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Rectangular distribution whose width is not exactly known: isocurvilinear trapezoidal distribution

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Abstract
After the Gaussian distribution, the probability distribution most commonly used in evaluation of uncertainty in measurement is the rectangular distribution. If the half-width of a rectangular distribution is specified, the mid-point is uncertain, and the probability distribution of the mid-point may be represented by another (narrower) rectangular distribution then the resulting distribution is an isosceles trapezoidal distribution. However, in metrological applications, it is more common that the mid-point is specified but the half-width is uncertain. If the probability distribution of the half-width may be represented by another (narrower) rectangular distribution, then the resulting distribution looks like an isosceles trapezoid whose sloping sides are curved. We can refer to such a probability distribution as an isocurvilinear trapezoidal distribution. We describe the main characteristics of an isocurvilinear trapezoidal distribution which arises when the half-width is uncertain. When the uncertainty in specification of the half-width is not excessive, the isocurvilinear trapezoidal distribution can be approximated by an isosceles trapezoidal distribution.

S This paper has associated online supplementary data files.

1. Introduction

If a variable X_k has a rectangular distribution on the interval $(-k, k)$ and another (independent) variable X_h has a rectangular distribution on the interval $(-h, h)$, where $k > h > 0$, then the sum $X = X_k + X_h$ has an isosceles trapezoidal distribution with parameters $-(k+h)$, $-(k-h)$, $(k-h)$ and $(k+h)$ [1, section 4.07]. Based on this result, the *Guide to the Expression of Uncertainty in Measurement* (GUM) [2, section 4.3.9, note 2] states that if the width of a rectangular distribution is uncertain and the probability distribution of the width can be represented by another (narrower) rectangular distribution then the resulting distribution is an isosceles trapezoidal distribution. We show that this statement is not correct; however, it is a reasonable approximation for many applications in metrology.

We can always express a rectangular distribution on an interval (α, β) as $(\mu - \delta, \mu + \delta)$, where $\mu = (\alpha + \beta)/2$ is the mid-point and $\delta = (\beta - \alpha)/2$ is the half-width. Suppose X_R is a variable with rectangular distribution on the interval $(\mu - \delta, \mu + \delta)$ with mid-point μ and half-width δ. As noted in [1, section 4.05] and [2, section 4.3.7], the expected value $E(X_R)$ and the standard deviation $S(X_R)$ of the probability density function (pdf) of X_R are, respectively,

$$E(X_R) = \mu, \tag{1.1}$$

and

$$S(X_R) = \sqrt{E(X_R - \mu)^2} = \sqrt{\frac{\delta^2}{3}} = \frac{\delta}{\sqrt{3}}. \tag{1.2}$$

In this paper, we use Greek letters, such as μ, δ and ε, for specified quantities.

We show in section 2 that if the half-width of a rectangular distribution is specified, the mid-point is uncertain and the probability distribution of the mid-point may be represented by another (narrower) rectangular distribution then the resulting

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distribution is an isosceles trapezoidal distribution. This result is of theoretical interest largely. In metrological applications, it is more common that the mid-point is specified but the half-width is uncertain. If the probability distribution of the half-width may be represented by another (narrower) rectangular distribution then we show in section 3 that the resulting distribution looks like an isosceles trapezoid whose sloping sides are curved. We can refer to such a probability distribution as an isocurvilinear trapezoidal distribution. In section 4, we describe the following characteristics of an isocurvilinear trapezoidal distribution which arises when the half-width is uncertain: cumulative distribution function (cdf), moment generating function (mgf), moments, expected value and standard deviation. We also discuss how random numbers from such a distribution may be generated. Finally, in section 5, we show that moderate uncertainty in the specification of half-width increases the standard deviation of the rectangular distribution only slightly. Also, if the uncertainty in the specification of half-width is not excessive, the isocurvilinear trapezoidal distribution can be approximated by an isosceles trapezoidal distribution.

2. The half-width is specified but the mid-point is uncertain

Suppose the probability distribution of a variable X_S is rectangular on the interval $(Y-\delta, Y+\delta)$, where the half-width δ is specified (fixed) but the mid-point Y is uncertain. It follows that the conditional pdf of X_S given $Y = y$ is rectangular on the interval $(y-\delta, y+\delta)$. Suppose the uncertainty concerning the mid-point Y may be represented by a rectangular distribution on the interval $(\mu-\varepsilon, \mu+\varepsilon)$, where $\delta > \varepsilon > 0$. Then, as shown in this section, the resulting unconditional distribution of X_S is an isosceles trapezoidal distribution with the pdf displayed in figure 1. The isosceles trapezoidal distribution of X_S is fully characterized by the three parameters δ, $(\mu-\varepsilon)$ and $(\mu+\varepsilon)$. However, following the parametrization of [3], it is more convenient to refer to the isosceles trapezoidal distribution of X_S as having the four parameters $\mu-(\delta+\varepsilon)$, $\mu-(\delta-\varepsilon)$, $\mu+(\delta-\varepsilon)$ and $\mu+(\delta+\varepsilon)$. As indicated in figure 1, the parameters $\mu-(\delta+\varepsilon)$ and $\mu+(\delta+\varepsilon)$ are the end-points and the parameters $\mu-(\delta-\varepsilon)$ and $\mu+(\delta-\varepsilon)$ identify the flat middle part of the isosceles trapezoid.

