu - \delta_1 + \varepsilon_1 = -0.75$, $d = \mu + \delta_2 - \varepsilon_2 = 0.5$ and $b = \mu + \delta_2 + \varepsilon_2 = 1.5$. The corresponding trapezoidal distribution is Trapezoid (a, c, d, b) with pdf $g(x)$ given in (21) where $a = -1.25$, $c = -0.75$, $d = 0.5$ and $b = 1.5$. Both pdfs $f(x)$ and $g(x)$ are plotted in figure 1. The expected value and the standard deviation of Curvilinear Trapezoid (–1.25, –0.75, 0.5, 1.5) are 0 and 0.6067, respectively. The expected value and the standard deviation of Trapezoid (–1.25, –0.75, 0.5, 1.5) are 0.0156 and 0.6206, respectively. We can use ε_1/δ_1 and ε_2/δ_2 as indicators of the relative uncertainties associated with δ_1 and δ_2, respectively. Even when ε_1/δ_1 is as large as 25 % and ε_2/δ_2 is as large as 50 % the expected values and the standard deviations of the curvilinear trapezoid and the corresponding trapezoid are not very different.

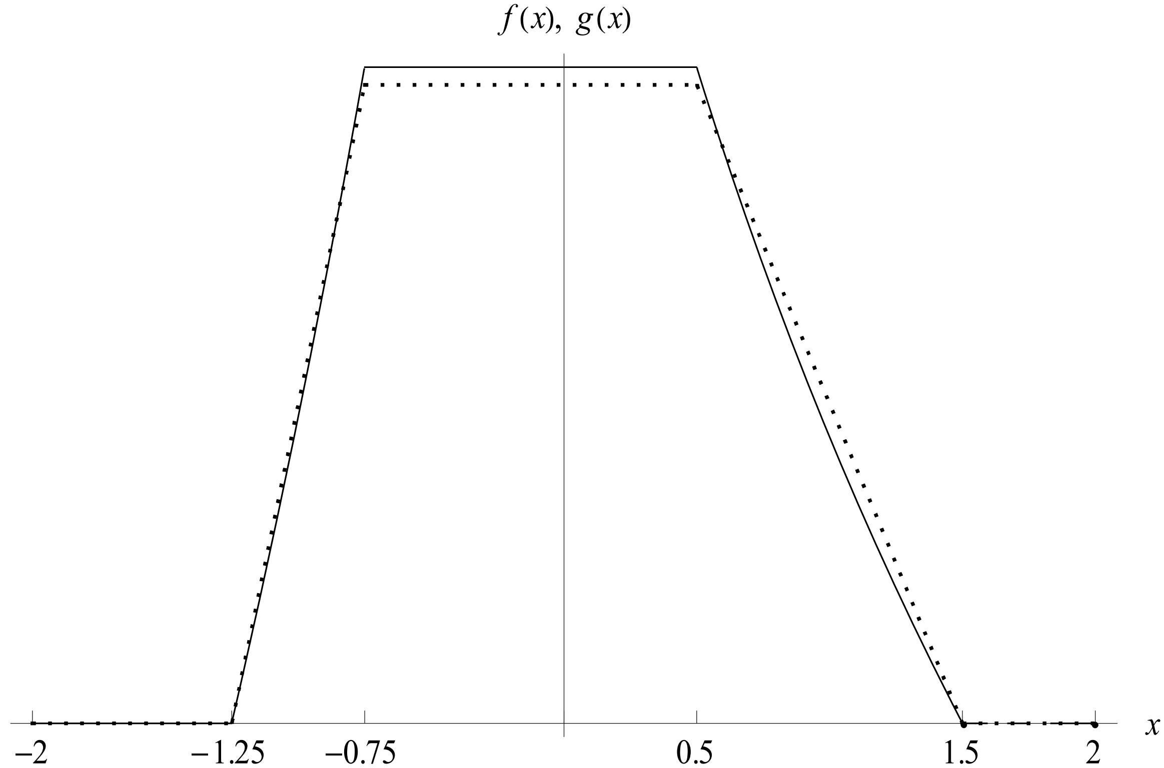

Figure 1: The pdf $f(x)$ of Curvilinear Trapezoid (a, c, d, b) displayed in solid lines and the pdf $g(x)$ of trapezoid (a, c, d, b) displayed in dotted lines both having the same parameters $a = \mu - \delta_1 - \varepsilon_1 = -1.25$, $c = \mu - \delta_1 + \varepsilon_1 = -0.75$, $d = \mu + \delta_2 - \varepsilon_2 = 0.5$ and $b = \mu + \delta_2 + \varepsilon_2 = 1.5$ corresponding to $\mu = 0$, $\delta_1 = \delta_2 = 1$, $\varepsilon_1 = 0.25$, and $\varepsilon_2 = 0.5$.

