# FALSE CHARACTERISTIC FUNCTIONS AND OTHER PATHOLOGIES IN VARIATIONAL BLIND DECONVOLUTION. A METHOD OF RECOVERY* 

ALFRED S. CARASSO ${ }^{\dagger}$


#### Abstract

Given a blurred image $g(x, y)$, variational blind deconvolution seeks to reconstruct both the unknown blur $k(x, y)$ and the unknown sharp image $f(x, y)$, by minimizing an appropriate cost functional. This paper restricts its attention to a rich and significant class of infinitely divisible isotropic blurs that includes Gaussians, Lorentzians, and other heavy-tailed densities, together with their convolutions. A recently developed highly efficient nonlinear variational approach is found to produce inadmissible reconstructions, consisting of partially deblurred images $f^{\dagger}(x, y)$, associated with physically impossible blurs $k^{\dagger}(x, y)$. Three basic flaws in this variational procedure are identified and shown to be the cause of this phenomenon. A method is then developed that can recover useful information from $k^{\dagger}(x, y)$, by constructing a physically valid rectified blur $h^{\#}(x, y)$, based on the low frequency part of $k^{\dagger}(x, y)$. A crucial step involves interpreting $h^{\#}(x, y)$ as the $p^{t h}$ convolution root of the true blur $k(x, y)$, for some postulated real number $p \geq 2$. Deconvolution is performed in slow motion, by solving an associated parabolic pseudo-differential equation backwards in time, with the blurred image $g(x, y)$ as data at $t=1$. Behavior of the evolution as $t \downarrow 0$ can be monitored and used to readjust the value of $p$. Previously developed APEX/SECB methodologies make such ill-posed continuation feasible. This recovery method is found highly effective in several instructive examples involving synthetically blurred images.


Key words. image deblurring, variational blind deconvolution, minimum norm solutions, false characteristic functions, infinitely divisible blurs, fractional diffusion equations, ill-posed continuation, APEX method, SECB method

AMS subject classifications. $35 \mathrm{R} 25,35 \mathrm{~B} 60,60 \mathrm{E} 07,60 \mathrm{E} 10,68 \mathrm{U} 10$
DOI. 10.1137/080737769

1. Introduction. In the simplest case, image deblurring requires solving the equation $k(x, y) \otimes f(x, y)=g(x, y)+n(x, y)$, where $k(x, y)$ is a shift-invariant point spread function ( psf ), $g(x, y)$ is the given blurred image, $f(x, y)$ is the desired true sharp image, $n(x, y)$, assumed small, represents system noise, and $\otimes$ denotes convolution. In many areas of application, $k(x, y)$ is only poorly known. Blind deconvolution seeks to obtain $f(x, y)$ from $g(x, y)$ without knowing $k(x, y)$. Rather, by formulating an appropriate variational principle in which a priori constraints are placed on each of the two unknowns $f(x, y)$ and $k(x, y)$, one seeks to obtain the pair $\{f, k\}$ as the unique minimizer of a cost functional. Three distinct examples of such procedures are discussed in [13], [21], and [29]. To the extent that such methods are successful, they appear to obviate the need for prior knowledge of $k(x, y)$. In reality, regularized variational blind deconvolution is a difficult mathematical problem that is not fully understood. The present paper draws attention to the unanticipated pitfalls that accompany these procedures and often invalidate the reconstructed solutions.

The methods in [13] and [29] are iterative in nature and time consuming for large images. In addition, significant limitations in these methodologies have been uncovered [3], [14], [19]. In [21], a direct noniterative method, centered on the concept of minimum norm solutions (MNS), is described and analyzed. This highly nonlinear

[^0]method is based on FFT algorithms and produces almost instantaneous deblurring of large size imagery. An instructive analysis is provided in [21], and the MNS method is shown to compare favorably with [13] and [29] for the case of zero phase kernels and low noise levels. One striking theoretical result in [21] deals with the nonlinear solution operator $\mathcal{L}$ that produces the MNS solution pair $\{f(x, y), k(x, y)\}$ when applied to the blurred image $g(x, y)$. It is shown in [21] that $\mathcal{L}$ is locally Hölder-continuous with exponent $1 / 2$. This is interpreted to mean that the blind deconvolution problem is less ill-posed than the nonblind problem. We show that that interpretation is illusory. In fact, the favorable Hölder exponent results from the anomaly in [21] that only partial deblurring is theoretically possible, full deblurring being precluded by the variational formulation. The blind problem remains seriously ill-posed.

More troubling is the fact that the MNS appproach generally produces inadmissible reconstructions $f(x, y)$, based on physically impossible detected blurs $k(x, y)$. Such reconstructions are of questionable value in astronomy, medical imaging, electron microscopy, and other scientific applications.

In this paper, Fourier space behavior is used to identify and document three basic flaws in the MNS procedure. We focus our attention on a rich class of isotropic zero phase kernels of prime significance in applications. This is the infinitely divisible class $\mathbf{L}$ defined below. That class includes Gaussians, Lorentzians, and other heavy-tailed blurs, together with their convolutions. Classical theorems of Bochner, Schoenberg, and Pólya are used to show that MNS-detected optical transfer functions (otf) are generally spurious, and are not characteristic functions [16], [23], [24]. Such detected otfs must therefore correspond to nonphysical probablity density functions (psfs) with negative values. Next, the origin of the anomalous limited deblurring property of the MNS procedure is isolated, and its connection with Hölder continuity is clarified. Lastly, we show that the MNS variational principle induces an erroneous relationship at high frequencies between the detected optical transfer function and the Fourier space reconstructed image. Each of these two objects becomes delusive at high frequencies. Unexpected spurious relationships between detected blurs and reconstructed images also occur in [13] and [29].

In the latter part of the paper, we show how to extract useful information from defective MNS reconstructions in the important special case of class $\mathbf{L}$ blurs. This intervention requires analytical considerations extraneous to the variational framework in [21], and it involves several steps. First, the high frequency portion of the nonphysical MNS-detected blur is discarded and a class $\mathbf{L}$ blur is associated with the low frequency part using least squares fitting. Next, this rectified blur is interpreted as the $p^{t h}$ convolution root of the true blur for some postulated real number $p \geq 2$. Infinite divisibility of this candidate true blur allows deconvolution to be performed in slow motion, by marching backwards in time in an associated parabolic equation, with the blurred image as data at $t=1$. Behavior of the image evolution as $t \downarrow 0$ can be monitored and used to readjust the value of $p$. Previously developed APEX/SECB methodologies [6], [8], [9], [10] provide the necessary computational tools to make such an approach feasible. In particular, the exceptional backwards stability provided by the SECB constraint [6], [7], [22] plays an essential role in this ill-posed continuation problem.

The recovery experiments associated with Figures 5 through 8 are a highlight of this paper. Five carefully selected synthetically blurred images are used to demonstrate the application of APEX/SECB methodology in improving MNS reconstructions. That intervention is decisive in all five cases. It replaces partially deblurred MNS images of uncertain scientific validity with fully deblurred images based on
physically meaningful point spread functions. The remarkable sequence in Figure 6 involving Uranus and its moons is particularly instructive.
2. Optical transfer functions are characteristic functions. We deal with deconvolution procedures which are formulated and implemented in the Fourier transform domain. The following observations will be important in the subsequent discussion. For $h(x, y) \in L^{1}\left(R^{2}\right)$, define its Fourier transform $\hat{h}(\xi, \eta)$ by

$$
\begin{equation*}
\hat{h}(\xi, \eta) \equiv \int_{R^{2}} h(x, y) e^{-2 \pi i(\xi x+\eta y)} d x d y \tag{1}
\end{equation*}
$$

For complex-valued $q(\xi, \eta)$, define

$$
\begin{equation*}
\operatorname{sign} q(\xi, \eta)=\exp (i \arg (q(\xi, \eta))) \tag{2}
\end{equation*}
$$

For real-valued $q(x, y)$, define

$$
\begin{equation*}
\operatorname{sign}^{+} q(x, y)=1, \quad q(x, y) \geq 0, \quad \operatorname{sign}^{+} q(x, y)=-1, \quad q(x, y)<0 \tag{3}
\end{equation*}
$$

A 2 D probability density function $k(x, y)$ is a nonnegative function in $L^{1}\left(R^{2}\right)$ with $\int_{R^{2}} k(x, y) d x d y=1$. Such an object has a Fourier transform $\hat{k}(\xi, \eta)$ with distinctive properties not shared by the Fourier transform of an arbitrary $h(x, y) \in L^{1}\left(R^{2}\right)$. In particular, $\hat{k}(\xi, \eta)$ must be a positive definite function in the sense of Bochner, which implies additional properties. For this reason, Fourier transforms of probability densities form a distinguished class with a well-developed theory [4], [15], [16], [23], [24]. Such functions are called characteristic functions.