The conditional pdf of X_S given $Y = y$ is

$$g_{X_s|Y}(x|y) = \frac{1}{2\delta} \qquad \text{if } y-\delta \leqslant x \leqslant y+\delta. \qquad (2.1)$$

The pdf of Y is

$$g_Y(y) = \frac{1}{2\varepsilon} \qquad \text{if } \mu-\varepsilon \leqslant y \leqslant \mu+\varepsilon. \qquad (2.2)$$

Therefore the joint pdf of X_S and Y is

$$g_{X_s,Y}(x, y) = \frac{1}{4\varepsilon\delta}$$
$$\text{if } y-\delta \leqslant x \leqslant y-\delta \text{ and } \mu-\varepsilon \leqslant y \leqslant \mu+\varepsilon, \qquad (2.3)$$

where $\delta > \varepsilon > 0$. The unconditional pdf $g(x)$ for a particular value x of X_S is obtained by integrating the joint pdf (2.3) with

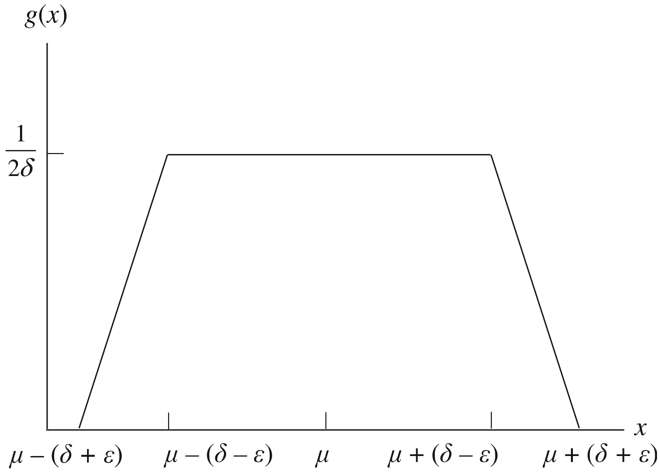

Figure 1. Probability density function $g(x)$ of an isosceles trapezoidal distribution with parameters $\mu-(\delta+\varepsilon)$, $\mu-(\delta-\varepsilon)$, $\mu+(\delta-\varepsilon)$ and $\mu+(\delta+\varepsilon)$.

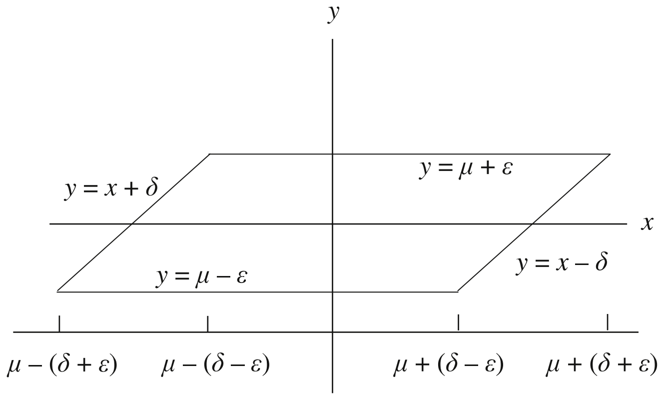

Figure 2. The region in the (x, y) plane where the joint pdf of X_S and Y has a positive value.

respect to the possible values of y corresponding to that x. The region in which the joint pdf of X_S and Y has positive value is indicated in figure 2 as the parallelogram bounded by the parallel lines $y = \mu \pm \varepsilon$ and the parallel lines $y = x \pm \delta$. The range of possible values of y for a given x depends on which of the three horizontal line segments in figure 2 contains that value x. If $\mu-(\delta+\varepsilon) \leqslant x \leqslant \mu-(\delta-\varepsilon)$, then $\mu-\varepsilon \leqslant y \leqslant x+\delta$. If $\mu-(\delta-\varepsilon) \leqslant x \leqslant \mu+(\delta-\varepsilon)$, then $\mu-\varepsilon \leqslant y \leqslant \mu+\varepsilon$. If $\mu+(\delta-\varepsilon) \leqslant x \leqslant \mu+(\delta+\varepsilon)$, then $x-\delta \leqslant y \leqslant \mu+\varepsilon$.