3.2 *Trapezoidal distribution with only left-side sloping*

The pdf of a variable Y_1 having the trapezoidal distribution Trapezoid (a, c, b) with only left-side sloping can be obtained from the pdf (21) by taking the limit as the point d approaches the point b [2, section 3.1]. The resulting pdf $g_1(y)$ of Y_1 with parameters a, c, and b is

$$g_1(y) = \begin{cases} \dfrac{2}{(b-a+b-c)} \dfrac{(y-a)}{(c-a)} & \text{if} \quad a \le y \le c \\ \dfrac{2}{(b-a+b-c)} & \text{if} \quad c \le y \le b \end{cases} . \tag{27}$$

The k-th moment $E(Y_1^k)$ of the pdf $g_1(y)$ is

$$E(Y_1^k) = \frac{1}{(k+2)(k+1)} \frac{2}{(b-a+d-c)} \left((k+2)b^{k+1} - \frac{c^{k+2} - a^{k+2}}{c-a} \right), \tag{28}$$

for $k = 1, 2, \ldots$ [2]. In particular, the expected value and the variance of the pdf $g_1(y)$ are, respectively,

$$E(Y_1) = \frac{3b^2 - a^2 - c^2 - ac}{3(2b-a-c)}, \tag{29}$$

and

$$V(Y_1) = \frac{(r+2s)^2}{48} + \frac{r^2}{24} - \frac{r^4}{144(r+2s)^2}, \tag{30}$$

where $r = (c - a)$ and $s = (b - c)$. The difference $E(Y_1) - E(X_1)$ (equation (29) minus equation (11)) between the expected values of the pdfs $g_1(y)$ and $f_1(x)$ is

$$E(Y_1) - E(X_1) = -\frac{(c-a)^2}{12(2b-a-c)} = -\frac{r^2}{12(r+2s)}. \tag{31}$$

The difference $V(Y_1) - V(X_1)$ (equation (30) minus equation (12)) between the variances of the pdfs $g_1(y)$ and $f_1(x)$ is

$$V(Y_1) - V(X_1) = \frac{r^2}{72} - \frac{r^4}{144(r+2s)^2}. \tag{32}$$

Metrology Example 3.2:

Suppose the maximum possible value of a quantity X_1 involved in measurement is known to be μ and the minimum possible value is not exactly known. In the absence of additional knowledge, suppose the state of knowledge about X_1 is represented by a rectangular distribution on the interval $(\mu - D_1, \mu)$, where D_1 is not exactly known. This is a special case of example 3.1 corresponding to $D_2 \equiv 0$. Suppose the state of knowledge about D_1 is represented by a rectangular distribution on the interval $(\delta - \varepsilon, \delta + \varepsilon)$ with expected value δ and half-width ε, where $\delta \geq \varepsilon > 0$.

Let us consider the case $\mu = 0$, $\delta = 1$, $\varepsilon = 0.75$. A corollary of the section 2.2 is that the unconditional state of knowledge distribution of X_1 is Curvilinear Trapezoid (a, c, b) with only left-side curving having the pdf $f_1(x)$ given in (9), where $a = \mu - \delta - \varepsilon = -1.75$ $c = \mu - \delta + \varepsilon = -0.25$ and $b = \mu = 0$. The corresponding trapezoidal distribution is Trapezoid (a, c, b) with only left-side sloping with the pdf $g_1(x)$ given in (27) where $a = -1.25$, $c = -0.25$, and $b = 0$. Both pdfs $f_1(x)$ and $g_1(x)$ are plotted in figure 2. The expected value and the standard deviation of Curvilinear Trapezoid (–1.75, –0.25, 0) with only left-side curving are –0.5 and 0.3819, respectively. The expected value and the standard deviation of Trapezoid (–1.75, –0.25, 0) with only left-side sloping are –0.5938 and 0.4102, respectively. Even when ε/δ is as large as 75 % the differences between the expected values and standard deviations of the curvilinear trapezoid and the corresponding trapezoid are not very large.