In an optical system, a shift-invariant point spread function $k(x, y)$ is also a nonnegative function which integrates to unity, and hence such a psf is a probability density function. The Fourier transform of the shift-invariant psf $k(x, y)$ is the optical transfer function $\hat{k}(\xi, \eta)$. True otfs are necessarily 2 D characteristic functions and must obey Bochner's theorem and its consequences. A candidate otf that is not a characteristic function must correspond to a nonphysical psf with negative values. All of the methods discussed in this paper eventually result in symmetric psfs where $k(x, y)=k(-x,-y)$. In that case, $\hat{k}(\xi, \eta)$ is real-valued and symmetric. The otf for Gaussian blur is positive, and sign ${ }^{+} \hat{k}=1$. The otf for uniform defocus blur oscillates about zero, and is given by the "sombrero function," [17, p. 72]

$$
\begin{equation*}
\hat{k}(\xi, \eta)=2 J_{1}(R \rho) /(R \rho), \quad \rho=\sqrt{\xi^{2}+\eta^{2}} \tag{4}
\end{equation*}
$$

where $J_{1}$ is the Bessel function of the first kind of order 1 , and $R>0$ is the radius of the "circle of confusion." Here, $\operatorname{sign}^{+} \hat{k}$ takes the value +1 or -1 .
3. $\boldsymbol{H}^{1}$ blind deconvolution and artificial Fourier space relation. The idea of formulating the blind problem as a joint regularization problem for both the image and the psf originates in [29]. That approach seeks the pair $\left(f^{H}, k^{H}\right)$ that minimizes the functional

$$
\begin{equation*}
\mathcal{F}_{\mathcal{H}}(f, k)=\|f \otimes k-g\|_{2}^{2}+\sigma_{1} \int_{R^{2}}|\nabla f|^{2} d x d y+\sigma_{2} \int_{R^{2}}|\nabla k|^{2} d x d y \tag{5}
\end{equation*}
$$

over all $f, k \in H^{1}\left(R^{2}\right)$. The regularization parameter $\sigma_{1}$ is the image signal to noise parameter, while $\sigma_{2}$ controls the blur point spread. The two variables functional $\mathcal{F}_{\mathcal{H}}$ is not jointly convex, and there are infinitely many solutions, most of which are not
physically useful. However, if one variable is held fixed, the functional is convex in the other variable. This leads to an alternating minimization (AM) algorithm, whereby a useful local minimum may be found by choosing a good initial guess, although different initial choices will produce different solutions. In [29], the blurred image is used as initial guess, $f^{0}=g$, and the AM algorithm takes the form

$$
\begin{align*}
k^{n} & =\arg \min _{k \in H^{1}} \mathcal{F}_{\mathcal{H}}\left(f^{n-1}, k\right),  \tag{6}\\
f^{n} & =\arg \min _{f \in H^{1}} \mathcal{F}_{\mathcal{H}}\left(f, k^{n}\right), \quad n=1,2, \ldots
\end{align*}
$$

The analysis in [14] reveals the surprising result that the above iteration converges to the limit pair $\left(f^{H}, k^{H}\right)$, where

$$
\begin{align*}
& \operatorname{sign} \hat{f}^{H}(\xi, \eta)=\operatorname{sign} \hat{g}(\xi, \eta), \quad \operatorname{sign}^{+} \hat{k}^{H}(\xi, \eta)=1 \\
& \left|\hat{k}^{H}(\xi, \eta)\right|=\sqrt{\left(\sigma_{1} / \sigma_{2}\right)}\left|\hat{f}^{H}(\xi, \eta)\right|, \quad(\xi, \eta) \neq(0,0) \tag{7}
\end{align*}
$$

Therefore, the detected otf $\hat{k}^{H}(\xi, \eta)$ is symmetric and has zero phase, whether or not this is the case for the true otf. In particular, uniform defocus blurs cannot be detected. The mysterious false proportionality relation between $\left|\hat{k}^{H}(\xi, \eta)\right|$ and $\left|\hat{f}^{H}(\xi, \eta)\right|$ in (7) is quite unlikely in practice, and difficult to explain from a priori analytical considerations. Further observations regarding (6) may be found in [19, Chapter 3]. Only limited success has been recorded with this method.
4. Total variation blind deconvolution. In [13], the authors propose the use of the total variation ( $T V$ ) norm, instead of the $H^{1}$ norm, to arrive at a more effective cost functional. The pair $\left(f^{V}, k^{V}\right)$ is sought that minimizes the functional

$$
\begin{equation*}
\mathcal{F}_{\mathcal{V}}(f, k)=\|f \otimes k-g\|_{2}^{2}+\gamma_{1} \int_{R^{2}}|\nabla f| d x d y+\gamma_{2} \int_{R^{2}}|\nabla k| d x d y \tag{8}
\end{equation*}
$$

over all $f, k \in B V\left(R^{2}\right)$. As in the previous section, the functional $\mathcal{F}_{\mathcal{V}}(f, k)$ is convex in one variable while the other is held fixed, but it is not jointly convex. Only a local minimum is possible, whose usefulness will depend on the initial guess. As before, starting with $f^{0}=g$, we are led to the AM algorithm,

$$
\begin{align*}
k^{n} & =\arg \min _{k \in B V} \mathcal{F}_{\mathcal{V}}\left(f^{n-1}, k\right),  \tag{9}\\
f^{n} & =\arg \min _{f \in B V} \mathcal{F}_{\mathcal{V}}\left(f, k^{n}\right), \quad n=1,2, \ldots
\end{align*}
$$

While numerical experiments in [13] and [14] indicate some improvement over the results in [29], a complete convergence analysis of $T V$ blind deconvolution is not available. Moreover, further carefully documented experiments reported in [3] reveal significant limitations in this methodology. In [3, Figure 1], an image synthetically blurred with an isotropic Gaussian psf is shown to be very poorly reconstructed using this technique. One major reason is that the $T V$-detected psf $k^{V}(x, y)$ is highly nonisotropic. Indeed, the shape of $k^{V}(x, y)$ appears to mimic the characteristics of the image. This is vividly confirmed in an independent second experiment described in [3, Figure 2], which shows conclusively that $T V$ psfs are significantly and improperly influenced by the distribution of edges in the image. These experiments raise the question as to whether an unsuspected spurious relation exists between $k^{V}(x, y)$ and $f^{V}(x, y)$, similar to that found in (7) for $H^{1}$ blind deconvolution. Inaccurate $T V$ image and psf reconstructions are also noted in [21, Figures 4 and 5]. Additional remarks concerning blind deconvolution methods may be found in [12].
5. Noniterative minimum norm blind deconvolution (MNS). A significant new approach is developed in [21]. Given the blurred image $g \in L^{2}\left(R^{2}\right)$, and initial guesses $\bar{f}, \bar{k} \in L^{2}\left(R^{2}\right)$, the pair $\left(f^{\dagger}, k^{\dagger}\right)$ is called an $(\bar{f}, \bar{k})$ minimum norm solution (MNS) of $f \otimes k=g$ if and only if

$$
\begin{align*}
& f^{\dagger} \otimes k^{\dagger}=g \\
& \left\|f^{\dagger}-\bar{f}\right\|_{2}^{2}+\left\|k^{\dagger}-\bar{k}\right\|_{2}^{2}=\min \left\{\|f-\bar{f}\|_{2}^{2}+\|k-\bar{k}\|_{2}^{2} \quad \mid f \otimes k=g\right\} \tag{10}
\end{align*}
$$

the minimum being taken over all $f, g \in L^{2}\left(R^{2}\right)$. We have the following fundamental result.

ThEOREM 1 ([21]). Let $\mathcal{Y}$ be the space of functions with Fourier transforms in $L^{1}\left(R^{2}\right)$, and let $g \in \mathcal{Y} \cap L^{2}\left(R^{2}\right)$. Let $\bar{k}(x, y)$ be a symmetric $2 D$ probability density function, and let

$$
\begin{equation*}
\hat{\bar{f}}=\hat{g} \operatorname{sign}^{+} \hat{\bar{k}} \tag{11}
\end{equation*}
$$

Then, there exists a unique $(\bar{f}, \bar{k})$ minimum norm solution $\left(f^{\dagger}, k^{\dagger}\right)$. That solution has the following properties:

$$
\begin{gather*}
\operatorname{sign} \hat{f}^{\dagger}=\operatorname{sign} \hat{\bar{f}}, \quad \operatorname{sign}^{+} \hat{k}^{\dagger}=\operatorname{sign}^{+} \hat{\bar{k}},  \tag{12}\\
\hat{f}^{\dagger}(\xi, \eta)=r^{\dagger}(\xi, \eta) \operatorname{sign} \hat{\bar{f}}(\xi, \eta),  \tag{13}\\
\hat{k}^{\dagger}(\xi, \eta)=\left\{|\hat{g}(\xi, \eta)| / r^{\dagger}(\xi, \eta)\right\} \operatorname{sign}^{+} \hat{\bar{k}}(\xi, \eta), \quad r^{\dagger}(\xi, \eta) \neq 0, \\
\hat{k}^{\dagger}(\xi, \eta)=\hat{\bar{k}}(\xi, \eta), \quad r^{\dagger}(\xi, \eta)=0, \tag{14}
\end{gather*}
$$

where $r^{\dagger}(\xi, \eta)$ is the unique nonnegative root of the polynomial

$$
\begin{equation*}
p(r)=r^{3}(r-c)+c(r b-c), \quad b(\xi, \eta)=|\hat{\bar{k}}(\xi, \eta)|, \quad c(\xi, \eta)=|\hat{g}(\xi, \eta)| \tag{15}
\end{equation*}
$$

Remarks. An explicit formula for the root $r^{\dagger}(\xi, \eta)$ is given in [21]. Using this, together with FFT algorithms, the computation of MNS solutions is almost instantaneous, even for $1024 \times 1024$ pixels imagery.

The detected psf $k^{\dagger}(x, y)$ depends on the initial guess $\bar{k}(x, y)$, and $k^{\dagger}(x, y)$ is symmetric, whether or not the true psf is symmetric. However, $\hat{k}^{\dagger}(\xi, \eta)$ in (14) need not be a characteristic function. Although $\int_{R^{2}} k^{\dagger}(x, y) d x d y=1, k^{\dagger}(x, y)$ may develop negative values. Likewise, although $\int_{R^{2}} f^{\dagger}(x, y) d x d y=\int_{R^{2}} \bar{f}(x, y) d x d y, f^{\dagger}(x, y)$ may develop negative values.

If the true psf is a uniform defocus blur with a radius $R_{0}>0$, and $\bar{k}(x, y)$ is selected to be a Gaussian or some other type of zero phase kernel, $k^{\dagger}(x, y)$ will not be a defocus psf in view of (4) and (12). For the same reason, if $\bar{k}(x, y)$ is selected to be a defocus blur with a radius $R_{1} \neq R_{0}, k^{\dagger}(x, y)$ will not be a defocus psf with radius $R_{0}$. Clearly, in general, the MNS algorithm cannot recover the true psf, given a plausible initial guess $\bar{k}(x, y)$. Indeed, for this reason, defocus blur experiments in [21] were unsuccessful.
6. Radial characteristic functions. In fact, the MNS approach fails more generally as will be shown below. A useful starting point for understanding the limitations of the MNS approach involves consideration of the simplest and most common types of blurs, namely, isotropic shift-invariant kernels. A review of the basic properties of isotropic characteristic functions is instructive.