Thus

$$g(x) = \begin{cases} 0 & \text{if } x \leqslant \mu-(\delta+\varepsilon), \\ \dfrac{[x-(\mu-(\delta+\varepsilon))]}{4\varepsilon\delta} & \text{if } \mu-(\delta+\varepsilon) \leqslant x \leqslant \mu-(\delta-\varepsilon), \\ \dfrac{1}{2\delta} & \text{if } \mu-(\delta-\varepsilon) \leqslant x \leqslant \mu+(\delta-\varepsilon), \\ \dfrac{[(\mu+(\delta+\varepsilon))-x]}{4\varepsilon\delta} & \text{if } \mu+(\delta-\varepsilon) \leqslant x \leqslant \mu+(\delta+\varepsilon), \\ 0 & \text{if } \mu+(\delta+\varepsilon) \leqslant x. \end{cases} \qquad (2.4)$$

By comparing (2.4) with the pdf of a trapezoidal distribution given in [3, section 2], we see that $g(x)$ is the pdf of an isosceles trapezoidal distribution with parameters $\mu-(\delta+\varepsilon)$, $\mu-(\delta-\varepsilon)$, $\mu+(\delta-\varepsilon)$ and $\mu+(\delta+\varepsilon)$. Thus the unconditional probability distribution of X_S is an isosceles trapezoidal distribution with the pdf $g(x)$ indicated in figure 1. The properties of a general trapezoidal distribution are described in [3]. In particular from [3, section 2] the expected value $E(X_S)$ and the standard deviation $S(X_S)$ of the pdf $g(x)$ given in (2.4) are, respectively,

$$E(X_S) = \mu \tag{2.5}$$

and

$$S(X_S) = \sqrt{E(X_S-\mu)^2} = \sqrt{\frac{\delta^2}{3}+\frac{\varepsilon^2}{3}}. \tag{2.6}$$

The situation discussed in this section is of theoretical interest largely. In metrology, one rarely encounters situations where the state of knowledge about a quantity is described by a rectangular distribution whose half-width is specified but the mid-point is uncertain. Generally, the mid-point is specified but the half-width is not known exactly. In the next section we show that if the uncertainty about the half-width may be represented by a rectangular distribution then the resulting distribution is an isocurvilinear trapezoidal distribution.

3. The mid-point is specified but the half-width is uncertain

Suppose the probability distribution of a variable X_C is rectangular on the interval $(\mu - Z,\ \mu + Z)$, where the mid-point μ is specified (fixed) but the half-width Z is uncertain. It follows that the conditional pdf of X_C given $Z = z$ is rectangular on the interval $(\mu - z,\ \mu + z)$. Suppose the uncertainty concerning the half-width Z may be represented by a rectangular distribution on the interval $(\delta - \varepsilon,\ \delta + \varepsilon)$, where $\delta > \varepsilon > 0$. Then, as shown in this section, the resulting unconditional distribution of X_C is an isocurvilinear trapezoidal distribution with the pdf displayed in figure 3. The isocurvilinear trapezoidal distribution of X_C is fully characterized by the three parameters μ, $(\delta-\varepsilon)$ and $(\delta+\varepsilon)$. However, it is more convenient to refer to the isocurvilinear trapezoidal distribution of X_C as having the four parameters $\mu-(\delta+\varepsilon)$, $\mu-(\delta-\varepsilon)$, $\mu+(\delta-\varepsilon)$ and $\mu+(\delta+\varepsilon)$. As indicated in figure 3, the parameters $\mu-(\delta+\varepsilon)$ and $\mu+(\delta+\varepsilon)$ are the end-points and the parameters $\mu-(\delta-\varepsilon)$ and $\mu+(\delta-\varepsilon)$ identify the flat middle part of the isocurvilinear trapezoid.

The conditional pdf of X_C given $Z = z$ is

$$f_{X_C|Z}(x|z) = \frac{1}{2z} \qquad \text{if } \mu - z \leqslant x \leqslant \mu + z. \tag{3.1}$$

The pdf of Z is

$$f_Z(z) = \frac{1}{2\varepsilon} \qquad \text{if } \delta-\varepsilon \leqslant z \leqslant \delta+\varepsilon. \tag{3.2}$$

Therefore the joint pdf of X_C and Z is

$$f_{X_C,Z}(x,z) = \frac{1}{4\varepsilon z} \qquad \text{if } \mu - z \leqslant x \leqslant \mu + z \quad \text{and} \quad \delta-\varepsilon \leqslant z \leqslant \delta+\varepsilon, \tag{3.3}$$

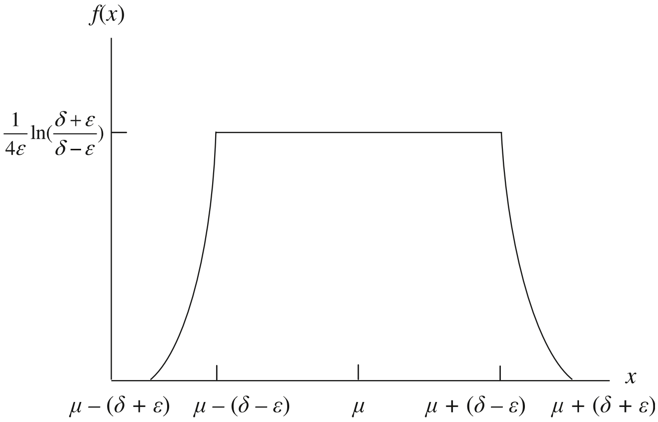

Figure 3. Probability density function $f(x)$ of an isocurvilinear trapezoidal distribution with parameters $\mu-(\delta+\varepsilon)$, $\mu-(\delta-\varepsilon)$, $\mu+(\delta-\varepsilon)$ and $\mu+(\delta+\varepsilon)$.