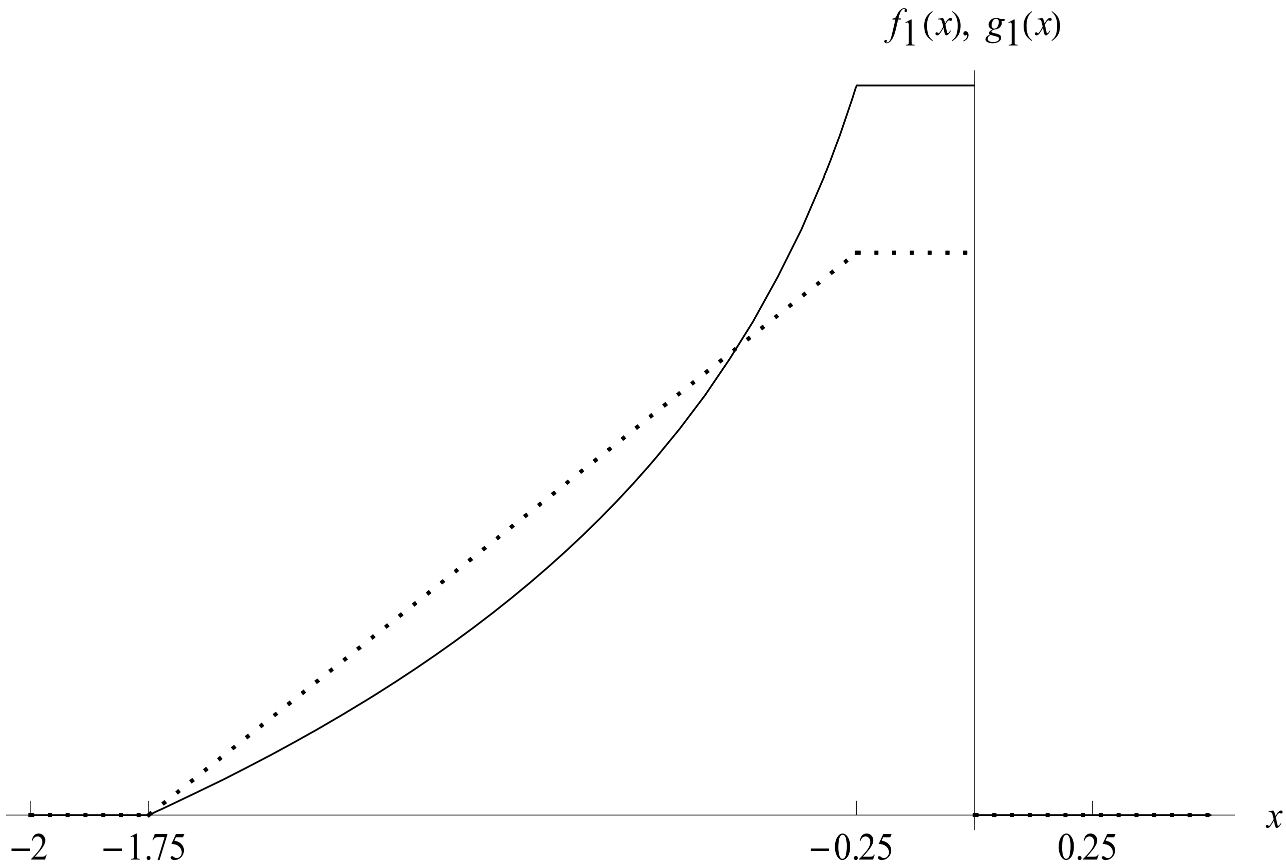

Figure 2: The pdf $f_1(x)$ of Curvilinear Trapezoid (a, c, b) with only left-side curving displayed in solid lines and the pdf $g_1(x)$ of trapezoid (a, c, b) with only left-side sloping displayed in dotted lines both having the same parameters $a = \mu - \delta - \varepsilon = -1.75$ $c = \mu - \delta + \varepsilon = -0.25$ and $b = \mu = 0$ corresponding to $\mu = 0$, $\delta = 1$, $\varepsilon = 0.75$.

3.2 *Trapezoidal distribution with only right-side sloping*

The pdf of a variable Y_2 having the trapezoidal distribution Trapezoid (a, d, b) with only right-side sloping can be obtained from the pdf (21) by taking the limit as the point c

approaches the point a [2, section 3.2]. The resulting pdf $g_2(y)$ of Y_2 with parameters a, d, and b is

$$g_2(y) = \begin{cases} \dfrac{2}{(b-a+d-a)} & \text{if} \quad a \le y \le d \\ \dfrac{2}{(b-a+d-a)} \dfrac{(b-y)}{(b-d)} & \text{if} \quad d \le y \le b \end{cases}. \tag{33}$$

The k-th moment $E(Y_2^k)$ of the pdf $g_2(y)$ is

$$E(Y_2^k) = \frac{1}{(k+2)(k+1)} \frac{2}{(b-a+d-c)} \left(\frac{b^{k+2} - d^{k+2}}{b-d} - (k+2)a^{k+1} \right), \tag{34}$$

for $k = 1, 2, \ldots$ [2]. In particular, the expected value and the variance of the pdf $g_2(y)$ are, respectively,

$$E(Y_2) = \frac{b^2 + d^2 + bd - 3a^2}{3(b+d-2a)}, \tag{35}$$

and

$$V(Y_2) = \frac{(2s+t)^2}{48} + \frac{t^2}{24} - \frac{t^4}{144(2s+t)^2}, \tag{36}$$

where $s = (d - a)$ and $t = (b - d)$. The difference $E(Y_2) - E(X_2)$ (equation (35) minus equation (15)) between the expected values of the pdfs $g_2(y)$ and $f_2(x)$ is

$$E(Y_2) - E(X_2) = \frac{(b-d)^2}{12(b-d-2a)} = \frac{t^2}{12(2s+t)}. \tag{37}$$

The difference $V(Y_2) - V(X_2)$ (equation (36) minus equation (16)) between the variances of the pdfs $g_2(y)$ and $f_2(x)$ is

$$V(Y_2) - V(X_2) = \frac{t^2}{72} - \frac{t^4}{144(2s+t)^2}. \tag{38}$$

Metrology Example 3.3:

As in example 3.2, if the state of knowledge about a quantity X_2 is represented by a rectangular distribution on the interval $(\mu, \mu + D_2)$, where about D_2 has a rectangular distribution on $(\delta - \varepsilon, \delta + \varepsilon)$ where $\delta \geq \varepsilon > 0$ then a corollary of the section 2.3 is that the unconditional state of knowledge distribution of X_2 is a Curvilinear Trapezoid (a, d, b) with only-right-side curving having the pdf $f_2(x)$ given in (13), where $a = \mu$, $d = \mu + \delta - \varepsilon$, and $b = \mu + \delta + \varepsilon$. The corresponding trapezoidal distribution is Trapezoid (a, d, b) with only right-side sloping having the pdf $g_2(x)$ given in (33). Let us consider the case $\mu = 0$, $\delta = 1$, $\varepsilon = 0.75$. The expected value and the standard deviation of the Curvilinear Trapezoid (0, 0.25, 1.75) with only-right-side curving having the pdf $f_2(x)$ are respectively 0.5 and 0.3819. The expected value and the standard deviation of the Trapezoid (0, 0.25, 1.75) with only right-side sloping having the pdf $g_2(x)$ are respectively 0.5938 and 0.4102. Thus even when ε/δ is as large as 75 % the differences between the expected values and standard deviations of the curvilinear trapezoid and the corresponding trapezoid are not very large.