Let $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right)$ be a point in $R^{n}$, and let $\rho=\left(x_{1}^{2}+\cdots+x_{n}^{2}\right)^{1 / 2}$. A realvalued function $\Phi(\mathbf{x})$ on $R^{n}$ is called radial, or isotropic, if $\Phi(\mathbf{x})=\phi(\rho)$. The Fourier transform of a radial function is again radial. While the optical transfer functions associated with isotropic shift-invariant blurring kernels are radial characteristic functions on $R^{2}$, the mathematical theory of characteristic functions is more transparent on $R^{n}$, with $n$ arbitrary.

Fundamental early work on characteristic functions by Bochner [4], Schoenberg [26], and Pólya [24] has spawned a large literature. See [2], [16], [18], [23], [27], and the references therein. The recent monograph [15] includes a useful survey of modern results, together with an extensive bibliography.

Definition 1. A function $\phi:[0, \infty) \rightarrow R$, which lies in $C[0, \infty) \cap C^{\infty}(0, \infty)$, and which satisfies

$$
\begin{equation*}
(-1)^{l} \phi^{(l)}(\rho) \geq 0, \quad \rho>0, l=0,1,2, \ldots \tag{16}
\end{equation*}
$$

is called completely monotone on $[0, \infty)$.
THEOREM $2([26])$. A function $\Phi(\mathbf{x})=\phi\left(\rho^{2}\right)$ is a radial characteristic function on $R^{n}$ for every $n>0$, if and only if $\phi(\rho)$ is positive and completely monotone on $[0, \infty)$, with $\phi(0)=1$.

Examples of such characteristic functions include the family of inverse multiquadrics, $\left\{1+\gamma \rho^{2}\right\}^{-\lambda}, \gamma, \lambda>0$, as well as Gaussians, $\exp \left\{-\alpha \rho^{2}\right\}$, Lorentzians, $\exp \{-\alpha \rho\}$, and, more generally, the family of Lévy stable laws, $\exp \left\{-\alpha \rho^{2 \beta}\right\}$, with $\alpha>0$ and $0<\beta \leq 1$. These functions have support extending over all of $R^{n}$.

The notion of multiply monotone function [15, p. 49] generalizes the Pólya criterion [24], and leads to examples of characteristic functions with compact support.

Definition 2. A function $\phi:[0, \infty) \rightarrow R$, which lies in $C^{k-2}(0, \infty),(k \geq$ $2)$, and is such that $(-1)^{l} \phi^{(l)}(\rho)$ is nonnegative, nonincreasing, and convex for $l=$ $0,1,2, \ldots, k-2$, is called $k$-times monotone on $(0, \infty)$

Theorem 3 ([15]). Let $s$ be a positive integer, let $k=[s / 2]+2$, and let $\phi(\rho)$ be $k$ times monotone on $(0, \infty)$, with $\phi(0)=1$. Then $\Phi(\mathbf{x})=\phi(\rho)$ is a radial characteristic function on $R^{s}$.

The truncated power function $\phi(\rho)=\left\{(1-\alpha \rho)_{+}\right\}^{k}$, where $\alpha>0$ and $k=[s / 2]+2$, satisfies the conditions of Theorem 3. Here, $f(t)_{+}=H(t) f(t)$, where $H(t)$ is the Heaviside unit step function. The truncated power function has compact support, and is a radial characteristic function on $R^{n}$ for $n \leq s$. However, functions satisfying the conditions of Theorem 3 need not have compact support.

Examples of radial characteristic functions not covered by either of the above theorems include oscillatory functions such as $J_{0}(\alpha \rho)$ on $R^{2}$, and $J_{1}(\alpha \rho) /(\alpha \rho)$ on $R^{s}$, for $s \leq 4$. Such functions are characteristic functions on $R^{n}$ only for a restricted range of $n$ and have global support.

Multiplication of any two radial characteristic functions produces a third such object. Hence, radial characteristic functions form a very diverse collection.
7. Infinitely divisible subclass $\mathbf{L}$. The variational blind deconvolution algorithms, [29], [13] and [21], generally aim at reconstructing arbitrary psfs from limited prior information. This seems unlikely, given the variability that already exists in the isotropic case. A more feasible program might be based on a useful, but restricted class of blurs. The family of isotropic Lévy stable laws, with characteristic function $\exp \left\{-\alpha \rho^{2 \beta}\right\}$, where $\alpha>0$ and $0<\beta \leq 1$, includes the ubiquitous Gaussians and Lorentzians, as well as the heavy-tailed densities that characterize a wide variety of electro-optical devices [20], [25], [28]. Complementary behavior is found in the family
of inverse multiquadrics $\left\{1+\gamma \rho^{2}\right\}^{-\lambda}, \gamma, \lambda>0$. The product of finitely many individual members from each of these two families, with appropriately chosen parameter values, may provide useful approximations to the lumped optical transfer functions describing multicomponent imaging systems. With $\lambda_{i}, \alpha_{i}, \gamma_{i} \geq 0, \rho=\left(\xi^{2}+\eta^{2}\right)^{1 / 2}$, $0<\beta_{i} \leq 1$, and arbitrary $N$, consider

$$
\begin{equation*}
\hat{k}_{L}(\xi, \eta) \equiv \phi(\rho)=\exp \left(-\Sigma_{i=1}^{N}\left\{\alpha_{i} \rho^{2 \beta_{i}}+\lambda_{i} \log \left(1+\gamma_{i} \rho^{2}\right)\right\}\right) \tag{17}
\end{equation*}
$$

We may define the class $\mathbf{L}$ to be the class of all objects $\hat{k}_{L}(\xi, \eta)$ of the form (17). This very rich class of isotropic otfs excludes oscillatory otfs, as well as otfs with compact support. In particular, motion and defocus blurs are not included in $\mathbf{L}$. Each $\phi(\rho) \in \mathbf{L}$ is monotone decreasing, has support on all of $R^{2}$, and is everywhere positive. The class $\mathbf{L}$ is a subclass of the class of infinitely divisible characteristic functions [16], [23]. Thus, if $\hat{k}(\xi, \eta) \in \mathbf{L}$, and $n$ is a positive integer, $\{\hat{k}(\xi, \eta)\}^{1 / n}$ is also a characteristic function. This could not happen if $\hat{k}(\xi, \eta)$ were oscillatory or had compact support.

The class $\mathbf{L}$ is much richer than the class $\mathbf{G} \subset \mathbf{L}$, consisting of all objects of the form (17) with $\lambda_{i}=0, i=1, N$. Blind deconvolution based on the class $\mathbf{G}$ has been successfully applied to sharpen blurred imagery from MRI and PET brain scans, from scanning electron microscopes, and from the Hubble space telecope and other earth-bound instruments [9], [10], [11]. In [3], a much smaller class of blurs, isotropic Gaussians, forms the basis for a novel blind approach based on minimizing the Mumford-Shah functional.
8. Spurious characteristic functions and nonphysical MNS psfs. As noted in [21] the MNS procedure described in Theorem 1 is not useful for motion or defocus blurs. Accordingly, this paper focuses exclusively on zero phase kernels, and specifically on class $\mathbf{L}$ otfs. Fourier domain behavior is the key to unlocking the several pertinent facts which are documented below. Using synthetically blurred images with no added noise, we study the MNS procedure under idealized conditions, and underscore the fact that pathological behavior occurs even in the absence of data noise. Images with multiplicative noise are considered in section 14.

Our first example involves a $1024 \times 1024$ pixel USS Kittyhawk image. This was synthetically blurred by Fourier multiplication with a class $\mathbf{L}$ otf to form $\hat{g}(\xi, \eta)$. For $\hat{\bar{k}}(\xi, \eta)$, we selected the isotropic Lévy stable characteristic function given by

$$
\begin{equation*}
\hat{\bar{k}}(\xi, \eta)=\exp \left\{-\alpha\left(\xi^{2}+\eta^{2}\right)^{\beta}\right\}, \quad \alpha=0.20, \beta=0.27 \tag{18}
\end{equation*}
$$

With $\hat{\bar{f}}(\xi, \eta)=\hat{g}(\xi, \eta)$, the algorithm in Theorem 1 produces the detected otf $\hat{k}^{\dagger}(\xi, \eta)$. The 1D slice $\hat{k}^{\dagger}(\xi, 0)$ is shown as the solid curve in Figure 1(A), while the dashed curve represents the initial guess $\hat{\bar{k}}(\xi, 0)$. The detected otf $\hat{k}^{\dagger}(\xi, 0)$ displays nonmonotone behavior at variance with Theorems 2 and 3 , and is not a characteristic function. The associated psf $k^{\dagger}(x, 0)$, shown in Figure 1(B), is highly oscillatory and exhibits sustained negativity, extending over the entire image domain, $-512 \leq x, y \leq 512$. Indeed, while $\int_{R^{2}} k^{\dagger}(x, y) d x d y=1$, the negative part of $k^{\dagger}(x, y)$ integrates to -395 , while the positive part integrates to +396 . Such a psf cannot represent a physical blur, and the physical validity of the deblurred image $f^{\dagger}(x, y)$ is open to question.

Similar nonmonotone otf behavior is found in the vast majority of cases. Some choices for $\hat{\bar{k}}(\xi, \eta)$ produce psfs $k^{\dagger}(x, y)$ with more muted negative excursions, but such negativity is almost always present. The integrated negative part of $k^{\dagger}(x, y)$ is

## A



B

MNS NON PHYSICAL PSF IN BLURRED USS KITTYHAWK IMAGE.


FIG. 1. Failure in MNS approach. (A) $1024 \times 1024$ USS Kittyhawk image was synthetically blurred with monotone decreasing class $\mathbf{L}$ otf. However, MNS detected otf (solid line) displays nonmonotone behavior at variance with Theorems 2 and 3, and is not a characteristic function. (B) Defective MNS otf leads to physically impossible highly oscillatory psf $k^{\dagger}(x, y)$, exhibiting sustained negative values. While $\int_{R^{2}} k^{\dagger}=1$, the negative part of $k^{\dagger}$ integrates to -395 .


Fig. 2. MNS deconvolution experiment in Figure 1. (A) Blurred image $g(x, y)$. (B) Partially deblurred MNS reconstruction $f^{\dagger}(x, y)$, obtained using invalid detected psf $k^{\dagger}(x, y)$. (C) Original sharp image $f(x, y)$. Full deblurring of image $A$ into sharp image $C$ is mathematically impossible in MNS approach.