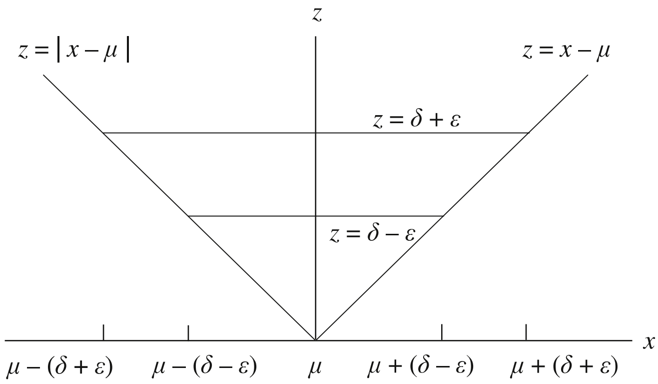

Figure 4. The region in the (x, z) plane where the joint pdf of X_C and Z has a positive value.

where $\delta > \varepsilon > 0$. The unconditional pdf $f(x)$ for a particular value x of X_C is obtained by integrating the joint pdf (3.3) with respect to the possible values of z corresponding to that x. The region in which the joint pdf of X_C and Z has positive value is indicated in figure 4 as the trapezoid (wider side up) bounded by the parallel lines $z = \delta\pm\varepsilon$ and the V-shaped lines $z = x-\mu$ and $z = |x-\mu|$. The range of the possible values of z for a given x depends on which of the three horizontal line segments in figure 4 contains that value x. If $\mu-(\delta+\varepsilon) \leqslant x \leqslant \mu-(\delta-\varepsilon)$, then $|x-\mu| \leqslant z \leqslant \delta+\varepsilon$. If $\mu-(\delta-\varepsilon) \leqslant x \leqslant \mu+(\delta-\varepsilon)$, then $\delta-\varepsilon \leqslant z \leqslant \delta+\varepsilon$. If $\mu+(\delta-\varepsilon) \leqslant x \leqslant \mu+(\delta+\varepsilon)$, then $x-\mu \leqslant z \leqslant \delta+\varepsilon$.

Thus

$$f(x) = \begin{cases} 0 & \text{if } x \leqslant \mu-(\delta+\varepsilon), \\ \dfrac{1}{4\varepsilon}\ln\left(\dfrac{\delta+\varepsilon}{|x-\mu|}\right) & \text{if } \mu-(\delta+\varepsilon) \leqslant x \leqslant \mu-(\delta-\varepsilon), \\ \dfrac{1}{4\varepsilon}\ln\left(\dfrac{\delta+\varepsilon}{\delta-\varepsilon}\right) & \text{if } \mu-(\delta-\varepsilon) \leqslant x \leqslant \mu+(\delta-\varepsilon), \\ \dfrac{1}{4\varepsilon}\ln\left(\dfrac{\delta+\varepsilon}{x-\mu}\right) & \text{if } \mu+(\delta-\varepsilon) \leqslant x \leqslant \mu+(\delta+\varepsilon), \\ 0 & \text{if } \mu+(\delta+\varepsilon) \leqslant x. \end{cases} \tag{3.4}$$

The range of possible values of z for a given x can succinctly be expressed as follows: if $\mu-(\delta+\varepsilon) \leqslant x \leqslant \mu+(\delta+\varepsilon)$, then $\max\{|x-\mu|, \delta\varepsilon\} \leqslant z \leqslant \delta+\varepsilon$. Thus the pdf $f(x)$ of X_C given in (3.4) can be expressed as

$$f(x)=\frac{1}{4\varepsilon}\ln\left(\frac{\delta+\varepsilon}{\max\{|x-\mu|,\delta-\varepsilon\}}\right)$$

$$\text{for}\quad \mu-(\delta+\varepsilon)\leqslant x\leqslant \mu+(\delta+\varepsilon). \tag{3.5}$$

The pdf $f(x)$ has the shape of an isocurvilinear trapezoid with parameters $\mu-(\delta+\varepsilon)$, $\mu-(\delta-\varepsilon)$, $\mu+(\delta-\varepsilon)$ and $\mu+(\delta+\varepsilon)$ illustrated in figure 3.

The *Supplement 1 to the GUM* (GUM S1) [4, section 6.4.3] states the following. Suppose a quantity X is known to lie between limits A and B with $A<B$, where the mid-point $(A+B)/2$ is fixed (specified) and the width $(B-A)$ is not known exactly (uncertain). Further suppose that A and B are known to lie in the known intervals $(a\pm d)$ and $(b\pm d)$, respectively, where $d>0$ and $a+d<b-d$. If no other information is available concerning X, A and B, then a probability distribution for X which has the maximum entropy (information uncertainty) is the curvilinear trapezoidal distribution with pdf

$$g_X(\xi)=\frac{1}{4d}\ln\left(\frac{w+d}{\max\{|\xi-x|,w-d\}}\right)$$

$$\text{for } a-d\leqslant\xi\leqslant b+d. \tag{3.6}$$

If we make the following substitutions in (3.6) $a=(\mu-\delta)$, $b=(\mu+\delta)$, $x=(a+b)/2=\mu$, $w=(b-a)/2=\delta$, $d=\varepsilon$ and $\xi=x$, we get the pdf $f(x)$ given in (3.5).