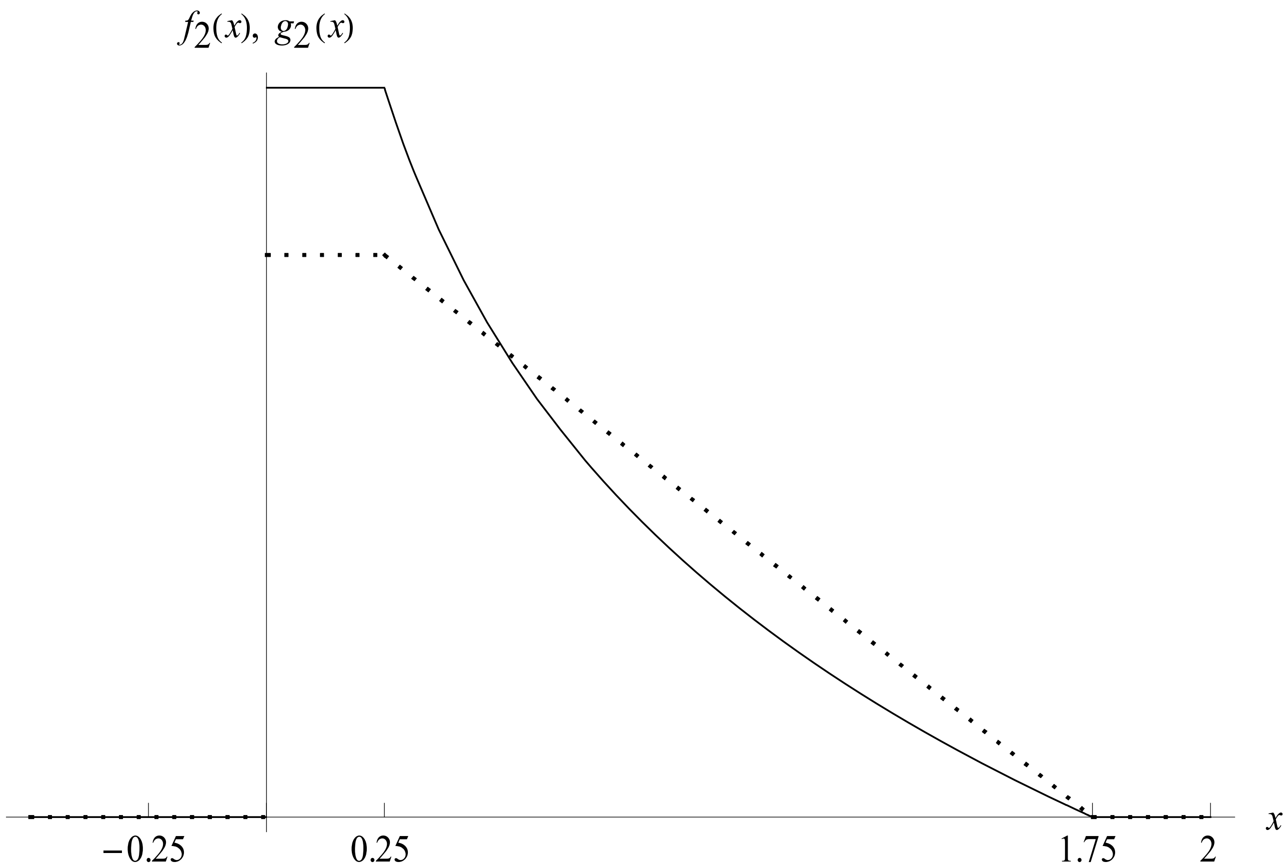

Figure 3: The pdf $f_2(x)$ of Curvilinear Trapezoid (a, d, b) with only right-side curving displayed in solid lines and the pdf $g_2(x)$ of Trapezoid (a, d, b) with only right-side sloping displayed in dotted lines both having the same parameters $a = \mu = 0$, $d = \mu + \delta - \varepsilon = 0.25$ and $b = \mu + \delta + \varepsilon = 1.75$ corresponding to $\mu = 0$, $\delta = 1$, $\varepsilon = 0.75$.

The expected values and variances of curvilinear trapezoidal distributions which arise when one or both end-points of a rectangular distribution are inexactly known are summarized in Table 1. Also included in Table 1 are the expected values and variances of the corresponding trapezoidal distributions.

Table 1: Expected values and variances of curvilinear trapezoidal distribution and corresponding trapezoidal distribution

	Expected value $E(X)$	Variance $V(X)$
Curvilinear Trapezoid (a, c, d, b), pdf (4), figure 1	$\frac{a+b+c+d}{4}$	$\frac{(r+2s+t)^2}{48}+\frac{(r^2+t^2)}{36}$ where $r = (c - a)$, $s = (d - c)$, $t = (b - d)$
Trapezoid (a, c, d, b), pdf (21), figure 1	$\left(\frac{b^3-d^3}{b-d}-\frac{c^3-a^3}{c-a}\right)\div 3(b-a+d-c)$	$\frac{(r+2s+t)^2}{48}+\frac{(r^2+t^2)}{24}-\frac{(r^2-t^2)^2}{144(r+2s+t)^2}$ where $r = (c - a)$, $s = (d - c)$, $t = (b - d)$
Curvilinear Trapezoid (a, c, b) with only left-side curving, pdf (9), figure 2	$\frac{a+2b+c}{4}$	$\frac{(r+2s)^2}{48}+\frac{r^2}{36}$ where $r = (c - a)$ and $s = (b - c)$
Trapezoid (a, c, b) with only left-side sloping, pdf (27), figure 2	$\frac{3b^2-a^2-c^2-ac}{3(2b-a-c)}$	$\frac{(r+2s)^2}{48}+\frac{r^2}{24}-\frac{r^4}{144(r+2s)^2}$ where $r = (c - a)$ and $s = (b - c)$
Curvilinear Trapezoid (a, d, b) with only right-side curving, pdf (13), figure 3	$\frac{2a+b+d}{4}$	$\frac{(2s+t)^2}{48}+\frac{t^2}{36}$ where $s = (d - a)$ and $t = (b - d)$