FIG. 3. Limitations of partial deblurring in MNS become apparent when zooming on selected parts of USS Kittyhawk image. (A) Planes on deck barely identifiable in blurred image. (B) Only modest improvement is provided by MNS deblurring. (C) Same scene in original sharp image $f(x, y)$.
commonly found to have an absolute value exceeding a third of the integrated positive part. In the above MNS experiment, Figure 2(A) is the blurred image $g(x, y)$, and Figure $2(\mathrm{~B})$ is the reconstruction $f^{\dagger}(x, y)$. Figure $2(\mathrm{C})$ is the original sharp image $f(x, y)$. While $f^{\dagger}(x, y)$ appears to be a plausible partial deblurring of $g(x, y)$, the limitations of MNS deconvolution become apparent on zooming over selected parts of the USS Kittyhawk image, as shown in Figure 3. Full deblurring that can retrieve Figures $2(\mathrm{C})$ and $3(\mathrm{C})$ is unattainable with MNS, no matter how $\hat{\bar{k}}(\xi, \eta)$ is chosen. This is discussed next.

## 9. Enforced limited MNS deblurring and Hölder continuity. The MNS

 procedure described in Theorem 1 precludes recovery of the original sharp image $f(x, y)$, given the blurred image $g(x, y)$. Only a partially deblurred image is possible. One of several important consequences of Theorem 1 is the following inequality.$$
\begin{equation*}
|\hat{g}(\xi, \eta)| \leq\left|\hat{f}^{\dagger}(\xi, \eta)\right| \leq|\hat{g}(\xi, \eta)|+|\hat{g}(\xi, \eta)|^{1 / 2} \tag{19}
\end{equation*}
$$

This follows from the polynomial expression $p(r)$ in (15). Since $0 \leq b=|\hat{\bar{k}}| \leq 1$, we have $p(c) \leq 0 \leq p(c+\sqrt{c})$, where $c=|\hat{g}(\xi, \eta)|$. The inequality (19) imposes pointwise
a priori bounds on the Fourier transform of the reconstructed image $f^{\dagger}$, independently of whatever choice was made for $\bar{k}$. The upper bound in (19) is unrealistic whenever the true blurring otf $\hat{k}(\xi, \eta)$ decays exponentially. Indeed, consider a Lorentzian otf $\hat{k}(\xi, \eta)=\exp \left\{-2 \alpha \sqrt{\xi^{2}+\eta^{2}}\right\}$. In that case, $k \otimes f=g$, together with (19), gives

$$
\begin{equation*}
\left|\hat{f}^{\dagger}(\xi, \eta)\right| \leq|\hat{f}(\xi, \eta)| e^{-2 \alpha \sqrt{\xi^{2}+\eta^{2}}}+|\hat{f}(\xi, \eta)|^{1 / 2} e^{-\alpha \sqrt{\xi^{2}+\eta^{2}}} \tag{20}
\end{equation*}
$$

The inequality (20) expresses the relation that exists between the MNS reconstruction $\left|\hat{f}^{\dagger}(\xi, \eta)\right|$ and the true image $|\hat{f}(\xi, \eta)|$. The exponential decay on the right-hand side in (20) indicates that the amplitudes of moderate and high-frequency components in the MNS reconstruction $\hat{f}^{\dagger}(\xi, \eta)$ must remain severely attenuated. These components carry valuable information regarding texture and other small scale structures, but cannot be restored to their true values in $\hat{f}(\xi, \eta)$, no matter how sagacious the choice for $\bar{k}$. This is vividly illustrated in Figure 4 . The sharp $512 \times 512$ USAF chart image $f(x, y)$ on the right is blurred by Fourier multiplication with $\exp \left\{-0.075 \sqrt{\xi^{2}+\eta^{2}}\right\}$, to create the image $g(x, y)$ on the left. The plots below the images show four distinct traces. Trace $A$ is a plot of $\log |\hat{f}(\xi, 0)|$ vs $\xi$. Trace $B$ is a plot of $0.075|\xi|$ vs $\xi$. Trace $C$ is a plot of $\log |\hat{g}(\xi, 0)|$ vs $\xi$. Traces $A$ and $C$ express the very substantial Fourier space differences between images $f$ and $g$. Full deblurring of $g$ must bring trace $C$ in close agreement with trace $A$ over a wide frequency range $|\xi| \leq \xi_{\max }$. That is impossible. Trace $D$ is a plot of $\log \left\{|\hat{g}(\xi, 0)|+|\hat{g}(\xi, 0)|^{1 / 2}\right\}$ vs $\xi$, and represents the upper limit of any possible MNS deblurring.

Estimating $L^{1}$ norms by $L^{2}$ norms on a finite 2D image domain, the pointwise Fourier space inequality (19) implies

$$
\begin{equation*}
\left\|\hat{f}^{\dagger}\right\|_{2}^{2} \leq 2\left\{\|\hat{g}\|_{1}+\|\hat{g}\|_{2}^{2}\right\} \leq C\left\{\|\hat{g}\|_{2}+\|\hat{g}\|_{2}^{2}\right\} \tag{21}
\end{equation*}
$$

Using Parseval's formula, this gives

$$
\begin{equation*}
\left\|f^{\dagger}\right\|_{2} \leq C\left\{\|g\|_{2}+\|g\|_{2}^{2}\right\}^{1 / 2} \tag{22}
\end{equation*}
$$

If the MNS deconvolution process were linear, the above inequality would remain valid for the difference of two reconstructions, $f_{1}^{\dagger}, f_{2}^{\dagger}$, corresponding to slightly different inputs, $g_{1}, g_{2}$. Assuming this to be the case, if $\left\|g_{1}-g_{2}\right\|_{2} \leq \epsilon$ for small $\epsilon>0$, (22) implies

$$
\begin{equation*}
\left\|f_{1}^{\dagger}-f_{2}^{\dagger}\right\|_{2} \leq C \sqrt{\epsilon} \tag{23}
\end{equation*}
$$

It follows that $f^{\dagger}$ depends Hölder-continuously on $g$ with exponent $1 / 2$. In reality, MNS deconvolution is highly nonlinear. In [21], substantial effort is necessary to establish the inequality (22) for the difference of any two solutions, and obtain the result in (23).

The fact that MNS blind deconvolution has a Hölder exponent of $1 / 2$ does not imply that the blind deconvolution problem is less ill-posed than the nonblind problem. The next section discusses an ill-posed continuation problem for the heat equation for which Hölder continuity holds. Positive Hölder exponents are possible only if continuation is terminated prior to reaching the boundary of the continuation region. That ill-posed heat flow problem is mathematically equivalent to deblurring Gaussian blurred imagery. A more general ill-posed parabolic problem is associated with class $\mathbf{L}$ blurs. In both cases, positive exponents imply partial deblurring. Clearly, the positive Hölder exponent in MNS blind deconvolution simply reflects the fact that only limited deblurring is possible using (13)-(15) in Theorem 1.


MNS DEBLURRING LIMTT IN $512 X 512$ USAF CHART IMAGE.


Fig. 4. Limited MNS deblurring enforced by Theorem 1. Blurred $512 \times 512$ USAF chart image $g(x, y)$ (left) was obtained by Fourier space multiplication of sharp image $f(x, y)$ (right), with Lorentzian otf $\exp \left\{-0.075 \sqrt{\xi^{2}+\eta^{2}}\right\}$. Trace $A$ is a plot of $\log |\hat{f}(\xi, 0)|$ vs $\xi$. Trace $B$ is a plot of $0.075|\xi|$ vs $\xi$. Trace $C$ is a plot of $\log |\hat{g}(\xi, 0)|$ vs $\xi$. Full deblurring of $g(x, y)$ must bring trace $C$ in close agreement with trace A. However, trace D, which is a plot of $\log \left\{|\hat{g}(\xi, 0)|+|\hat{g}(\xi, 0)|^{1 / 2}\right\}$ vs $\xi$, represents the upper limit of any possible MNS deblurring. See (19), (20).
10. Ill-posed continuation, Hölder continuity, and class $L$ blurs. The classical well-posed problems in linear evolution equations exhibit continuous dependence on the initial data, typically expressed by

$$
\begin{equation*}
\|w(., t)\| \leq C\|w(., 0)\|, \quad 0<t \leq T \tag{24}
\end{equation*}
$$

where $w(., t)$ denotes the difference of any two solutions at time $t$, and $C$ is a constant that may depend on $T$. The above inequality implies Lipschitz continuous dependence on the data in the Banach space norm $\|\|$. Two solutions that differ by $\epsilon$ in norm initially will differ by no more than $C \epsilon$ on the interval $[0, T]$.

Stabilized ill-posed continuation problems are generally characterized by Hölder continuity with respect to the data [1], [7]. A typical example is backwards in time continuation in the heat equation from data at time $t=1$ [1, pp. 17-20]. Here, the corresponding result is

$$
\begin{equation*}
\|w(., t)\|_{2} \leq 2\|w(., 0)\|_{2}^{1-t}\|w(., 1)\|_{2}^{t}, \quad 0 \leq t \leq 1 \tag{25}
\end{equation*}
$$

for the $L^{2}$ difference of any two solutions at time $t$. We may stabilize the backwards problem by prescribing an a priori $L^{2}$ bound $M$ for the desired solution at time $t=0$. If two such solutions differ in norm by $\epsilon$ at time $t=1$, they differ by no more than $2 M^{1-t} \epsilon^{t}$ on the interval $[0, T]$. The Hölder exponent tends to zero as $t \downarrow 0$.