4. Characteristics of an isocurvilinear trapezoidal distribution which arises from uncertain half-width

We will describe the cdf, mgf, moments, expected value and standard deviation of an isocurvilinear trapezoidal distribution having the pdf $f(x)$ given in (3.4). We will also describe how random numbers from the pdf $f(x)$ may be generated.

4.1. Cumulative distribution function

The cdf of a random variable W is defined as the probability $\Pr(W\leqslant x)$ and it is obtained by integrating the pdf of W within the limits from $-\infty$ to x. The cdf $F(x)$ of an isocurvilinear trapezoidal distribution with the pdf $f(x)$ given in (3.4) is

$$F(x)=\begin{cases} 0 & \text{if } x\leqslant\mu-(\delta+\varepsilon),\\ \dfrac{1}{4\varepsilon}\left[(x-\mu)\left(\ln\dfrac{\delta+\varepsilon}{|x-\mu|}+1\right)+\delta+\varepsilon\right] & \text{if } \mu-(\delta+\varepsilon)\leqslant x\leqslant\mu-(\delta-\varepsilon),\\ \dfrac{1}{4\varepsilon}\left[(x-\mu)\ln\dfrac{\delta+\varepsilon}{\delta-\varepsilon}+2\varepsilon\right] & \text{if } \mu-(\delta-\varepsilon)\leqslant x\leqslant\mu+(\delta-\varepsilon),\\ \dfrac{1}{4\varepsilon}\left[(x-\mu)\left(\ln\dfrac{\delta+\varepsilon}{x-\mu}+1\right)+3\varepsilon-\delta\right] & \text{if } \mu+(\delta-\varepsilon)\leqslant x\leqslant\mu+(\delta+\varepsilon),\\ 1 & \text{if } \mu+(\delta+\varepsilon)\leqslant x \end{cases} \tag{4.1}$$

(see appendix 1, available from the online version of this journal).

4.2. Moment generating function and moments

The mgf of a random variable W denoted by $M_W(t)$ is defined as the expected value $E(e^{tW})$ with respect to the pdf $p(w)$ of W. If the expected value $E(e^{tW})$ exists for t in some neighbourhood of zero, then the mgf $M_W(t)$ exists. The mgf for a pdf is unique. If the expected value $E(e^{tW})$ exists then the mgf can be expressed as

$$M_W(t)=\int\left(\sum_{k=0}^{\infty}\frac{t^k w^k}{k!}\right)p(w)\,dw$$
$$=\sum_{k=0}^{\infty}\left(\int w^k p(w)\,dw\right)\frac{t^k}{k!}=1+\sum_{k=1}^{\infty}E(W^k)\frac{t^k}{k!}. \tag{4.2}$$

The expected value $E(X_C)$ of an isocurvilinear trapezoidal distribution with the pdf $f(x)$ given in (3.4) is

$$E(X_C)=\mu \tag{4.3}$$

(see appendix 2, available from the online version of this journal).

We will first determine the mgf of $X_C-\mu$ and then determine the mgf of X_C. The mgf of $X_C-\mu$ is

$$M_{X_C-\mu}(t)=1+\sum_{j=1}^{\infty}\frac{(\delta+\varepsilon)^{2j+1}-(\delta-\varepsilon)^{2j+1}}{2\cdot\varepsilon\cdot(2j+1)^2}\,\frac{t^{2j}}{(2j)!} \tag{4.4}$$

(see appendix 3, available from the online version of this journal).

The central moments $E(X_C-\mu)^k$ about the expected value (mean) μ can be easily determined from the mgf of $X_C-\mu$. By comparing the coefficients of $t^k/k!$ in (4.2) and (4.4) we note that all moments about the mean μ of odd order (that is, $k = 2j - 1$ for $j = 1, 2, \ldots$) are zero, thus

$$E(X_{\rm C} - \mu)^k = 0 \qquad \text{for} \quad k = 1, 3, 5 \ldots \tag{4.5}$$

and if k is even (that is, $k = 2j$ for $j = 1, 2, \ldots$), then

$$E(X_{\rm C} - \mu)^k = \frac{(\delta + \varepsilon)^{k+1} - (\delta - \varepsilon)^{k+1}}{2 \cdot \varepsilon \cdot (k+1)^2} \quad \text{for } k = 2, 4, 6 \ldots. \tag{4.6}$$

In particular,

$$E(X_{\rm C} - \mu)^2 = \frac{(\delta + \varepsilon)^3 - (\delta - \varepsilon)^3}{2 \cdot \varepsilon \cdot (2+1)^2} = \frac{\delta^2}{3} + \frac{\varepsilon^2}{9}. \tag{4.7}$$

Thus the standard deviation $S(X_{\rm C})$ of an isocurvilinear trapezoidal distribution with the pdf $f(x)$ given in (3.4) is

$$S(X_{\rm C}) = \sqrt{E(X_{\rm C} - \mu)^2} = \sqrt{\frac{\delta^2}{3} + \frac{\varepsilon^2}{9}}. \tag{4.8}$$