Trapezoid (a, d, b) with only right-side sloping, pdf (33), figure 3	$\frac{b^2+d^2+bd-3a^2}{3(b+d-2a)}$	$\frac{(2s+t)^2}{48}+\frac{t^2}{24}-\frac{t^4}{144(2s+t)^2}$ where $s = (d - a)$ and $t = (b - d)$

4. Comparison with isocurvilinear distribution arising from uncertain half-width
If the mid-point of a rectangular distribution is known but the half-width is uncertain and the state of knowledge about the half-width may be represented by another (narrower) rectangular distribution then it is shown in [4] that the resulting distribution is isocurvilinear trapezoidal. We will compare the isocurvilinear trapezoidal distribution which arises from uncertain half-width with the isocurvilinear trapezoidal which arises when the end points are uncertain.

Suppose the probability distribution of a variable X_C is rectangular on the interval ($\mu - D$, $\mu + D$), whose mid-point μ is known and the half-width D has a rectangular distribution on the interval ($\delta - \varepsilon$, $\delta + \varepsilon$), where $\delta > \varepsilon > 0$, then as shown in [4] the distribution of X_C is Isocurvilinear Trapezoid (a, c, d, b), where $a = \mu - \delta - \varepsilon$, $c = \mu - \delta + \varepsilon$, $d = \mu + \delta - \varepsilon$ and $b = \mu + \delta + \varepsilon$. The pdf of X_C is

$$h(x) = \frac{1}{4\varepsilon}\ln\left(\frac{\delta+\varepsilon}{\max\{|x-\mu|, \delta-\varepsilon\}}\right) \quad \text{for} \quad \mu-(\delta+\varepsilon) \le x \le \mu+(\delta+\varepsilon)\,. \tag{39}$$

As discussed in [4], the pdf $h(x)$ has expected value

$$E(X_C) = \mu \tag{40}$$

and

$$V(X_C) = \frac{\delta^2}{3} + \frac{\varepsilon^2}{9} \tag{41}$$

Suppose the probability distribution of a variable X is rectangular on the interval ($\mu - D_1$, $\mu + D_2$) where D_1 and D_2 are uncertain. If D_1 and D_2 are independent but have the same rectangular distribution on the interval ($\delta - \varepsilon$, $\delta + \varepsilon$), where $\delta > \varepsilon > 0$, then the distribution of X is Curvilinear Trapezoid (a, c, d, b) with pdf $f(x)$ given in (4), where $a = \mu - \delta - \varepsilon$, $c = \mu - \delta + \varepsilon$, $d = \mu + \delta - \varepsilon$ and $b = \mu + \delta + \varepsilon$. Since D_1 and D_2 have identical distributions the pdf $f(x)$ is symmetric about = μ, i.e. the distribution of X is Isocurvilinear Trapezoid (a, c, d, b). The expected value and the variance of this isocurvilinear trapezoidal distribution arising from uncertain end points can be obtained by substituting $a = \mu - \delta - \varepsilon$, $c = \mu - \delta + \varepsilon$, $d = \mu + \delta - \varepsilon$, and $b = \mu + \delta + \varepsilon$ in (7) and (8), respectively. Thus

$$E(X) = \mu \tag{42}$$

and

$$V(X) = \frac{\delta^2}{3} + \frac{2\varepsilon^2}{9} \tag{43}$$

Suppose the probability distribution of a variable X_S is Isosceles Trapezoid (a, c, d, b) with the pdf $g(x)$ given in (21), where $a = \mu - \delta - \varepsilon$, $c = \mu - \delta + \varepsilon$, $d = \mu + \delta - \varepsilon$ and $b = \mu + \delta + \varepsilon$. The expected value and variance of Isosceles Trapezoid (a, c, d, b) with the pdf $g(x)$ can be obtained by substituting $a = \mu - \delta - \varepsilon$, $c = \mu - \delta + \varepsilon$, $d = \mu + \delta - \varepsilon$, and $b = \mu + \delta + \varepsilon$ in (23) and (24) respectively. Thus

$$E(X_S) = \mu \tag{44}$$

and

$$V(X_S) = \frac{\delta^2}{3} + \frac{\varepsilon^2}{3} \tag{45}$$

Figure 7 displays the pdfs $f(x)$, $g(x)$, and $h(x)$ all with the same parameters a, c, d, and b, where $a = \mu - \delta - \varepsilon = -1.75$, $c = \mu - \delta + \varepsilon = -0.25$, $d = \mu + \delta - \varepsilon = 0.25$, and $b = \mu + \delta + \varepsilon = 1.75$ corresponding to $\mu = 0$, $\delta = 1$ and $\varepsilon = 0.75$.

The probability distributions of X_C, X, and X_S have the same parameters and the same expected values but the variances are different. The variance $V(X)$ of the pdf $f(x)$ arising from uncertain end-points is larger than the variance $V(X_C)$ of the pdf $h(x)$ arising from uncertain half-width because two end points are independently uncertain rather than one parameter half-width. The variance $V(X_S)$ of the pdf $g(x)$ of isosceles trapezoidal distribution is largest. But the differences in variances are not large.