Image deblurring with Gaussian psfs is mathematically equivalent to continuation backwards in time in a heat conduction equation where the conduction coefficient is proportional to the Gaussian spread. More generally, we may consider class $\mathbf{L}$ otfs $\hat{k}_{L}(\xi, \eta)$ as in (17). Here, with $\sigma_{i}=\alpha_{i} /\left(4 \pi^{2}\right)^{\beta_{i}}, \delta_{i}=\gamma_{i} / 4 \pi^{2}, i=1, N$, the associated linear evolution equation is

$$
\begin{equation*}
u_{t}=-A u \equiv-\left\{\Sigma_{i=1}^{N} \sigma_{i}(-\Delta)^{\beta_{i}}+\lambda_{i}\left\{\log \left(I-\delta_{i} \Delta\right)\right\}\right\} u, \quad t>0 \tag{26}
\end{equation*}
$$

This is a well-posed forward parabolic equation in that the linear pseudo-differential operator $-A$ in (26) is the infinitesimal generator of a holomorphic semigroup on $L^{2}\left(R^{2}\right)$ [5]. When $\hat{k}_{L}(\xi, \eta)$ is known explicitly, solving the image deconvolution problem $k_{L}(x, y) \otimes f(x, y)=g(x, y)$ is mathematically equivalent to solving the evolution equation (26) backwards in time, using the blurred image $g(x, y)$ as data at time $t=1$. The fully deblurred sharp image $f(x, y)$ corresponds to $u(x, y, 0)$, while $u(x, y, t)$, the solution at some intermediate time $t, 0<t<1$, corresponds to the partially deblurred image $f_{t}(x, y)$. The Hölder continuity result in (26) is identical to that for the heat equation in (25). Thus, if $f(x, y)$ satisfies the prescribed $L_{2}$ bound $M$, and if $\epsilon>0$ is an $L^{2}$ bound for the noise in the blurred image $g(x, y)$, then any two partially deblurred images $f_{t}^{1}(x, y), f_{t}^{2}(x, y)$ satisfy

$$
\begin{equation*}
\left\|f_{t}^{1}-f_{t}^{2}\right\|_{2} \leq 2 M^{1-t} \epsilon^{t}, \quad 0 \leq t \leq 1 \tag{27}
\end{equation*}
$$

For class $\mathbf{L}$ blurs, the partially deblurred image $f_{t}(x, y)$ depends Hölder-continuously on the blurred data $g(x, y)$ with exponent $t>0$. Such partial deblurring is clearly less ill-conditioned than full deblurring, where the exponent is zero.
11. Erroneous MNS Fourier behavior at high frequencies. In the case of $H^{1}$ blind deconvolution discussed in section 3, the governing variational principle creates a false relationship between $\left|\hat{k}^{H}(\xi, \eta)\right|$ and $\left|\hat{f}^{H}(\xi, \eta)\right|$, as shown in (7). In section 4, dealing with total variation blind deconvolution, experimental evidence strongly suggests a deleterious spurious coupling between $k^{V}(x, y)$ and $f^{V}(x, y)$. The MNS variational principle likewise induces a false relationship at high frequencies between $\left|\hat{k}^{\dagger}(\xi, \eta)\right|$ and $\left|\hat{f}^{\dagger}(\xi, \eta)\right|$.

We restrict attention to class $\mathbf{L}$ otfs as in (17). With $\rho=\left(\xi^{2}+\eta^{2}\right)^{1 / 2}$, such otfs have the form $\exp (-\omega(\rho))$, where $\omega \uparrow \infty$ as $\rho \uparrow \infty$. It follows that $|\hat{g}(\xi, \eta)| \ll 1$ for large $\rho$. Returning to the inequality (19), namely,

$$
\begin{equation*}
|\hat{g}(\xi, \eta)| \leq\left|\hat{f}^{\dagger}(\xi, \eta)\right| \leq|\hat{g}(\xi, \eta)|+|\hat{g}(\xi, \eta)|^{1 / 2} \tag{28}
\end{equation*}
$$

we first note that $\left|\hat{f}^{\dagger}(\xi, \eta)\right|=0$ if and only if $|\hat{g}(\xi, \eta)|=0$. If $\left|\hat{f}^{\dagger}(\xi, \eta)\right| \neq 0$,

$$
\begin{equation*}
\frac{|\hat{g}(\xi, \eta)|}{\left|\hat{f}^{\dagger}(\xi, \eta)\right|}=\left|\hat{k}^{\dagger}(\xi, \eta)\right| \geq \frac{|\hat{g}(\xi, \eta)|^{1 / 2}}{1+|\hat{g}(\xi, \eta)|^{1 / 2}} \tag{29}
\end{equation*}
$$

In (29), the lower bound for $\left|\hat{k}^{\dagger}(\xi, \eta)\right|$ remains valid even if $\left|\hat{f}^{\dagger}(\xi, \eta)\right|=0$. Consider values of $\rho$ such that $|\hat{g}(\xi, \eta)|^{1 / 2} \ll 1$. Then, $|\hat{g}(\xi, \eta)| \ll|\hat{g}(\xi, \eta)|^{1 / 2}$, and hence, from (28), (29),

$$
\begin{equation*}
\left|\hat{k}^{\dagger}(\xi, \eta)\right| \geq|\hat{g}(\xi, \eta)|^{1 / 2}+o(1) \geq\left|\hat{f}^{\dagger}(\xi, \eta)\right|, \quad \rho \uparrow \infty \tag{30}
\end{equation*}
$$

The inequalities (30) remain valid independently of the choice for $\hat{\bar{k}}(\xi, \eta)$. However, the choice of $\hat{\bar{k}}(\xi, \eta)$ also plays a role in that we always have [21],

$$
\begin{equation*}
\left|\hat{k}^{\dagger}(\xi, \eta)\right| \geq|\hat{\bar{k}}(\xi, \eta)| \tag{31}
\end{equation*}
$$

Combining (30) and (31), we find

$$
\begin{equation*}
\left|\hat{k}^{\dagger}(\xi, \eta)\right| \geq \max \left\{|\hat{\bar{k}}(\xi, \eta)|,|\hat{g}(\xi, \eta)|^{1 / 2}+o(1)\right\} \geq\left|\hat{f}^{\dagger}(\xi, \eta)\right|, \quad \rho \uparrow \infty \tag{32}
\end{equation*}
$$

Such high frequency behavior is the complete opposite of what is found in practice. Indeed, for most natural images $f(x, y)$ with realistic class $\mathbf{L}$ blurs $k_{L}(x, y)$, the relation $k_{L} \otimes f=g$ implies

$$
\begin{equation*}
\left|\hat{k}_{L}(\xi, \eta)\right| \ll|\hat{g}(\xi, \eta)|^{1 / 2} \ll|\hat{f}(\xi, \eta)|, \quad \rho \uparrow \infty \tag{33}
\end{equation*}
$$

This is exemplified in Figure 4, where, at high frequencies, trace $B$ (otf) lies well below trace $D\left(|\hat{g}(\xi, \eta)|^{1 / 2}\right)$, which lies well below trace $A(|\hat{f}(\xi, \eta)|)$. Clearly, high frequency behavior in the MNS pair $\left\{\hat{f}^{\dagger}(\xi, \eta), \hat{k}^{\dagger}(\xi, \eta)\right\}$ is deceptive and untrustworthy.
12. Recovery using APEX/SECB techniques. The MNS otf $\hat{k}^{\dagger}(\xi, \eta)$ does not appear to play an explicit role in the solution process. Rather, as is clear from Theorem 1, given $\hat{g}(\xi, \eta)$ and $\hat{k}(\xi, \eta)$, the MNS procedure obtains $\hat{f}^{\dagger}(\xi, \eta)$ directly by solving the polynomial equation (15). The determination of $\hat{k}^{\dagger}(\xi, \eta)$ follows from the already constructed $\hat{f}^{\dagger}(\xi, \eta)$ by considering $\hat{k}^{\dagger}(\xi, \eta) \hat{f}^{\dagger}(\xi, \eta)=\hat{g}(\xi, \eta)$. When the MNS procedure results in a questionable partially deblurred image associated with a physically impossible point spread function, there is no recourse available. We arrive at a mathematical dead end.

A way out of this impasse is possible in the case of class $\mathbf{L}$ blurs, and it involves an approach to deconvolution that is distinctly different from the variational formulation in [21]. A key element is the infinite divisibility of class $\mathbf{L}$ otfs. For such otfs, the deblurring problem can be reformulated into the equivalent mathematical problem of solving the parabolic equation (26) backwards in time, using the blurred image as data at $t=1$. Extensions of previously developed APEX/SECB methodologies, [6], [8], [9], [10], provide the necessary computational tools to make such ill-posed continuation feasible. In particular, the improved backwards stability provided by the SECB constraint [6], [7], [22] plays an essential role. In a blind deconvolution context, an evolutionary approach provides significant new capabilities, as will be seen below.

Given that the image was blurred by a class $\mathbf{L}$ otf, we now view the nonphysical MNS detected otf $\hat{k}^{\dagger}(\xi, \eta)$ as a potentially salvageable failed attempt to detect the true otf $\hat{k}_{L}(\xi, \eta)$. The next several steps are informed by the preceding analysis.

First, we must discard the unreliable high-frequency portion of $\hat{k}^{\dagger}(\xi, \eta)$, as indicated by (32). On the finite interval $\rho \leq \rho^{\#}$, we best fit $\hat{k}^{\dagger}(\xi, \eta)$ with the expression on the right-hand side of (17), using nonlinear least squares. More precisely, the least squares fitting is applied to the natural logarithm of $\hat{k}^{\dagger}(\xi, \eta)$. Here, $\rho^{\#}$ is an adjustable constant, subject to subsequent fine-tuning. The fitted parameter values $\alpha_{i}^{\#}, \beta_{i}^{\#}, \lambda_{i}^{\#}$, and $\gamma_{i}^{\#}, i=1$, and $N^{\#}$ are used to define a new class $\mathbf{L}$ otf $\hat{h}^{\#}(\xi, \eta)$, which is then assumed valid for all $\rho>0$. The use of $\hat{h}^{\#}(\xi, \eta)$ in a nonblind deconvolution algorithm results in a physically valid deblurred image $f^{\#}(x, y)$ that can replace the faulty MNS image $f^{\dagger}(x, y)$. This first step is already a significant improvement of [21].