The mgf of $X_{\rm C}$ is related to the mgf of $X_C - \mu$ by the following equation:

$$M_{X_{\rm C}}(t) = E({\rm e}^{tX_{\rm C}}) = {\rm e}^{t\mu} E({\rm e}^{t(X_{\rm C}-\mu)}) = {\rm e}^{t\mu} M_{X_{\rm C}-\mu}(t). \tag{4.9}$$

Therefore from (4.4) the mgf of the pdf $f(x)$ given in (3.4) is

$$M_{X_{\rm C}}(t) = {\rm e}^{t\mu} M_{X_{\rm C}-\mu}(t)$$

$$= {\rm e}^{t\mu}\left(1 + \sum_{j=1}^{\infty} \frac{(\delta + \varepsilon)^{2j+1} - (\delta - \varepsilon)^{2j+1}}{2 \cdot \varepsilon \cdot (2j+1)^2} \frac{t^{2j}}{(2j)!}\right). \tag{4.10}$$

The kth moment $E(X_{\rm C}^k)$ for $k = 1, 2, \ldots$, is the kth derivative of (4.10) evaluated at $t = 0$. The first and second derivatives of (4.10) evaluated at $t = 0$ yield

$$E(X_{\rm C}) = \mu \qquad \text{and} \qquad E(X_{\rm C}^2) = \mu^2 + \frac{\delta^2}{3} + \frac{\varepsilon^2}{9}. \tag{4.11}$$

Since $E(X_{\rm C} - \mu)^2 = E(X_{\rm C}^2) - (E(X_{\rm C}))^2$, expressions (4.7) and (4.11) agree.

4.3. Generation of random numbers

The GUM [2] approach to determine the expected value and the variance for an output variable (measurand) is to propagate the expected values and the variances of incompletely specified state-of-knowledge probability distributions for the input variables through a linear approximation of the measurement function. An alternative approach proposed in the GUM S1 [4] is to propagate probability distributions by numerical simulation of the measurement equation. The latter approach requires generation of random numbers from the probability distributions specified for the input variables. Random numbers from an isocurvilinear trapezoidal distribution with the pdf $f(x)$ given in (3.4) can be easily generated as follows.

Suppose $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ are two independent sets of random numbers obtained by a random number generator from a rectangular distribution on the interval [0, 1]. Then $\{z_1, \ldots, z_n\}$, where $z_i = (\delta - \varepsilon) + u_i \times 2\varepsilon$, is a set of random numbers for the half-width from a rectangular distribution on the interval $(\delta - \varepsilon, \delta + \varepsilon)$ and $\{x_1, \ldots, x_n\}$, where $x_i = (\mu - z_i) + v_i \times 2z_i$ is a set of random numbers from the isocurvilinear trapezoidal distribution with the pdf $f(x)$.

5. Conclusion

The GUM S1 [4, section 6.4.3] identifies the pdf $f(x)$ of an isocurvilinear trapezoidal distribution given in (3.5) as a maximum entropy distribution. Our derivation of the pdf $f(x)$ is closely linked to the metrological practice. Metrologists often use a rectangular distribution to describe the state of knowledge about a quantity for which very little reliable specific information is available. They identify the best available estimate for the quantity as the mid-point and set the half-width by subjective judgment. Therefore the half-width is uncertain (not exactly known). Thus an isocurvilinear trapezoidal distribution, derived by assuming that the inexact knowledge about the half-width may be represented by a rectangular distribution, is a more accurate representation of the available state of knowledge than a rectangular distribution which ignores the uncertainty in specification of the half-width.

If a variable $X_{\rm R}$ has a rectangular distribution with mid-point μ and half-width δ, then $E(X_{\rm R}) = \mu$ and $S(X_{\rm R}) = \sqrt{(\delta^2/3)}$. If a variable $X_{\rm C}$ is has a rectangular distribution with mid-point μ but uncertain half-width represented by a rectangular distribution on the interval $(\delta - \varepsilon,\ \delta + \varepsilon)$, where $\delta > \varepsilon > 0$, then the resulting distribution of $X_{\rm C}$ is isocurvilinear trapezoidal with $E(X_{\rm C}) = \mu$ and $S(X_{\rm C}) = \sqrt{(\delta^2/3 + \varepsilon^2/9)}$. If the uncertainty concerning the half-width is ignored then the standard deviation $S(X_{\rm C})$ is underestimated by an amount quantified by the relative difference $[S(X_{\rm C}) - S(X_{\rm R})]/S(X_{\rm R}) = \sqrt{(1 + \varepsilon^2/3\delta^2)} - 1$.

The ratio ε/δ indicates the uncertainty concerning the specification of half-width. The values of the relative difference $[S(X_{\rm C}) - S(X_{\rm R})]/S(X_{\rm R})$ for $\varepsilon/\delta = 0.25$, 0.50, 0.75 and in the limit as ε/δ tends to 1.0 are 1%, 4%, 9% and 15%, respectively. Therefore, even if the ratio ε/δ is as large as 0.50, the underestimation of the standard deviation of the isocurvilinear trapezoidal distribution will not be more than 4%. Thus in many metrology applications the uncertainty in the specification of the half-width of a rectangular distribution may be ignored. In applications where thorough evaluation of the uncertainty is desired, the larger standard deviation $S(X_{\rm C})$ of the isocurvilinear trapezoidal distribution should be used.