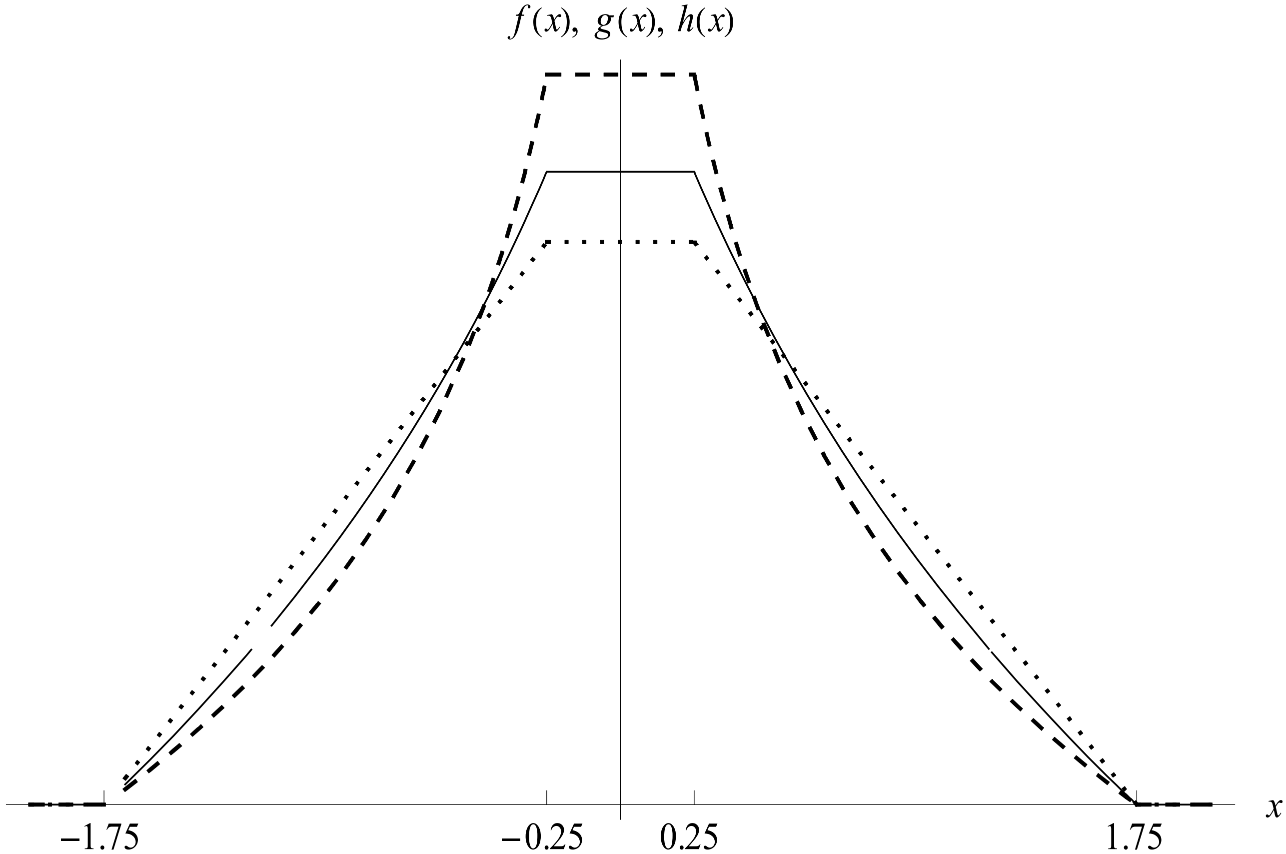

Figure 7: The pdf $f(x)$ of Isocurvilinear Trapezoid (a, c, d, b) which arises from uncertain end-points displayed in solid lines, the pdf $h(x)$ of Isocurvilinear Trapezoid (a, c, d, b) which arises from uncertain half-width displayed in dashed lines, and the pdf $g(x)$ of Isosceles Trapezoid (a, c, d, b) displayed in dotted lines, all having the same parameters where $a = \mu - \delta - \varepsilon = -1.75$, $c = \mu - \delta + \varepsilon = -0.25$, $d = \mu + \delta - \varepsilon = 0.25$, and $b = \mu + \delta + \varepsilon = 1.75$ corresponding to $\mu = 0$, $\delta = 1$ and $\varepsilon = 0.75$.

5. Conclusion

When meager specific information is available about a quantity involved in measurement, a rectangular distribution is often used to represent the state of knowledge and to quantify the uncertainty associated with the best estimate of the quantity value. The GUM suggests that in some cases an isosceles trapezoidal distribution represents the state of knowledge better than a rectangular distribution. In a previous paper we showed that if the mid-point of a rectangular distribution is known and the state of knowledge about the half-width may be represented by another (narrower) rectangular distribution then the resulting distribution is isocurvilinear trapezoidal. In some metrology applications, the extent of knowledge concerning the end points of a rectangular distribution is different. In particular one of the end-points may be known and only one end-point may not be exactly known. In this paper, we showed that when the available information about the end-points of a rectangular distribution may be represented by other (narrower) rectangular distributions, then the resulting distribution is curvilinear trapezoidal which may be asymmetric. We described the moments of a curvilinear trapezoidal distribution which arises when one or both end-points of a rectangular distribution are inexactly known. In particular, we gave compact algebraic expressions for the expected value and the variance. We compared the curvilinear trapezoidal distribution which arises from inexactly known end-points with the corresponding trapezoidal distribution whose sloping sides are straight. In many metrology applications the differences between the expected values and variances of a curvilinear trapezoidal distribution arising from inexactly known end-points and the corresponding trapezoidal distribution are likely to be small.