We can go a good deal further if we properly interpret $f^{\#}(x, y)$ as a partially deblurred image associated with (26). In that context, for some real number $p>1$, we consider $f^{\#}(x, y)$ to be the solution of $(26)$ at some fixed time $t^{\#}=1-1 / p$, well away from $t=0$. Comparing the Hölder continuity result for $f^{\dagger}(x, y)$ in (23), with the Hölder continuity property (27) that is characteristic of the self-adjoint parabolic equation (26), we conclude that $t^{\#}=1-1 / p \geq 1 / 2$. Thus, $p \geq 2$. Accordingly, we now hypothesize $\hat{h}^{\#}(\xi, \eta)$ to be the $p^{t h}$ root of the true otf $\hat{k}_{L}(\xi, \eta)$,

$$
\begin{equation*}
\hat{k}_{L}(\xi, \eta) \equiv\left\{\hat{h}^{\#}(\xi, \eta)\right\}^{p}, \quad p \geq 2 . \tag{34}
\end{equation*}
$$

From (17) and (34), $\hat{k}_{L}(\xi, \eta)$ has the parameters $\alpha_{i}=p \alpha_{i}^{\#}, \beta_{i}=\beta_{i}^{\#}, \lambda_{i}=p \lambda_{i}^{\#}$, and $\gamma_{i}=\gamma_{i}^{\#}, i=1, N^{\#}$. As was the case with $\rho^{\#}$, the real number $p$ is subject to subsequent fine-tuning. Also, as shown in Example 4 below, extending the definition of $\hat{h}^{\#}(\xi, \eta)$ to all $\rho>0$ need not produce a useful $p^{t h}$ root candidate.

Given $\hat{k}_{L}(\xi, \eta)$ and $\hat{g}(\xi, \eta)$, the "slow evolution from the continuation boundary" method (SECB), is a well-regularized, fast, direct FFT method for solving the parabolic equation (26) backwards in time, [6], [7]. The SECB method approximates the solution $u(x, y, t)$ of $(26)$, with $u^{S}(x, y, t)$, defined as follows in Fourier space

$$
\begin{equation*}
\hat{u}^{S}(\xi, \eta, t)=\frac{\hat{k}_{L}^{t}(\xi, \eta) \hat{k}_{L}^{*}(\xi, \eta) \hat{g}(\xi, \eta)}{\left|\hat{k}_{L}(\xi, \eta)\right|^{2}+K^{-2}\left|1-\hat{k}_{L}^{s}(\xi, \eta)\right|^{2}}, \quad 0 \leq t<1 . \tag{35}
\end{equation*}
$$

Here, $\hat{k}_{L}^{*}$ denotes the complex conjugate of $\hat{k}_{L}$. The positive constants $K$ and $s$, with $s \ll 1$, are regularization parameters [8]. Typical values used in section 13 are $K=1000, s=0.001$, reflecting synthetically blurred imagery with slight noise. In the noisy images in section $14, K$ is reduced to 10 . Note that $\hat{k}_{L}^{t}(\xi, \eta) \equiv\left\{\hat{k}_{L}(\xi, \eta)\right\}^{t}$ is well-defined for any $t>0$, and is a characteristic function. An inverse FFT in (35) returns $u^{S}(x, y, t)$. All negative values in $u^{S}(x, y, t)$ are reset to the value 0 .

The approximation (35) incorporates a stabilizing "slow evolution" constraint that is significantly more effective at suppressing the growth of noise as $t \downarrow 0$ than is the case with the prescribed $L^{2}$ bound $M$ constraint used in (27). This is clearly demonstrated in [6, Figures 4 and 5], and in [7], for a much wider class of problems. Further new observations regarding SECB are reported in [22]. In contrast to the Hölder estimate (27) which degenerates at $t=0$, the SECB constraint maintains continuity at $t=0$. In the case of (35), if $\epsilon>0$ is an $L^{2}$ bound for the uncertainty in the input data
$g(x, y)$ at $t=1$, the difference of any two solutions $u_{1}^{S}(x, y, t), u_{2}^{S}(x, y, t)$, satisfies

$$
\begin{equation*}
\left\|u_{1}^{S}(., t)-u_{2}^{S}(., t)\right\|_{2} \leq 2 \sqrt{2} \Gamma^{1-t} \epsilon, \quad 0 \leq t \leq 1, \tag{36}
\end{equation*}
$$

where $\Gamma \ll M / \epsilon$ is a computable constant.
With tentative initial values for $\rho^{\#}$ and $p$ in (34), we proceed to evaluate the resulting deconvolution, readjusting parameter values as required. In this exploratory APEX phase, great benefit derives from the ability to perform the deconvolution in slow motion. As $t \downarrow 0$ the partial restorations $u^{S}(x, y, t)$ become sharper and noisier; ringing and other artifacts may appear, possibly indicating that continuation has proceeded too far. Displaying the evolution of $u^{S}(x, y, t)$ as $t$ decreases from 1 to 0 allows for monitoring the deblurring process and selecting an optimal image, which may occur at some $\bar{t}>0$. Terminating continuation at $\bar{t}>0$ is equivalent to resetting $p$ to the value $\bar{p}=(1-\bar{t}) p$, and then selecting the image at $t=0$ as optimal. Diagnostic statistical information about $u^{S}(x, y, t)$ can also be calculated for selected values of $t$ as $t \downarrow 0$. Of particular interest are the discrete $L^{1}$ norm, $\left\|u^{S}(., t)\right\|_{1}$, and the discrete "total variation" norm $\left\|u^{S}(., t)\right\|_{T V}$. If $\hat{k}_{L}(\xi, \eta)$ in (34), (35) were the actual true otf, the deconvolution process would be well-behaved, image flux would be conserved, and $\left\|u^{S}(., t)\right\|_{1}$ would remain constant as $t \downarrow 0$. At the same time, $\left\|u^{S}(., t)\right\|_{T V}$ would increase monotonically, reflecting the gradual sharpening of edges and other localized singularities as $t \downarrow 0$. In practice, the true otf is seldom found, and the image $L^{1}$ norm may show a modest increase as $t \downarrow 0$. As emphasized in [10, Figure 1], given an image blurred with a class $\mathbf{L}$ otf, there are in general infinitely many distinct otfs $\hat{k}_{L}(\xi, \eta)$ that can competently deblur that image. We may enforce conservation of $L^{1}$ norm in $u^{S}(x, y, t)$, for any desired $t$, by rescaling $\left\|u^{S}(., t)\right\|_{1}$ to the value $\|g\|_{1}$.
13. Some examples. All examples in this paper involve 8 -bit images, blurred synthetically by Fourier space multiplication with class $\mathbf{L}$ otfs. No noise is added in the examples in this section. Images with multiplicative noise are considered in section 14. In applying the MNS algorithm to these images, the choice for $\hat{\bar{k}}(\xi, \eta)$ is the same as in (18), with various values of $\alpha, \beta$.

1. Uranus and its moons. The MNS algorithm was applied to a $512 \times 512$ blurred Uranus image. Spurious results were obtained, similar to those in the USS Kittyhawk experiment in Figure 1. Only that portion of the false MNS otf $\hat{k}^{\dagger}(\xi, \eta)$ on $\rho \leq \rho^{\#}=50$ was retained. This is shown as the solid curve in Figure 5, together with the blurred image. The remaining portion, on $50<\rho \leq 512$, was deemed unreliable and discarded. A nonlinear least squares fit on $\rho \leq \rho^{\#}$, with the expression on the right of (17) with $N=1$, produced the candidate $\hat{h}^{\#}(\rho)$ with parameters $\alpha^{\#}=0.00233511, \beta^{\#}=0.609951, \lambda^{\#}=0.798301$, and $\gamma^{\#}=0.0234441$. This is shown as the dashed curve in Figure 5, and this definition of $\hat{h}^{\#}(\rho)$ was assumed valid for all $\rho>0$. Note that the nonlinear least squares fitting is applied to the logarithm of $\hat{k}^{\dagger}(\xi, \eta)$ as indicated in Figure 5 .

The physically valid otf $\hat{h}^{\#}(\xi, \eta)$ provides a rectification of the false MNS otf $\hat{k}^{\dagger}(\xi, \eta)$. If we use $\hat{h}^{\#}(\xi, \eta)$ in lieu of $\hat{k}_{L}(\xi, \eta)$ in (35) with $t=0$, we obtain $f^{\#}(x, y)$, a physically valid replacement for the MNS image $f^{\dagger}(x, y)$.

It is more fruitful to identify the partially deblurred image $f^{\#}(x, y)$ with $u^{S}(x, y, t)$ in (35), evaluated at $t^{\#}=1-1 / p>0$. Choosing $p=2.5, K=3000, s=0.0005$, and letting $\hat{k}_{L}(\xi, \eta) \equiv\left\{\hat{h}^{\#}(\xi, \eta)\right\}^{p}$ in (35), we explore the evolution of $u^{S}(x, y, t)$ as $t$ decreases from 1 to 0 , while monitoring diagnostic statistical information. Conservation of $L^{1}$ norm is enforced at each $t$ value by rescaling $\left\|u^{S}(., t)\right\|_{1}$ to the value

LEASt SQUARES FIT of FALSE MNS otF WITH CLASS L otf


Fig. 5. Blurred Uranus image experiment. Least squares fit of false MNS otf $\hat{k}^{\dagger}(\xi, 0)$ (solid curve) with class $\mathbf{L}$ otf $\hat{h}^{\#}(\xi, 0)$ (dashed curve) on the interval $|\xi| \leq \rho^{\#}=50.0$. This leads to $2 D$ otf $\hat{h}^{\#}(\rho)=\exp \left(-\alpha \rho^{2 \beta}-\lambda \log \left(1+\gamma \rho^{2}\right)\right)$, with $\alpha=0.00233511, \beta=0.609951, \lambda=0.798301, \gamma=$ 0.0234441 , valid for all $\rho=\left(\xi^{2}+\eta^{2}\right)^{1 / 2}$. MNS partial deblurring property suggests $\hat{h}^{\#}(\rho)$ is some fractional power of true Uranus image otf $\hat{k}_{L}(\rho)$. Here, we hypothesize $\hat{k}_{L}(\rho)=\left\{\hat{h}^{\#}(\rho)\right\}^{2.5}$.
$\|g\|_{1}$, and all displayed images in Figure 6 have the same $L^{1}$ norm. However, prior to such rescaling, there is a $23 \%$ increase in $\left\|u^{S}(., t)\right\|_{1}$, from 9.3 at $t=1$ to 11.4 at $t=0.05$. This is accompanied by a thirtyfold increase in $\left\|u^{S}(., t)\right\|_{T V}$. Such large increases typically signal the development of noise and ringing artifacts.