Let us compare the isocurvilinear trapezoidal distribution with the corresponding isosceles trapezoidal distribution. If a variable $X_{\rm S}$ has an isosceles trapezoidal distribution with parameters $\mu - (\delta + \varepsilon)$, $\mu - (\delta - \varepsilon)$, $\mu + (\delta - \varepsilon)$ and $\mu + (\delta + \varepsilon)$, then $E(X_{\rm S}) = \mu$ and $S(X_{\rm S}) = \sqrt{(\delta^2/3 + \varepsilon^2/3)}$. The relative difference $[S(X_{\rm S}) - S(X_{\rm C})]/S(X_{\rm C})$ in the standard deviations of the isosceles and the isocurvilinear trapezoidal distributions with the same parameters $\mu - (\delta + \varepsilon)$, $\mu - (\delta - \varepsilon)$, $\mu + (\delta - \varepsilon)$ and $\mu + (\delta + \varepsilon)$ is $\sqrt{[(1 + \varepsilon^2/\delta^2)/(1 + \varepsilon^2/3\delta^2)]} - 1$. The values of the relative difference $[S(X_{\rm S}) - S(X_{\rm C})]/S(X_{\rm C})$ for $\varepsilon/\delta = 0.25$, 0.50, 0.75, and in the limit as ε/δ tends to 1.0 are, respectively, 2%, 7%, 15% and 22%. Therefore, even if the ratio ε/δ is as large as 0.50, the relative difference $[S(X_{\rm S}) - S(X_{\rm C})]/S(X_{\rm C})$ in the standard deviations will not be more than 7%. Thus it is reasonable to conclude that from the viewpoint of evaluating

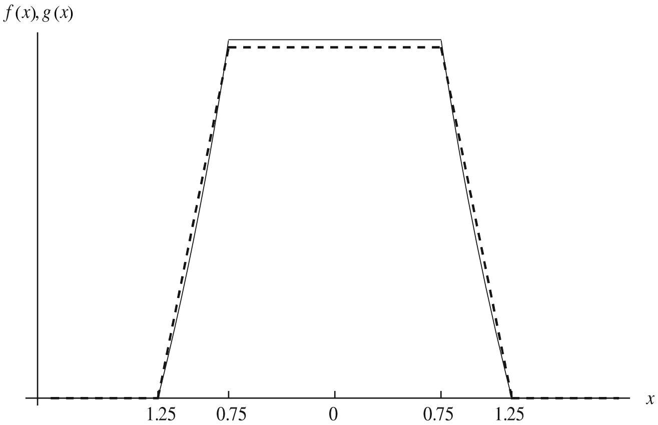

Figure 5. Comparison of the pdf $f(x)$ of isocurvilinear trapezoidal distribution indicated by solid lines and the pdf $g(x)$ of isosceles trapezoidal distribution indicated by dashed lines with the same parameters $\mu - (\delta + \varepsilon)$, $\mu - (\delta - \varepsilon)$, $\mu + (\delta - \varepsilon)$ and $\mu + (\delta + \varepsilon)$ for $\mu = 0$, $\delta = 1.0$ and $\varepsilon = 0.25$.

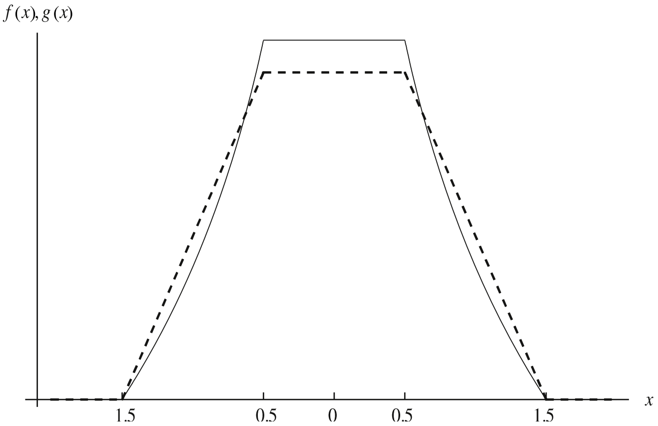

Figure 6. Comparison of the pdf $f(x)$ of isocurvilinear trapezoidal distribution indicated by solid lines and the pdf $g(x)$ of isosceles trapezoidal distribution indicated by dashed lines with the same parameters $\mu - (\delta + \varepsilon)$, $\mu - (\delta - \varepsilon)$, $\mu + (\delta - \varepsilon)$ and $\mu + (\delta + \varepsilon)$ for $\mu = 0$, $\delta = 1.0$ and $\varepsilon = 0.50$.

uncertainty in measurement, the differences between the standard deviations of the isosceles and the isocurvilinear trapezoidal distribution are not significant for the values of the ratio ε/δ likely to occur in many metrological applications. Thus note 2 of section 4.3.9 in the GUM [2] is a reasonable approximation for many applications in metrology.