Acknowledgment

References

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[3] ISO 1995 *Guide to the Expression of Uncertainty in Measurement* (GUM) 2nd ed (Geneva: International Organization for Standardization, ISO) ISBN 92-67-10188-9
[4] Kacker R N and Lawrence J F 2009 Rectangular distribution whose width is not exactly known: isocurvilinear trapezoidal *Metrologia* **46** 254-260

Appendix 1

From the expressions (1), (2), and (3), the joint pdf of X, Z_1, and Z_2 is

$$f_{X,Z_1,Z_2}(x, z_1, z_2) = \frac{1}{(c-a)(b-d)} \frac{1}{z_2 - z_1} \quad \text{if } z_1 \le x \le z_2, a \le z_1 \le c, \text{and } d \le z_2 \le b.$$

The unconditional pdf $f(x)$ for a particular value x of X is obtained by integrating the joint pdf of X, Z_1, and Z_2 with respect to the possible values of z_1 and z_2 corresponding to that x. The region of the possible values of z_1 and z_2 in the $z_1 \times z_2$ plane for a given x depends on which of the three line segments, $a \le x \le c$, $c \le x \le d$, and $d \le x \le b$, contains that value x. If $a \le x \le c$, the possible values of z_1 and z_2 are in the rectangle defined by: $a \le z_1 \le x$ and $d \le z_2 \le b$. If $c \le x \le d$, the possible values of z_1 and z_2 are in the rectangle defined by: $a \le z_1 \le c$ and $d \le z_2 \le b$. If $d \le x \le b$, the possible values of z_1 and z_2 are in the rectangle defined by: $a \le z_1 \le c$ and $x \le z_2 \le b$.

To determine the unconditional pdf $f(x)$, we will need to evaluate the integrals

$$\int \frac{1}{z_2 - z_1}\,\mathrm{d}z_1 \quad \text{and} \quad \int \ln(z_2 - t)\,\mathrm{d}z_2, \text{ for some constant } t.$$

In the first integral, putting $u = z_2 - z_1$ and $\mathrm{d}u = -\,\mathrm{d}z_1$, we have

$$\int \frac{1}{z_2 - z_1}\,\mathrm{d}z_1 = -\int \frac{1}{u}\,\mathrm{d}u = -\ln(u) = -\ln(z_2 - z_1).$$

In the second integral, putting $u = z_2 - t$ and $\mathrm{d}u = \mathrm{d}z_2$, we have

$$\int \ln(z_2 - t)\,\mathrm{d}z_2 = \int \ln(u)\,\mathrm{d}u = u\ln(u) - u = (z_2 - t)\ln(z_2 - t) - (z_2 - t).$$

If $a \le x \le c$, then the possible values of z_1 and z_2 are in the rectangle $[a \le z_1 \le x,\ d \le z_2 \le b]$ of the $z_1 \times z_2$ plane. Now

$$\begin{aligned}
&\int_d^b \int_a^x \frac{1}{z_2 - z_1}\,\mathrm{d}z_1\,\mathrm{d}z_2 = \int_d^b \left(-\ln(z_2 - z_1)\Big|_a^x\right)\mathrm{d}z_2 = \int_d^b (\ln(z_2 - a) - \ln(z_2 - x))\,\mathrm{d}z_2 \\
&= \int_d^b \ln(z_2 - a)\,\mathrm{d}z_2 - \int_d^b \ln(z_2 - x)\,\mathrm{d}z_2 = \big((z_2 - a)\ln(z_2 - a) - (z_2 - a)\big)\Big|_d^b - \big((z_2 - x)\ln(z_2 - x) - (z_2 - x)\big)\Big|_d^b \\
&= (b - a)\ln(b - a) - (d - a)\ln(d - a) - (b - x)\ln(b - x) + (d - x)\ln(d - x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(x) &= \int_d^b \int_a^x f_{X,Z_1,Z_2}(x, z_1, z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2 = \int_d^b \int_a^x \frac{1}{(c-a)(b-d)} \frac{1}{z_2 - z_1}\,\mathrm{d}z_1\,\mathrm{d}z_2 \\
&= \frac{1}{(c-a)(b-d)}\big((b-a)\ln(b-a) - (d-a)\ln(d-a) - (b-x)\ln(b-x) + (d-x)\ln(d-x)\big).
\end{aligned}$$

Alternatively if $a \le x \le c$, then $f(x) = \dfrac{1}{(c-a)(b-d)} \ln \dfrac{(b-a)^{(b-a)}(d-x)^{(d-x)}}{(d-a)^{(d-a)}(b-x)^{(b-x)}}$.

Similarly if $c \le x \le d$, then the possible values of z_1 and z_2 are in the rectangle $[a \le z_1 \le c,\ d \le z_2 \le b]$ of the $z_1 \times z_2$ plane and

$$\begin{aligned}
f(x) &= \int_d^b \int_a^c f_{X,Z_1,Z_2}(x, z_1, z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2 = \int_d^b \int_a^c \frac{1}{(c-a)(b-d)} \frac{1}{z_2 - z_1}\,\mathrm{d}z_1\,\mathrm{d}z_2 \\
&= \frac{1}{(c-a)(b-d)}\big((b-a)\ln(b-a) - (d-a)\ln(d-a) - (b-c)\ln(b-c) + (d-c)\ln(d-c)\big).
\end{aligned}$$

Alternatively if $c \le x \le d$, then $f(x) = \dfrac{1}{(c-a)(b-d)} \ln \dfrac{(b-a)^{(b-a)}(d-c)^{(d-c)}}{(d-a)^{(d-a)}(b-c)^{(b-c)}}$.