Indeed, such artifacts are quite pronounced near $t=0.05$ in the original full-size images, but are much less visible in reduced size on the printed page. Nevertheless, this example leads to the remarkable sequence exhibited in Figure 6, where, beginning with four barely visible moons at $t=1$, we end up with 11 moons at $t=0.05$ ! The rectified MNS deblurred image corresponds to $t=0.6$. We may elect to terminate continuation at $t=0.15$, where noise and ringing artifacts are less evident. There is a more moderate $8 \%$ increase in $\left\|u^{S}(., t)\right\|_{1}$ from $t=1$ to $t=0.15$. Ten moons are still clearly visible at $t=0.15$.
2. Marilyn Monroe image. The MNS algorithm was applied to a $1024 \times 1024$ blurred Marilyn Monroe image. The detected MNS otf $\hat{k}^{\dagger}(\xi, \eta)$ again exhibited false nonmonotone behavior. The corresponding psf $k^{\dagger}(x, y)$ was highly oscillatory and nonphysical, with a negative part that integrated to -2785 , while the positive part integrated to +2786 . The MNS deblurred image $f^{\dagger}(x, y)$, is a plausible partially deblurred image, although it is devoid of meaning. Initial attempts to fit $\hat{k}^{\dagger}(\xi, \eta)$ on $\rho \leq \rho^{\#}=50$, with the expression on the right of (17) with $N=1$, were unsuccessful.

SECB CONTINUATION IN URANUS MOONS IMAGE.


Fig. 6. SECB deblurring of blurred Uranus image by solving parabolic equation in Eq. (26) backwards in time, using Eq. (35) with $\hat{k}_{L}(\xi, \eta)$ identified in Figure 5, and regularization parameters $K=3000, s=0.0005$. Original blurred image is data at time $t=1.0$, while MNS partially deblurred image corresponds to solution at time $t=0.6$. Here, $A P E X / S E C B$ intervention enables ill-posed continuation all the way to $t=0$. Despite the presence of noise and ringing artifacts as tends to zero, 11 moons become visible at small values of $t$, compared to 5 moons at $t=0.6$. Uranus has 27 known moons.

Success was achieved after dropping the logarithmic term in (17). This produced the candidate $\hat{h}^{\#}(\rho)$ with parameters $\alpha^{\#}=0.05664, \beta^{\#}=0.439202$, and $\lambda^{\#}=0$, and this was assumed valid for all $\rho \geq 0$. We chose $p=3$, and let $\hat{k}_{L}(\xi, \eta) \equiv\left\{\hat{h}^{\#}(\xi, \eta)\right\}^{3}$ in (35).

A recognizable facial image exhibits subtle characteristics that are not apparent in images of inanimate objects, such as buildings, warships, or galaxies. As such, a familiar face image is a good vehicle for illustrating the shortcomings of blind deconvolution, even when the algorithm is highly successful. With $K=1000$ and $s=0.001$, the evolution of $u^{S}(x, y, t)$ as $t$ decreases from 1 to 0 is shown in Figure 7. The image at $t=0.67$ corresponds to $f^{\#}(x, y)$, the rectified version of the MNS partially deblurred image. Evidently, the APEX/SECB intervention provides considerable further sharpening, and fully deblurs the image. Conservation of $L^{1}$ norm was enforced at each displayed $t$ value, by rescaling $\left\|u^{S}(., t)\right\|_{1}$ to the value $\|g\|_{1}$. However, prior to rescaling, there is a $3 \%$ increase in $\left\|u^{S}(., t)\right\|_{1}$, from 107.4 at $t=1$ to 110.2 at $t=0$. This is accompanied by a thirteenfold increase in $\left\|u^{S}(., t)\right\|_{T V}$. The otf $\hat{k}_{L}(\xi, \eta)$ that produced Figure 7 was not the same as the one that synthetically blurred the Marilyn Monroe image.


Fig. 7. SECB deblurring of blurred Marilyn Monroe image by solving parabolic equation in Eq. (26) backwards in time, using Eq. (35) with $\hat{k}_{L}(\xi, \eta)$ identified using a process similar to Figure 5, and regularization parameters $K=1000, s=0.001$. Original blurred image is data at time $t=1.0$, while MNS partially deblurred image corresponds to $t=0.67$. APEX/SECB intervention continues solution to $t=0$. Here, best results occur near $t=0.2$. Unflattering oversharpening develops at smaller values of $t$.

While no ringing or noise arifacts are visible at $t=0$ in the reduced Figure 7, the full-size image exhibits unflattering oversharpening effects. A more faithful image is found at $t=0.2$, compared to which, the person at $t=0$ appears to be an older lookalike. This suggests resetting $p$ to the value $\bar{p}=0.8 \times p=2.4$. Indeed, this leads to a more realistic fivefold increase in $\left\|u^{S}(., t)\right\|_{T V}$. We perceive here one of the major difficulties in blind deconvolution. A simple mathematical characterization of the true image is needed, one that can easily be implemented computationally as an a priori constraint. That constraint should lead to a cost functional that is easily minimized, while having the true image, and not a look-alike, as its unique minimum. Such an algorithm has not yet been found. The APEX/SECB evolutionary approach allows continuation to be terminated at a user-determined optimal point. The selection of the optimal image may involve complex and possibly subjective prior knowledge, not necessarily expressible in mathematical terms.
3. USS Kittyhawk image. We now consider improving the partially deblurred MNS image in Figures 2(B) and 3(B). The false MNS otf in Figure 1(A) was restricted to the interval $\rho \leq \rho^{\#}=50$, and a nonlinear least squares fit to the expression on the right of (17) with $N=1$, was found. This produced $\hat{h}^{\#}(\rho)$ with parameters $\alpha^{\#}=0.001469, \beta^{\#}=0.733929, \lambda^{\#}=0.463893$, and $\gamma^{\#}=0.00022$, assumed valid for
all $\rho \geq 0$. With $K=1000, s=0.001, p=3.5$, and $\hat{k}_{L} \equiv\left\{\hat{h}^{\#}\right\}^{p}$ in (35), $\left\|u^{S}(., t)\right\|_{1}$ is conserved as $t$ decreases from $t=1$ to $t=0.05$. However, there is an unrealistically large increase in $\left\|u^{S}(., t)\right\|_{T V}$. In fact, the image at $t=0.05$ is seriously affected by ringing artifacts. Reducing the value of $\beta^{\#}$ to $0.9 \beta^{\#}$ in (35), while leaving all other parameter values unchanged, produces significantly better results. We again have $L^{1}$ norm conservation at $t=0.05$, with a ninefold increase in $\left\|u^{S}(., t)\right\|_{T V}$. Considerable improvement becomes apparent on zooming in on the control tower in the island part of Kittyhawk's deck. The original blurred image is shown as the leftmost image in the first row in Figure 8, while the middle image is the unrectified MNS partially deblurred image. The rightmost image is the SECB image at $t=0.05$.
4. Alphanumeric image. The blurred $512 \times 512$ alphanumeric image, shown as the leftmost image in the middle row in Figure 8, is an important example. Here, the initially obtained $\hat{h}^{\#}(\rho)$ was not useful. Applying the MNS algorithm produces a nonmonotone false otf. The corresponding psf is oscillatory, with sustained negativity as in Figure 1(B). The negative part of $k^{\dagger}(x, y)$ integrates to -160 , while the positive part integrates to +161 . The partially deblurred MNS image $f^{\dagger}(x, y)$ is shown as the middle image in Figure 8. The false MNS otf was again restricted to the interval $\rho \leq$ $\rho^{\#}=50$, and a nonlinear least squares fit to the expression on the right of (17) with $N=1$ was found. This produced $\hat{h}^{\#}(\rho)$ with parameters $\alpha^{\#}=0.000362529, \beta^{\#}=$ $0.80643992, \lambda^{\#}=0.756433752$, and $\gamma^{\#}=0.005456718$. As in previous cases, this definition of $\hat{h}^{\#}(\rho)$ was assumed valid for all $\rho \geq 0$. However, with $\hat{k}_{L} \equiv\left\{\hat{h}^{\#}\right\}^{p}$ in (35), only slight deblurring was achieved at $t=0$, even with $p$ values as high as $p=5$. An alternative least squares fit $\hat{h}_{1}^{\#}(\rho)$ on $\rho \leq \rho^{\#}=50$ was sought, without the logarithmic term in (17). This produced the parameters $\alpha_{1}^{\#}=0.0387745, \beta_{1}^{\#}=$ 0.523677 , and $\lambda_{1}^{\#}=0$. Again, this was assumed valid for all $\rho \geq 0$. While $\log \left\{\hat{h}^{\#}(\rho)\right\}$ and $\log \left\{\hat{h}_{1}^{\#}(\rho)\right\}$ roughly coincide on $\rho \leq \rho^{\#}=50$, the function $\log \left\{\hat{h}_{1}^{\#}(\rho)\right\}$ decays much faster on $\rho>50$. Now, with $K=1000, s=0.001, p=2$, and $\hat{k}_{L} \equiv\left\{\hat{h}_{1}^{\#}\right\}^{2}$ in (35), full deblurring is achieved at $t=0$, as shown in the rightmost image in the middle row of Figure 8. Prior to rescaling, we find a $6.6 \%$ increase in $\left\|u^{S}(., t)\right\|_{1}$, from 8.809 at $t=1$ to 9.388 at $t=0$. This is accompanied by a sixfold increase in $\left\|u^{S}(., t)\right\|_{T V}$. Clearly, in this example, extending the definition of the initial $\hat{h}^{\#}(\rho)$ to all $\rho \geq 0$ did not produce a viable $p^{\text {th }}$ root for the true otf, because $\log \left\{\hat{h}^{\#}(\rho)\right\}$ decayed too slowly on $\rho>50$.
5. Pyramids image. The MNS algorithm was applied to a $512 \times 512$ blurred Pyramids image, shown as the leftmost image in the last row in Figure 8. While the MNS otf $\hat{k}^{\dagger}(\xi, \eta)$ exhibited the usual false nonmonotone behavior, a plausible partially deblurred image was obtained, as shown in the middle of the last row in Figure 8. Considerable further sharpening is possible with APEX/SECB intervention. Least squares fitting of $\hat{k}^{\dagger}$ on $\rho \leq \rho^{\#}=50$ produced $\hat{h}^{\#}(\rho)$ with parameters $\alpha^{\#}=$ $0.011654282, \beta^{\#}=0.67079376, \lambda^{\#}=0.0282072$, and $\gamma^{\#}=2.789097 \times 10^{-7}$. With $K=1000, s=0.001, p=2.5$, and $\hat{k}_{L}(\xi, \eta) \equiv\left\{\hat{h}^{\#}(\xi, \eta)\right\}^{p}$ in $(35), u^{S}(x, y, t)$ is quite well-behaved as $t$ decreases from 1 to 0 . One has conservation of $\left\|u^{S}(., t)\right\|_{1}$ along with a sixfold increase in $\left\|u^{S}(., t)\right\|_{T V}$. The SECB image at $t=0$ is the rightmost image in the last row of Figure 8.
14. Images with multiplicative noise. The examples in the preceding section were reconsidered in the presence of noise. Each blurred $N \times N$ image $g(x, y)$ was replaced by the noisy image $g_{n}(x, y)=g(x, y)+0.001 \sigma(x, y) g(x, y)$, where $\sigma(x, y)$ is an $N \times N$ array of uniformly distributed random numbers in the range $[-1,1]$. Such nonlinear multiplicative noise can significantly curtail the reconstruction of fine scale