Let us compare the pdf $f(x)$ of the isocurvilinear trapezoidal distribution with the pdf $g(x)$ of the isosceles trapezoidal distribution with the same parameters $\mu - (\delta + \varepsilon)$, $\mu - (\delta - \varepsilon)$, $\mu + (\delta - \varepsilon)$ and $\mu + (\delta + \varepsilon)$. In figures 5–8, the pdf $f(x)$ of the isocurvilinear distribution is displayed by solid lines and the pdf $g(x)$ of the isosceles trapezoidal distribution is displayed by dashed lines.

In figure 5, $\mu = 0$, $\delta = 1.0$ and $\varepsilon = 0.25$. The standard deviation $S(X_S)$ of the isosceles trapezoidal distribution is 0.60 and the standard deviation of the isocurvilinear trapezoidal distribution $S(X_C)$ is 0.58. Therefore the relative difference is 2%.

In figure 6, $\mu = 0$, $\delta = 1.0$ and $\varepsilon = 0.50$. The standard deviation $S(X_S)$ of the isosceles trapezoidal distribution is 0.65

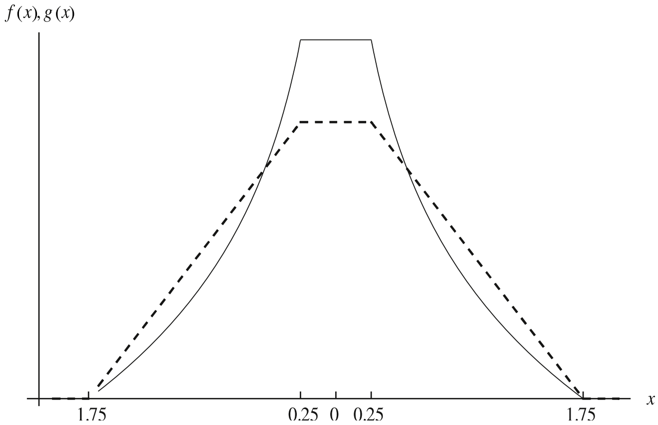

Figure 7. Comparison of the pdf $f(x)$ of isocurvilinear trapezoidal distribution indicated by solid lines and the pdf $g(x)$ of isosceles trapezoidal distribution indicated by dashed lines with the same parameters $\mu - (\delta + \varepsilon)$, $\mu - (\delta - \varepsilon)$, $\mu + (\delta - \varepsilon)$ and $\mu + (\delta + \varepsilon)$ for $\mu = 0$, $\delta = 1.0$ and $\varepsilon = 0.75$.

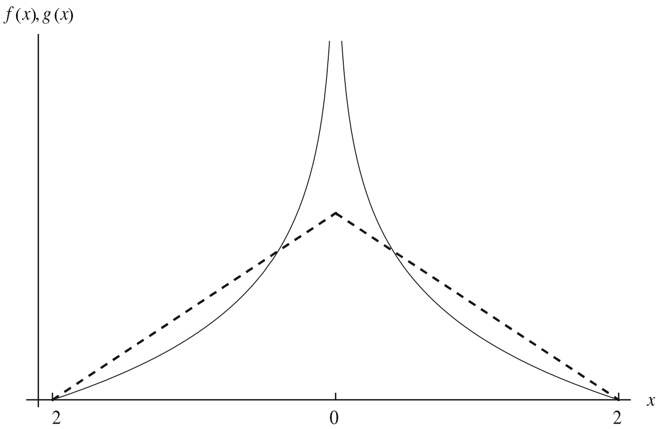

Figure 8. Comparison of the pdf $f(x)$ of isocurvilinear trapezoidal distribution indicated by solid lines and the pdf $g(x)$ of isosceles trapezoidal distribution indicated by dashed lines with the same parameters $\mu - (\delta + \varepsilon)$, $\mu - (\delta - \varepsilon)$, $\mu + (\delta - \varepsilon)$ and $\mu + (\delta + \varepsilon)$ for $\mu = 0$, $\delta = 1.0$ and ε tends to 1.0.

and the standard deviation of the isocurvilinear trapezoidal distribution $S(X_C)$ is 0.60. Therefore the relative difference is 7%.

In figure 7, $\mu = 0$, $\delta = 1.0$ and $\varepsilon = 0.75$. The standard deviation $S(X_S)$ of the isosceles trapezoidal distribution is 0.72 and the standard deviation of the isocurvilinear trapezoidal distribution $S(X_C)$ is 0.63. Therefore the relative difference is 15%.

In figure 8, $\mu = 0$, $\delta = 1.0$ and ε tends to 1.0. The standard deviation $S(X_S)$ of the isosceles trapezoidal distribution is 0.82 and the standard deviation of the isocurvilinear trapezoidal distribution $S(X_C)$ is 0.67. Therefore the relative difference is 22%.

Figures 5 and 6 indicate that the isosceles trapezoidal distribution is a good approximation for the isocurvilinear trapezoidal distribution even when ε is as large as 50% of δ. However, since the isocurvilinear trapezoidal distribution arising from uncertain half-width is fully characterized, there may be no need to approximate it, especially in the Monte Carlo method for propagating probability distributions, GUM S1 [4].

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