Likewise, if $d \le x \le b$, then the possible values of z_1 and z_2 are in the rectangle $[a \le z_1 \le c, x \le z_2 \le b]$ of the $z_1 \times z_2$ plane and

$$f(x) = \int_x^b \int_a^c f_{X,Z_1,Z_2}(x, z_1, z_2)\, \mathrm{d}z_1\, \mathrm{d}z_2 = \int_x^b \int_a^c \frac{1}{(c-a)(b-d)} \frac{1}{z_2 - z_1} \mathrm{d}z_1\, \mathrm{d}z_2$$
$$= \frac{1}{(c-a)(b-d)} \big((b-a)\ln(b-a) - (x-a)\ln(x-a) - (b-c)\ln(b-c) + (x-c)\ln(x-c) \big).$$

Alternatively if $d \le x \le b$, then $f(x) = \dfrac{1}{(c-a)(b-d)} \ln \dfrac{(b-a)^{(b-a)}(x-c)^{(x-c)}}{(b-c)^{(b-c)}(x-a)^{(x-a)}}$.

Appendix 2

The joint pdf of X, Z_1, and Z_2 is

$$f_{X,Z_1,Z_2}(x, z_1, z_2) = \frac{1}{(c-a)(b-d)} \frac{1}{z_2 - z_1} \quad \text{if } z_1 \le x \le z_2, a \le z_1 \le c, \text{and } d \le z_2 \le b.$$

Therefore the k-th moment $E(X^k)$ of the unconditional distribution of X, for $k = 1, 2, \dots$, is

$$E(X^k) = \int_a^c \int_d^b \int_{z_1}^{z_2} x^k\, f_{X,Z_1,Z_2}(x, z_1, z_2)\, \mathrm{d}x\, \mathrm{d}z_2 \mathrm{d}z_1 = \frac{1}{(c-a)(b-d)} \int_a^c \int_d^b \int_{z_1}^{z_2} \frac{x^k}{z_2 - z_1} \mathrm{d}x\, \mathrm{d}z_2 \mathrm{d}z_1.$$

Since

$$\int_{z_1}^{z_2} \frac{x^k}{z_2 - z_1} \mathrm{d}x = \frac{1}{k+1} \left(\frac{x^{k+1}}{z_2 - z_1} \right) \Bigg|_{z_1}^{z_2} = \frac{1}{k+1} \frac{z_2^{\,k+1} - z_1^{\,k+1}}{z_2 - z_1} = \frac{1}{k+1} \sum_{j=0}^{k} z_1^j z_2^{k-j} ,$$

therefore

$$E(X^k) = \frac{1}{(c-a)(b-d)}\frac{1}{k+1}\int_a^c\int_d^b\sum_{j=0}^{k} z_1^j z_2^{k-j}\mathrm{d}z_2\mathrm{d}z_1 = \frac{1}{(c-a)(b-d)}\frac{1}{k+1}\sum_{j=0}^{k}\int_a^c z_1^j\int_d^b z_2^{k-j}\mathrm{d}z_2\mathrm{d}z_1.$$

Since

$$\int_d^b z_2^{k-j}\mathrm{d}z_2 = \left.\frac{z_2^{k-j+1}}{k-j+1}\right|_d^b = \frac{b^{k-j+1}-d^{k-j+1}}{k-j+1}$$

and

$$\int_a^c z_1^j\mathrm{d}z_1 = \left.\frac{z_1^{j+1}}{j+1}\right|_a^c = \frac{c^{j+1}-a^{j+1}}{j+1}$$

we have

$$E(X^k) = \frac{1}{(c-a)(b-d)}\frac{1}{k+1}\sum_{j=0}^{k}\frac{c^{j+1}-a^{j+1}}{j+1}\frac{b^{k-j+1}-d^{k-j+1}}{k-j+1}.$$

The expected value $E(X)$ is obtained by substituting $k = 1$; thus,

$$E(X) = \frac{1}{(c-a)(b-d)}\frac{1}{2}\left(\frac{c-a}{1}\frac{b^2-d^2}{2}+\frac{c^2-a^2}{2}\frac{b-d}{1}\right) = \frac{a+b+c+d}{4}.$$

The second moment $E(X^2)$ is obtained by substituting $k = 2$; thus,

$$E(X^2) = \frac{1}{(c-a)(b-d)}\frac{1}{3}\left(\frac{c-a}{1}\frac{b^3-d^3}{3}+\frac{c^2-a^2}{2}\frac{b^2-d^2}{2}+\frac{c^3-a^3}{3}\frac{b-d}{1}\right),$$

which simplifies to

$$E(X^2) = \frac{1}{3}\left(\frac{b^2 + bd + d^2}{3} + \frac{(c+a)}{2}\frac{(b+d)}{2} + \frac{c^2 + ca + a^2}{3}\right).$$

The variance $V(X)$ is then

$$V(X) = E(X^2) - (E(X))^2 = \frac{1}{3}\left(\frac{b^2 + bd + d^2}{3} + \frac{(c+a)}{2}\frac{(b+d)}{2} + \frac{c^2 + ca + a^2}{3}\right) - \left(\frac{a+b+c+d}{4}\right)^2.$$

Let $r = (c - a)$, $s = (d - c)$, and $t = (b - d)$, then the variance $V(X)$ can be expressed as

$$V(X) = \frac{(r + 2s + t)^2}{48} + \frac{(r^2 + t^2)}{36}.$$