Fig. 8. APEX/SECB intervention as discussed in Examples 3-5 produces striking improvements of MNS results. Leftmost column contains synthetically blurred images $g(x, y)$. Middle column contains unrectified MNS partially deblurred images $f^{\dagger}(x, y)$. Rightmost column contains the improved APEX/SECB images. First row: Zooming on control tower in USS Kittyhawk image in Figure 2 (Example 3). Second row: Alphanumeric image (Example 4). Last row: Pyramids image (Example 5).
structure in many deconvolution methods. Behavior in the following two examples shown in Figure 9 is typical of what can be expected.

In the noisy $1024 \times 1024$ Marilyn Monroe image, the detected MNS otf $\hat{k}^{\dagger}(\xi, \eta)$ exhibits little change from the noiseless case on $\rho \leq \rho^{\#}=50$, although quite noticeable changes occur at high frequencies. However, such high frequency behavior is discarded in SECB intervention, and a nonlinear least squares fit on $\rho \leq \rho^{\#}$ yields parameter values for the rectified blur $\hat{k}_{L}(\xi, \eta)$ that differ very slightly from the noiseless case. What does change is the value of $K$ in the SECB constraint, which must be reduced to $K=10$, with $s=0.001$. Now, $\left\|u^{S}(., t)\right\|_{1}$ is conserved from $t=1$ to $t=0.2$, along with a fourfold increase in $\left\|u^{S}(., t)\right\|_{T V}$. The SECB image at $t=0.2$, shown as the rightmost image in the top row of Figure 9, is less sharp than the corresponding


Fig. 9. Noisy imagery. Perturbing blurred images in section 13 by low level multiplicative noise affect MNS-detected otfs at high frequencies, but cause little change on $\rho \leq \rho^{\#}=50$, and hence little change in previously constructed rectified blurs $\hat{k}_{L}(\xi, \eta)$. However, SECB regularization parameter $K$ must be reduced and continuation terminated further away from $t=0$ to mitigate noise-induced graininess in SECB images. Considerable improvement over MNS reconstruction is still evident.
image at $t=0.2$ in Figure 7. Nevertheless, the SECB image in Figure 9 is a very substantial improvement over the MNS image.

In the noisy $512 \times 512$ Pyramids image, one is again led to the same rectified blur $\hat{k}_{L}(\xi, \eta)$ as in the noiseless case. However, the effects of multiplicative noise are more apparent in this case. Even with $K$ reduced to the value 10 with $s=0.001$, there is more than a sevenfold increase in $\left\|u^{S}(., t)\right\|_{T V}$ as $t$ decreases from 1 to 0 . Some of that increase is undoubtedly due to noise in the reconstruction. At the same time, $\left\|u^{S}(., t)\right\|_{1}$ is conserved. The full-size image at $t=0$ is affected by noise-induced background graininness. Such graininess is reduced in the image at $t=0.05$, and becomes tolerable in the image at $t=0.1$. There is a more moderate fivefold increase in $\left\|u^{S}(., t)\right\|_{T V}$ from $t=1$ to $t=0.1$. The image at $t=0.1$ is shown as the rightmost image in the bottom row of Figure 9. While that image is less sharp than the corresponding image at $t=0$ in Figure 8, it is still noticeably sharper than the MNS image.
15. Concluding remarks. Formulating an appropriate variational principle that would result in full deblurring of images with unknown blurs is an unsolved problem in mathematical analysis. The great diversity of possible image blurs, together with the difficulty of encapsulating the subtleties of the unknown sharp image into a mathematical statement that can serve as an a priori constraint, makes it unlikely that a useful all-encompassing principle can be found. Rather, just as the study of hyperbolic equations requires a different set of mathematical tools than does the
study of elliptic equations, it is more likely that distinct classes of blurs will require distinct analytical approaches, variational or otherwise. Therefore, it should not be surprising to uncover pathological behavior in variational blind deconvolution.

The MNS procedure is partially successful on zero phase kernels, and not at all useful on kernels with phases [21]. The class $\mathbf{L}$ of blurs considered in this paper is intimately tied to diffusion processes, probability theory, and parabolic equations. The expression (17) is the Fourier transform of the Green's function for the evolution equation (26). The astonishing success of the recovery method in the above examples was unanticipated.

## REFERENCES

[1] K. A. Ames and B. Straughan, Non-Standard and Improperly Posed Problems, Academic Press, New York, 1997.
[2] R. Askey, Radial characteristic functions, MRC Report \# 1262, University of WisconsinMadison, 1973
[3] L. Bar, N. Sochen, and N. Kiryati, Semi-blind image restoration via Mumford-Shah regularization, IEEE Trans. Image Process., 15 (2006), pp. 483-493
[4] S. Bochner, Monotone funktionen, Stieltjes integrale und harmonische analyse, Math. Ann., 108 (1933), pp. 378-410.
[5] A. S. Carasso and T. Kato, On subordinated holomorphic semigroups, Trans. Amer. Math. Soc., 327 (1991), pp. 867-878.
[6] A. S. Carasso, Overcoming Hölder continuity in ill-posed continuation problems, SIAM J. Numer. Anal., 31 (1994), pp. 1535-1557.
[7] A. S. Carasso, Logarithmic convexity and the 'slow evolution' constraint in ill-posed initial value problems, SIAM J. Math. Anal., 30 (1999), pp. 479-496.
[8] A. S. Carasso, Linear and nonlinear image deblurring: A documented study, SIAM J. Numer. Anal., 36 (1999), pp. 1659-1689.
[9] A. S. Carasso, Direct blind deconvolution, SIAM J. Appl. Math., 61 (2001), pp. 1980-2007.
[10] A. S. Carasso, The APEX method in image sharpening and the use of low exponent Lévy stable laws, SIAM J. Appl. Math., 63 (2002), pp. 593-618.
[11] A. S. Carasso, APEX blind deconvolution of color Hubble space telescope imagery and other astronomical data, Optical Engineering, 45 (2006), p. 107004.
[12] T. F. Chan and J. Shen, Image Procesing and Analysis: Variational, PDE, Wavelet, and Stochastic Methods, SIAM, Philadelphia, 2005.
[13] T. F. Chan and C. K. Wong, Total variation blind deconvolution, IEEE Trans. Image Process., 7 (1998), pp. 370-375.
[14] T. F. Chan and C. K. Wong, Convergence of the alternating minimization algorithm for blind deconvolution, Linear Algebra Appl., 316 (2000), pp. 259-285.
[15] G. Fasshauer, Meshfree Approximation Methods with MATLAB, Interdisciplinary Mathematical Sciences, Vol. 6, World Scientific Publishers, Singapore, 2007.
[16] W. Feller, An Introduction to Probability Theory and its Applications, Vol. 2, 2nd ed., Wiley, New York, 1971.
[17] J. D. Gaskill, Linear Systems, Fourier Transforms, and Optics, Wiley, New York, (1978).
[18] T. Gneiting, Criteria of Pólya type for radial positive definite functions, Proc. Amer. Math. Soc., 129 (2001), pp. 2309-2317.
[19] L. He, Application and Generalizations of the Iterative Refinement Method, Ph.D. thesis, UCLA Computational and Applied Mathematics Report \# 06-34, June 2006.
[20] C. B. Johnson, Classification of electron-optical device modulation transfer function, Advances in Electronics and Electron Physics, Vol. 33B, (1972), pp. 579-584.
[21] L. Justen and R. Ramlau, A non-iterative regularization approach to blind deconvolution, Inverse Prob., 22 (2006), pp. 771-800.
[22] J. Lee and D. Sheen, F. John's stability conditions versus A. Carasso's SECB constraint for backward parabolic problems, Inverse Prob., 25 (2009), p. 055001.
[23] E. Lukacs, Characteristic Functions, 2nd ed., Griffin, London, 1970.
[24] G. Pólya, Remarks on characteristic functions, Proceedings of the First Berkeley Symposium on Mathematical Statistics and Probability, J. Neyman, ed., University of California Press, Berkeley, CA, 1949, pp. 115-123.
[25] G. Samorodnitsky and M. S. Taqqu, Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance, Chapman and Hall, New York, 1994.
[26] I. J. Schoenberg, Metric spaces and completely monotone functions, Ann. Math., 39 (1938), pp. 811-841.
[27] R. M. Trigub, A criterion for a characteristic function and a Pólya type criterion for radial functions of several variables, Theory Probab. Appl., 34 (1989), pp. 738-742.
[28] W. A. Woyczyński, Lévy processes in the physical sciences, in Lévy Processes-Theory and Applications, O. Barndorff-Nielsen, T. Mikosch, and S. Resnick eds., Birkhäuser, Boston, 2001.
[29] Y-L. You and M. Kaveh, A regularization approach to joint blur identification and image restoration, IEEE Trans. Image Process., 5 (1996), pp. 416-428.


[^0]:    *Received by the editors October 10, 2008; accepted for publication (in revised form) June 16, 2009; published electronically September 9, 2009.
    http://www.siam.org/journals/siap/70-4/73776.html
    ${ }^{\dagger}$ Mathematical and Computational Sciences Division, National Institute of Standards and Technology, Gaithersburg, MD 20899 (alfred.carasso@nist.gov).

